Asymptotic Spectrum Duality

Part 2

Jeroen Zuiddam

1. Shannon capacity and duality

independence number $\alpha(G)$

strong product $G \boxtimes H$

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Duality

Strassen 86

Z 18

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Duality

k	$\Theta(C_k)$		
5	2.23	2.23	Shannon 56, Lovasz 79
7	3.25	3.31	Polak-Schrijver 18
9	4.32	4.36	Baumert et al. 71
11	5.28	5.38	"
13	6.27	6.40	"
15	7.30	7.41	de Boer, Buys, Z 24

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Asymptotic behaviour

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$\leq \quad \rightsquigarrow \quad \lesssim$

(cohomomorphism order)

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$G \lesssim H \;:\Longleftrightarrow\; G^{\boxtimes n} \leq H^{\boxtimes (n+o(n))}$

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- Lovasz theta function
- fractional Haemer bound
- fractional orth. rank
- fractional clique cover number

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- Compactness
- Complexity/Universality Anna, Jim
- Convexity Péter Vrana
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New lower bounds for Θ

Algebraic geometry

Hilbert

Algebraic geometry	"Asymptotic" real geometry
Hilbert	Strassen

Algebraic geometry

Hilbert

$$\mathbb{C}[x_1, \ldots, x_n] \supseteq x_1^2 + x_2 x_3 + 1$$

"Asymptotic" real geometry

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$I \subseteq \sqrt{I}$

"Asymptotic" real geometry

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proper maximal ideals

$(x_1 - a_1, x_2 - a_2, \dots)$

evaluation in $\mathbb{C}^n$

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$$\mathbb{N}[G_1,\dots,G_n] \ni G_1^{\boxtimes 2} \sqcup G_2 \boxtimes G_3 \sqcup \bullet$$

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"maximal Strassen preorders"

$$\phi \in X : G_1 \mapsto a_1, G_2 \mapsto a_2, \dots$$

evaluation in $\mathbb{R}^n_{\geq 0}$

Duality

$\mathcal{G}$:= semiring of graphs with $\sqcup$, $\boxtimes$, $\underset{\underset{0}{\shortparallel}}{E_0}$, $\underset{\underset{1}{\shortparallel}}{E_1}$

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$$\phi(G \sqcup H) = \phi(G)+\phi(H),\ G \leq H \implies \phi(G) \leq \phi(H)\}.$$

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Convexity Theorem $X \longleftrightarrow X_{\text{prob}}$ convex!

2. Proof of Duality Theorem

Strassen preorder Preorder $\lesssim$ such that:

(1) $\forall G,H,K,\ G \lesssim H \Rightarrow GK \lesssim HK$ and $G \sqcup K \lesssim H \sqcup K$

(2) $\forall n,m \in \mathbb{N},\quad n \leq m$ in $\mathbb{N} \iff n \lesssim m$ (i.e. $E_n \lesssim E_m$)

(3) $\forall G \neq 0, \exists n,\ 1 \lesssim G \lesssim n$ "Archimedean"

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Lemma $\underset{\sim}{\leqslant}$ is closed (i.e. $\underset{\approx}{\leqslant} = \underset{\sim}{\leqslant}$)

Lemma If $\curlywedge$ is closed, then $\forall\, G, H, K$

(1) $G \sqcup K \curlywedge H \sqcup K \Rightarrow G \curlywedge H$ additive cancellation

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(3) $\forall n,\ nG \preccurlyeq nH \sqcup K \Rightarrow G \preccurlyeq H$ gap property

Lemma If $\lesssim$ is closed, then $\forall\, G, H, K$

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Proof: $U \leqslant' V :\Leftrightarrow \exists W,\ U \sqcup WG \leqslant V \sqcup WH$

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Lemma (One-step extension) If $\lesssim$ is closed and $G \not\gtrsim H$, then $\exists$ Strassen preorder $\lesssim\ \subseteq\ \lesssim'$ such that $H \lesssim' G$

Proof: $U \lesssim' V :\Leftrightarrow \exists W,\ U \sqcup WG \lesssim V \sqcup WH$

Lemma (Maximal extension) Every Strassen preorder $\lesssim$ has a maximal Strassen extension (which is then closed and total).

Lemma If $\lesssim$ is closed, then $\forall\, G, H, K$

(1) $G \sqcup K \lesssim H \sqcup K \Rightarrow G \lesssim H$ — additive cancellation

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Lemma (Maximal extension) Every Strassen preorder $\lesssim$ has a maximal Strassen extension (which is then closed and total).

Proof: Zorn. Maximal $\Rightarrow$ closed. Maximal & closed $\Rightarrow$ total.

Totality

Let $\precsim$ be a Strassen preorder

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Lemma.
- $\sigma \le \rho$

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Lemma. • $\sigma \leq \rho$

• ρ is sub-add, sub-mult, norm., $\precsim$-mon.

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Theorem If $\lesssim$ is total, then $\sigma = \rho$ and is thus a $\lesssim$-mon. hom.

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Proof Suppose $\sigma(G) < \frac{n}{m} < \rho(G)$

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Fractional rank $\rho(G) := \inf \left\{ \frac{n}{m} : mG \leqslant n \right\}$

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Lemma. • $\sigma \leq \rho$

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Theorem If $\leqslant$ is total, then $\sigma = \rho$ and is thus a $\leqslant$-mon. hom.

Proof Suppose $\sigma(G) < \frac{n}{m} < \rho(G)$

Then $n \not\leqslant mG$

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Let $\preccurlyeq$ be a Strassen preorder

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- $\sigma \leq \rho$
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Theorem If $\preccurlyeq$ is total, then $\sigma = \rho$ and is thus a $\preccurlyeq$-mon. hom.

Proof Suppose $\sigma(G) < \frac{n}{m} < \rho(G)$

Then $n \not\preccurlyeq mG$

So $mG \preccurlyeq n$ (totality)

Totality

Let $\precsim$ be a Strassen preorder

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Theorem If $\precsim$ is total, then $\sigma = \rho$ and is thus a $\precsim$-mon. hom.

Proof Suppose $\sigma(G) < \frac{n}{m} < \rho(G)$

Then $n \not\precsim mG$

So $mG \precsim n$ (totality) so $\rho(G) \leq \frac{n}{m}$ $\lightning$ $\square$.

Duality Theorem $\quad G \lneqq H \iff \forall \phi \in X,\ \phi(G) \leq \phi(H).$

Duality Theorem $G \lesssim H \iff \forall \phi \in X,\ \phi(G) \leq \phi(H)$.

Proof $\Rightarrow$ is easy

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Proof $\Rightarrow$ is easy

$\Leftarrow$ Suppose $G \not\lesssim H$. (To prove: $\exists \phi \in X,\ \phi(G) \not\leq \phi(H)$.)

Duality Theorem $G \precsim H \iff \forall \phi \in X,\ \phi(G) \leq \phi(H)$.

Proof $\Rightarrow$ is easy

$\Leftarrow$ Suppose $G \not\precsim H$. (To prove: $\exists \phi \in X,\ \phi(G) \not\leq \phi(H)$.)

By the gap property, $\exists n$, $nG \not\precsim nH + 1$

Duality Theorem $G \lesssim H \iff \forall \phi \in X,\ \phi(G) \leq \phi(H)$.

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Extend $\lesssim$ to a Strassen preorder $\lesssim'$ such that $nG + 1 \lesssim' nH$

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Extend $\lesssim$ to a Strassen preorder $\lesssim'$ such that $nG + 1 \lesssim' nH$

Extend $\lesssim'$ to a (maximal, closed) total Strassen preorder $\lesssim''$.

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$\mathcal{G} = \{ \text{graphs} \}$,

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n-tuples of type P

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- bijection $X \leftrightarrow X_{prob}$

$X \longrightarrow X_{prob}$

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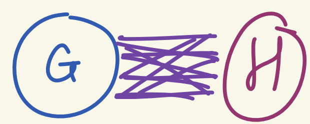
graph join



Formally verified in Lean:

- Strassen's asymptotic spectrum duality [S], for graphs [Z] ✓✓
- Asymptotic spectrum distance [dBBZ], Limit theorem [BPZ] ✓✓
- Universality [Anna, Jim] ✓
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relies on

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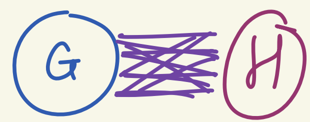

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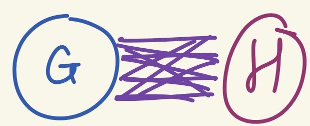

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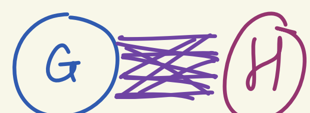

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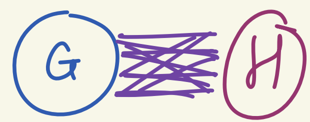

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