Asymptotic Spectrum Duality

Asymptotic spectrum duality: proof.

1. Shannon capacity
2. Duality
3. Proof
   - 3.1 Strassen preorder
   - 3.2 Closure
   - 3.3 Extension
   - 3.4 Totality
4. Compactness

1. Shannon capacity and duality

$G \leq H :\Leftrightarrow \exists$ cohom. $G \to H$

$\alpha(G) :=$ ind. nr.

$\alpha(C_5) = |\{0, 2\}|$

$G \boxtimes H$ strong product

$\Theta(G) := \sup_n \alpha(G^{\boxtimes n})^{1/n}$

$\alpha(C_5^{\boxtimes 2}) = |\{(0,0), (1,2), (2,4), (3,1), (4,3)\}|$

k	$\Theta(C_k)$			
5	2 23	2.23	2.5	[Shannon 56, Lovasz 79]
7	3.25	3.31	3.5	[PS 18]
9	4.32	4.36	4.5	[Baumert et al. 71]
11	5.28	5.38	5.5	"
13	6.27	6.40	6.5	"
15	7.30	7.41	7.5	[de Boer, Buys, Zuiddam 24]

constant "C_g" in Tao's list of important constants

Asymptotic behaviour

"Asymptotic spectrum"

Constructing elements of X

- Lovasz theta function
- fractional Haemers bound
- fractional orth. rank
- fractional clique cover number

$\alpha(G) \rightsquigarrow \Theta(G) \longleftrightarrow X$

$\leq \rightsquigarrow \lesssim$

Duality

Structure of X

- Compactness
- Convexity Péter Vrana
- Complexity Anna
- Distance David, Pjotr, me

Algebraic geometry

Hilbert

$\mathbb{C}[x_1, \_, x_n] \ni 3 + x_1^2 + x_2 x_3$

$I = (p_1 = 0, p_2 = 0, \dots)$ eqs.

$Z(I) = \{a : p_1(a) = 0, p_2(a) = 0, \dots\}$

$\subseteq \mathbb{C}^n$ zero locus

- $I(Z(I)) = \sqrt{I}$
- $q = 0$ on $Z(I)$

$\Rightarrow \exists k, q^k \in I$

"Asymptotic" real geometry

Strassen

$\mathbb{N}[G_1, G_2, \dots, G_n] \ni E_3 \sqcup G_1^{\boxtimes 2} \sqcup G_2 \boxtimes G_3$

$I = (p_1 \leq q_1, p_2 \leq q_2, \dots) = \leq$ ineqs.

$N(I) = \{a : p_1(a) \leq q_1(a), p_2(a) \leq q_2(a), \dots\}$

$\subseteq \mathbb{R}^n_{\geq 0}$ nonnegative locus $= X$

- $I(N(I)) = \lesssim$
- $q_1 \leq q_2$ on $N(I)$

$\Rightarrow q_1^k \leq q_2^{k+o(k)}$ in I

Algebraic geometry

Hilbert

$$I \subseteq \sqrt{I} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$$

proper maximal ideals

$$(x_1 - a_1, x_2 - a_2, \ldots)$$

evaluation in $\mathbb{C}^n$

"Asymptotic" real geometry

Strassen

$$\leq \;\subseteq\; \lesssim \;=\; \bigcap_{\preceq \,\supseteq\, \leq} \preceq$$

"maximal Strassen preorders"

$$\phi \in X : G_1 \mapsto a_1, G_2 \mapsto a_2, \ldots$$

evaluation in $\mathbb{R}^n_{\geq 0}$

Challenge: Find X !

2. Duality

Notation

(commutative) — isomorphism classes of

- $(\mathcal{R}, +, \cdot, 0, 1)$ = semiring of graphs with $\sqcup, \boxtimes, \underset{\substack{\| \\ 0}}{E_0}, \underset{\substack{\| \\ 1}}{E_1}$
- $G \leq H$ iff cohomomorphism $G \to H$
- $\alpha(G) = \max\{n \in \mathbb{N} : n \leq G\}$.
- $\lesssim$ asymptotic cohom. $\quad G \lesssim H \;:\Leftrightarrow\; G^k \leq 2^{o(k)} H^k$

$$\Leftrightarrow \; G^k \leq H^{k+o(k)}$$

Asymptotic spectrum

$$X = \{\phi : R \to \mathbb{R}_{\geq 0} : \; \phi(0)=0,\ \phi(1)=1,\ \forall G,H \;\; \phi(GH) = \phi(G)\phi(H),$$
$$\phi(G+H) = \phi(G)+\phi(H),\ G \leq H \Rightarrow \phi(G) \leq \phi(H)\}.$$

Main Theorem $\forall G,H,\quad G \lesssim H \iff \forall \phi \in X, \ \phi(G) \leq \phi(H)$

$(\Rightarrow)$ $G^k \leq 2^{o(k)} H^k \Rightarrow \phi(G^k) \leq \phi(2^{o(k)} H^k) \Rightarrow \phi(G)^k \leq 2^{o(k)} \phi(H)^k \Rightarrow \phi(G) \leq \phi(H)$

$\square$

Main Theorem $\forall G, H,\ G \lesssim H \iff \forall \phi \in X, \phi(G) \leq \phi(H)$

Corollary $\forall G,\ \Theta(G) = \min_{\phi \in X} \phi(G)$

Proof $\leq$ $m \leq H \Rightarrow m \leq \phi(H) \Rightarrow \alpha(H) \leq \phi(H)$

$$\Rightarrow \alpha(G^n) \leq \phi(G^n) = \phi(G)^n \Rightarrow \sup_n \alpha(G^n)^{1/n} \leq \phi(G).$$

$\geq$

- Suppose $\forall \phi \in X,\ \phi(G) \geq r$
- $\phi(G^n) \geq \lfloor r^n \rfloor$ make RHS integer
- $G^n \gtrsim \lfloor r^n \rfloor$ duality theorem
- $G^{nm + o_n(m)} \geq \lfloor r^n \rfloor^m$ definition of $\lesssim$
- $\Theta(G^n) \geq \lfloor r^n \rfloor^{1/1+o_n(1)} \to \lfloor r^n \rfloor$ $m \to \infty$
- $\Theta(G) \geq \lfloor r^n \rfloor^{1/n} \to r$ $n \to \infty$.

3. Proof.

3.1 Strassen preorder

Commutative semiring $\mathcal{R}$

Preorder

Reflexive and transitive relation $\leqslant$

Strassen preorder

Preorder $\leqslant$ such that:

(1) $\forall G, H, K,\ G \leqslant H \Rightarrow GK \leqslant HK$ and $G + K \leqslant H + K$

(2) $\forall n, m \in \mathbb{N},\ n \leq m$ in $\mathbb{N} \Longleftrightarrow n \leqslant m$ (i.e. $E_n \leqslant E_m$)

(3) $\forall G \neq 0, \exists n,\ 1 \leqslant G \leqslant n$ "Archimedean"

Example

Cohomomorphism order on graphs

3.2 Closure

Closure For any Strassen preorder $\leqslant$ define the **closure** $\underset{\sim}{\leqslant}$ by

$$G \underset{\sim}{\leqslant} H \;:\Longleftrightarrow\; G^n \leqslant 2^{o(n)} H^n$$

Closed We call $\leqslant$ **closed** if $\underset{\sim}{\leqslant} = \leqslant$.

Lemma $\underset{\sim}{\leqslant}$ is closed. (i.e. $\underset{\approx}{\leqslant} = \underset{\sim}{\leqslant}$)

Lemma If $\leqslant$ is closed, then $\forall\, G, H, K$

- $G^n K \leqslant H^n K$
- $G^n \leqslant 2^{o(n)} H^n$
- $G \underset{\sim}{\leqslant} H$

(1) $G + K \leqslant H + K \Rightarrow G \leqslant H$ — additive cancellation

(2) $K \neq 0$ and $GK \leqslant HK \Rightarrow G \leqslant H$ — multiplicative cancellation

(3) $\forall n,\ nG \leqslant nH + K \Rightarrow G \leqslant H$ — gap property

$\exists n,\ nG \not\leqslant nH + 1 \Leftarrow G \not\leqslant H$

3.3 Extension

Lemma (One-step extension) Let $\lesssim$ be closed. Suppose $G \not\lesssim H$.
Then $\exists$ Strassen preorder $\lesssim \subseteq \lesssim'$ such that $H \lesssim' G$

Proof Define $\lesssim'$ by: $U \lesssim' V :\Leftrightarrow \exists W,\ U + WG \lesssim V + WH$

Check:

- extension: $\lesssim \subseteq \lesssim'$ (pick $W = 0$)
- $H \lesssim' G$ (pick $W = 1$, $H + G \lesssim G + H \Rightarrow H \lesssim' G$)
- Strassen preorder (exercise)

e.g. to prove $n \lesssim' m \Rightarrow n \leq m$ in $\mathbb{N}$:

$\exists W,\ n + WG \lesssim m + WH$. Suppose $n > m$.
Then $1 + WG \lesssim WH$ (additive cancellation for closed preorder)
So $WG \lesssim WH$. So $G \lesssim H$ (multiplicative cancellation $\not\leftarrow$) $\square$

Lemma (Maximal extension) Every Strassen preorder has a maximal Strassen extension (which is then closed and total).

Proof Let $\lesssim$ be a Strassen preorder.

We will use Zorn's lemma.

Let $\mathcal{P}$ be the set of all Strassen preorders extending $\lesssim$ ordered by inclusion.

Let $\mathcal{C} \subseteq \mathcal{P}$ be any chain. Then $\bigcup \mathcal{C}$ is in $\mathcal{P}$ and contains all preorders in $\mathcal{C}$.

Then by Zorn's lemma $\mathcal{P}$ contains a maximal element.

Maximal $\Rightarrow$ closed (otherwise the closure is larger).

Maximal & closed $\Rightarrow$ total (otherwise can do one-step extension) $\square$.

3.4 Totality

Let $\lesssim$ be a Strassen preorder

Fractional rank $\rho(G) := \inf \{ \frac{n}{m} : mG \lesssim n \}$

Fractional subrank $\sigma(G) := \sup \{ \frac{n}{m} : n \lesssim mG \}$

Lemma. • $\sigma \leq \rho$

• ρ is sub-add, sub-mult, norm., $\lesssim$-mon.

• σ is super-add, super-mult, norm., $\lesssim$-mon.

Theorem If $\lesssim$ is total, then $\sigma = \rho$ and is thus a $\lesssim$-mon. hom.

Proof Suppose $\sigma(G) < \frac{n}{m} < \rho(G)$

Then $n \not\lesssim mG$

So $mG \lesssim n$ (totality) so $\rho(G) \leq \frac{n}{m}$ $\lightning$ $\square$.

Theorem $G \lesssim H \iff \forall \phi \in X,\ \phi(G) \le \phi(H)$.

Proof $\Rightarrow$ is easy

$\Leftarrow$ Suppose $G \not\lesssim H$. (To prove: $\exists \phi \in X,\ \phi(G) \not\le \phi(H)$.)

By the gap property, $\exists n$, $nG \not\lesssim nH + 1$

Extend $\lesssim$ to a Strassen preorder $\lesssim'$ such that $nG+1 \lesssim' nH$

Extend $\lesssim'$ to a (maximal, closed) total Strassen preorder $\lesssim''$.

Then $nG + 1 \lesssim'' nH$

Let $\phi = \kappa = \rho$ be the corresponding $\lesssim''$-mon. hom, $\phi \in X$.

Then $\phi(nG+1) \le \phi(nH)$ so $\phi(G) + \frac{1}{n} \le \phi(H)$ so $\phi(G) < \phi(H)$ $\square$.

4. Compactness

Topology • $X = \{\phi \cdot \text{mon hom.}\} \subseteq \mathbb{R}_{\geq 0}^{\mathcal{R}}$

• $\forall G \in \mathcal{R}$, $ev_G : X \to \mathbb{R}_{\geq 0} : \phi \mapsto \phi(G)$

• Give X coarsest topology making all ev_G continuous.

Theorem. X is compact

Proof $X \subseteq \underbrace{\prod_G \underbrace{[0, |G|]}_{\text{compact}}}_{\text{compact (Tychonoff)}} \subseteq \mathbb{R}_{\geq 0}^{\mathcal{R}}$. To prove: X is closed.

$X = X_{mon} \cap X_{norm} \cap X_{submult} \cap \cdots$ intersection of closed sets

e.g. $X_{norm} := ev_0^{-1}(\{0\}) \cap ev_1^{-1}(\{1\})$ closed $\square$