recent paper by Pascal Koiran

Tensor rank and commuting matrices

https://arxiv.org/abs/2006.02374v1

Matrices generally don't commute

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\nparallel$$

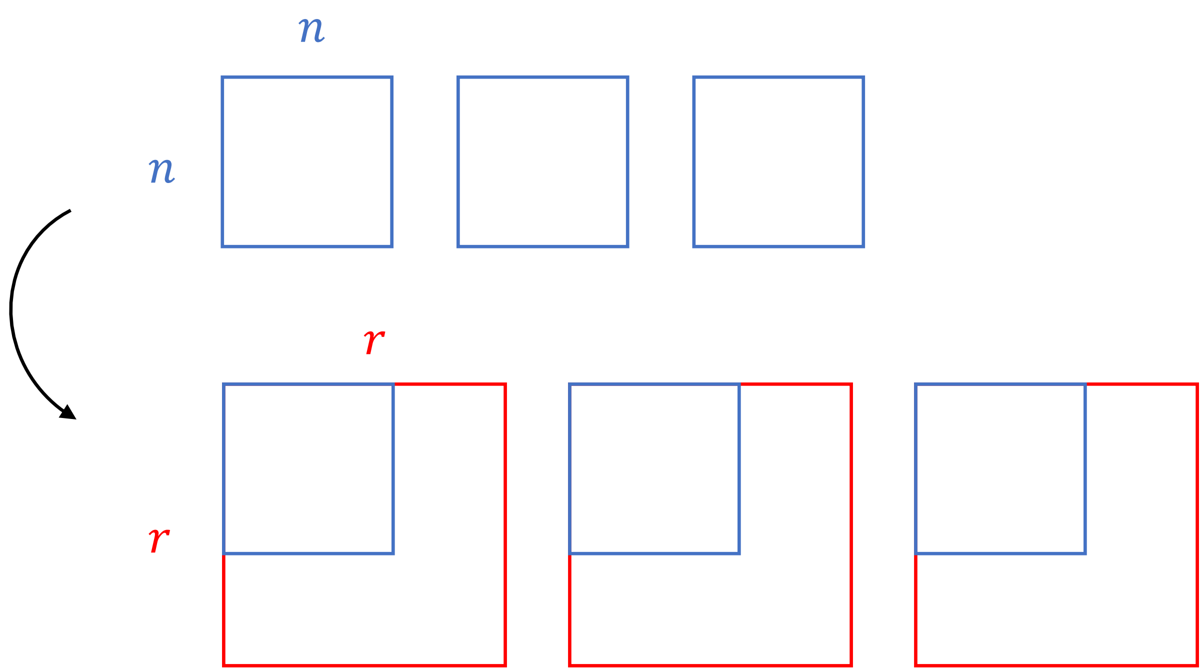
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

But we may extend them to make them commute

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

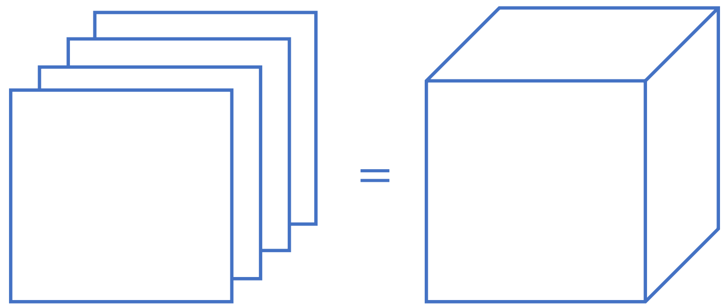
$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Problem 1: Extend matrices until they commute

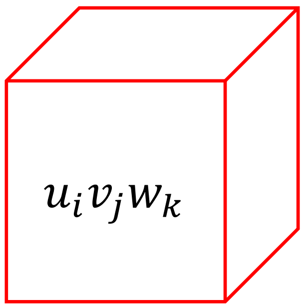


Commuting extension

Tensor

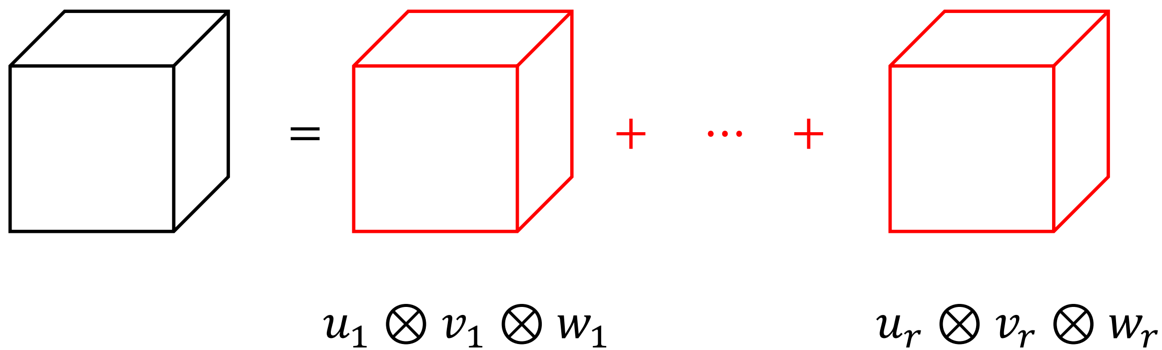


A <u>simple tensor</u> is the outer product of three vectors

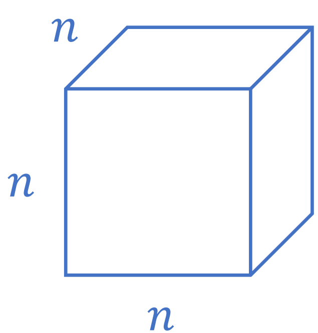


$$u \otimes v \otimes w$$

Smallest number of simple tensors that sum to T is the tensor rank



Problem 2: Random tensors have high rank; find <u>explicit</u> ones



	tensor rank
random tensor	$= c\, n^2$
explicit tensor	$\geq d\, n$

Explicit tensors of high rank, in complexity theory:

1. Explicit tensors of high rank imply lower bounds on the size of arithmetic formulas [Raz]

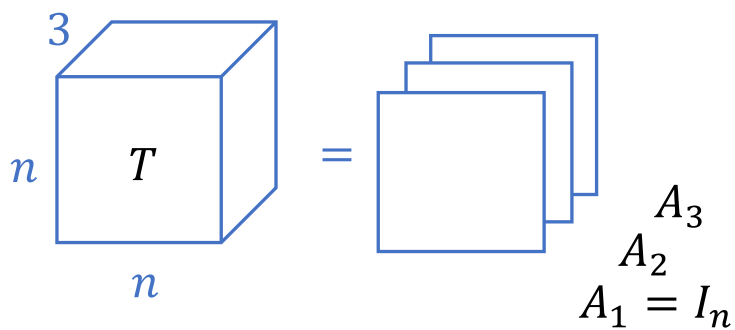
2. Tools to prove explicit tensor rank lower bounds may prove non-trivial lower bounds on the exponent of matrix multiplication

$$\omega > 2$$

Inspiration: Strassen lower bound on tensor rank

Theorem 1. (Strassen)

Given:



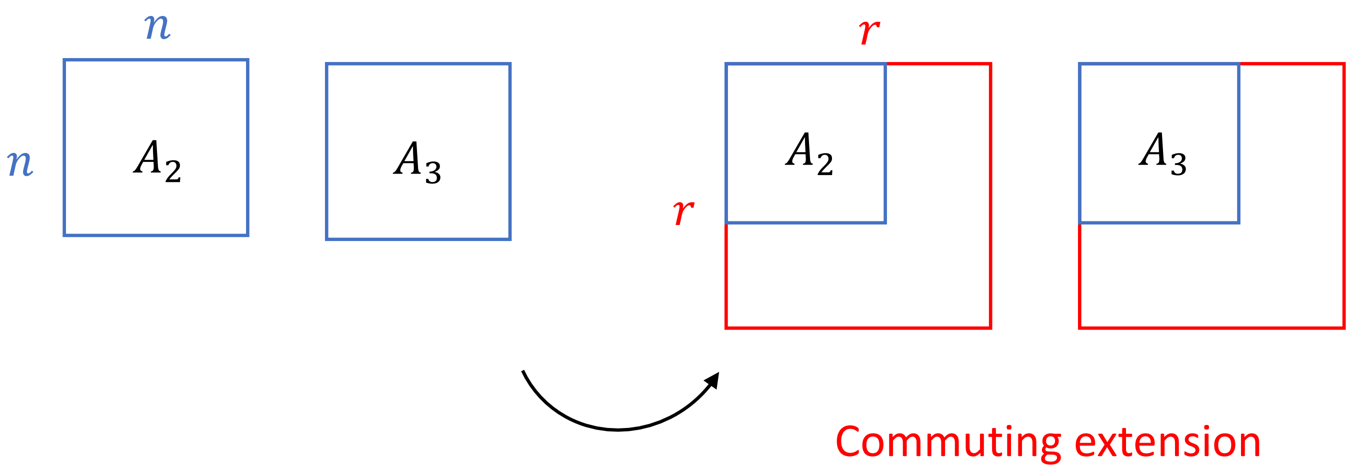
Then:

$$\mathrm{rank}(T) \geq n + \frac{1}{2}\mathrm{rank}(A_2A_3 - A_3A_2)$$

Koiran interpretation: Lower bound on commuting extensions

Theorem 2. (Strassen, Koiran)

Given:

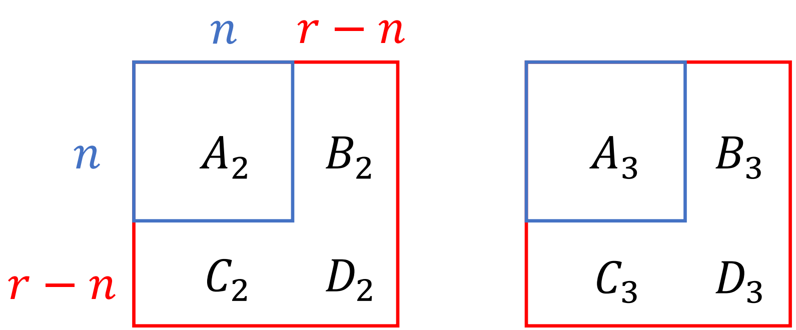


Then:

$$r \geq n + \frac{1}{2}\operatorname{rank}(A_2 A_3 - A_3 A_2)$$

Proof of the Koiran interpretation

Given: Commuting extension

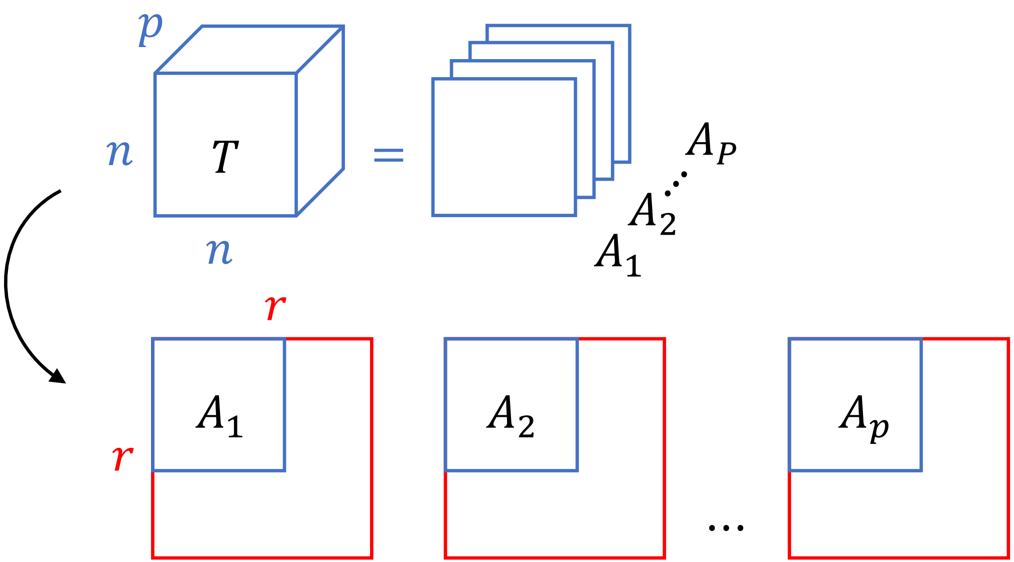


1. Commutativity: $A_2A_3 - A_3A_2 = B_3C_2 - B_2C_3$
2. $2(r-n) \geq \mathrm{rank}(B_3C_2 - B_2C_3)$
3. $r \geq n + \frac{1}{2}\mathrm{rank}(A_2A_3 - A_3A_2)$ ∎

Matrix extensions characterize tensor rank

Theorem 3. (Koiran)

Given:



Commuting and diagonalizable extension

Then:

$$\mathrm{rank}(T) + n \geq r_{\min} \geq \mathrm{rank}(T)$$

Some words on commuting and diagonalizable matrices

$B_1, \dots, B_p \quad n \times n$ matrices

- Simultaneously diagonalizable $\Longrightarrow$ Diagonalizable

$$B_k = V^{-1} D_k V \qquad\qquad B_k = V_k^{-1} D_k V_k$$

- Simultaneously diagonalizable $\Longrightarrow$ Commuting

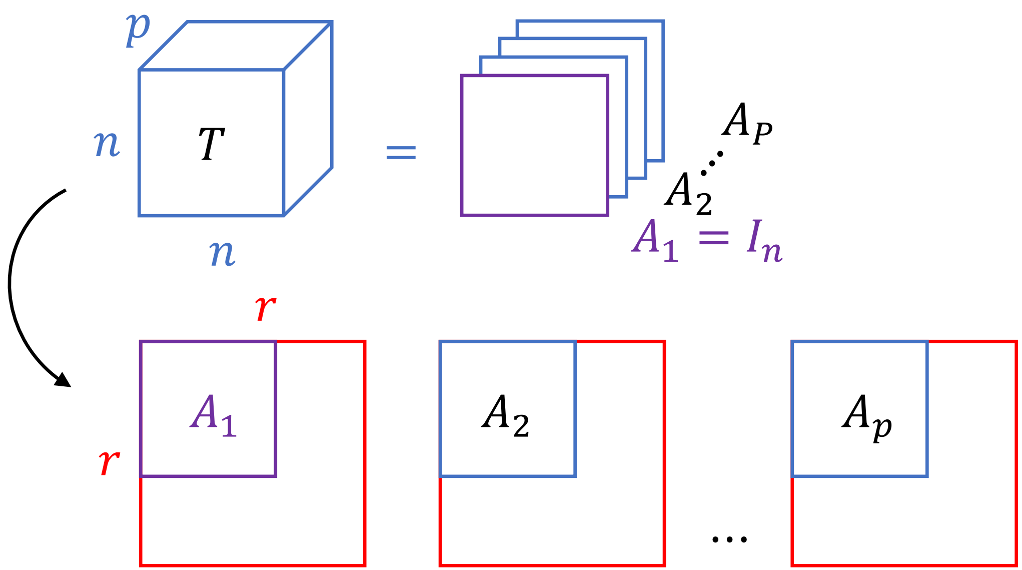
$$\begin{aligned} B_k B_l &= V^{-1} D_k V V^{-1} D_l V \\ &= V^{-1} D_k D_l V \\ &= V^{-1} D_l D_k V \end{aligned}$$

- Simultaneously diagonalizable $\Longleftrightarrow$ Commuting and diagonalizable

Simpler version of the theorem

Theorem 3'. (Koiran)

Given:



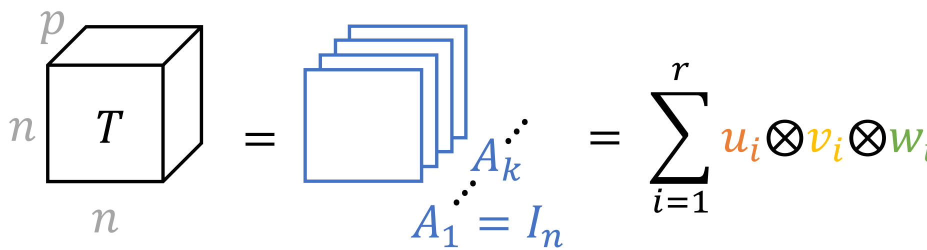
Commuting and diagonalizable extension

Then:

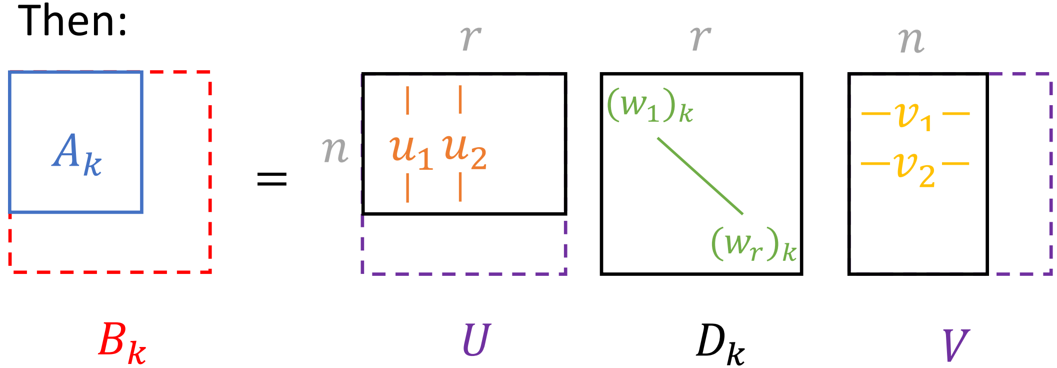
$r_{\min} = \text{rank}(T)$

Proof sketch of $r_{\min} \leq \mathrm{rank}(T)$

Given: T as a sum of $r = \mathrm{rank}(T)$ many simple tensors



Then:



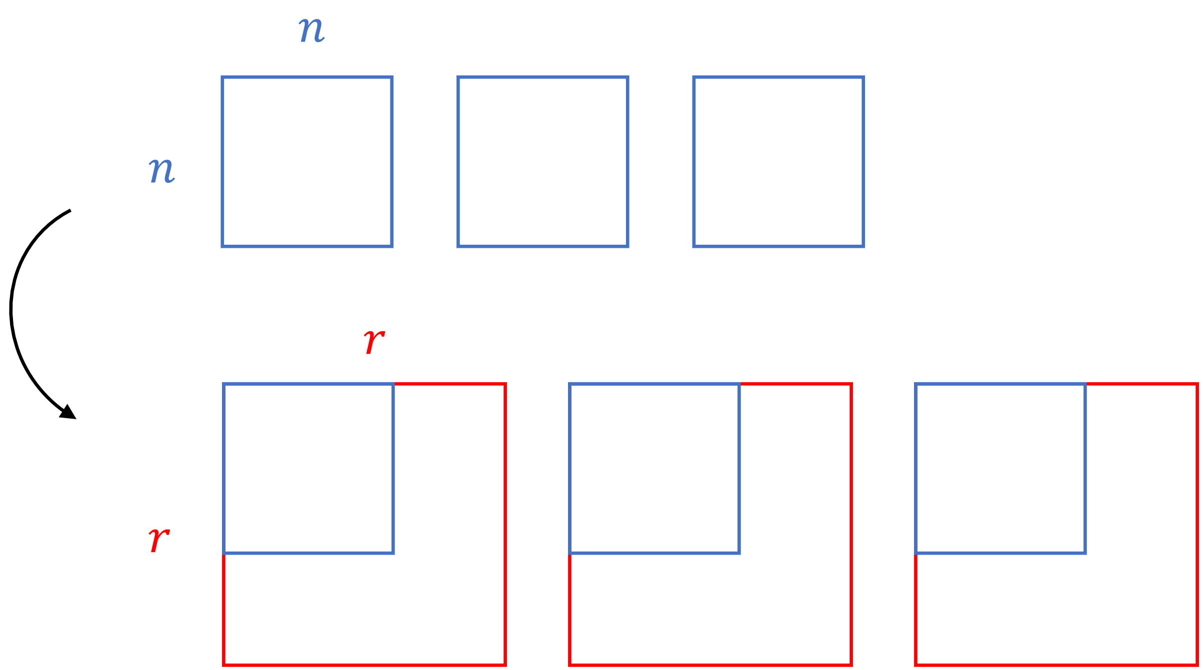
B_k simultaneously diagonalizable:

$B_1 = I_r$

$B_k = B_1^{-1} B_k = (V^{-1} D_1^{-1} U^{-1})(U D_k V) = V^{-1} D_1^{-1} D_k V$

■

Problem 1: Extend matrices until they commute



Commuting extension

(or: Commuting and diagonalizable extension)

Example: W-tensor

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Example: matrix multiplication tensor

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1&0&0&0&1&1&1\\0&1&0&0&0&1&0\\2&-2&0&0&2&0&2\\-2&1&0&0&-2&-1&-2\\-1&1&0&0&-1&0&-1\end{pmatrix} \quad \begin{pmatrix}0&0&1&0&0&1&-1\\0&0&0&1&1&1&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&-1&-1&-1&0\\0&0&-1&1&1&0&1\end{pmatrix} \quad \begin{pmatrix}0&0&0&0&-\frac{1}{2}&0&-1\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&\frac{1}{2}&0&0\\-1&0&0&-1&-\frac{3}{2}&-1&-1\\1&0&0&0&1&1&1\\1&0&0&1&2&1&2\end{pmatrix}$$

Some questions

How can we lower bound commutative extensions?

commutative and diagonalizable extensions?