Asymptotic spectra: Theory, applications and extensions

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Strassen, in his seminal 1969 paper

"Gaussian Elimination is Not Optimal"

sent a clear message to the scientific community:

Natural, obvious and centuries-old methods for solving important computational problems may be far from the fastest.

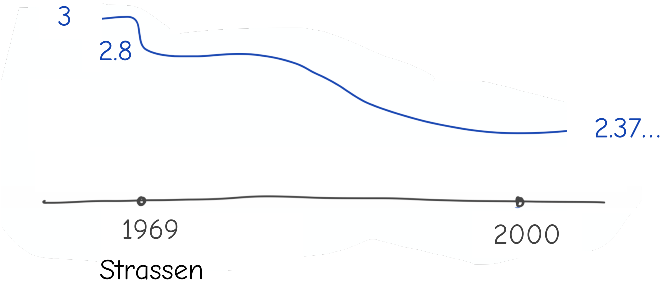
"Gaussian elimination is not optimal"

- multiplying $n \times n$ matrices
- inverting $n \times n$ matrices
- solving a system of n linear equations in n unknowns
- computing the determinant of an $n \times n$ matrix

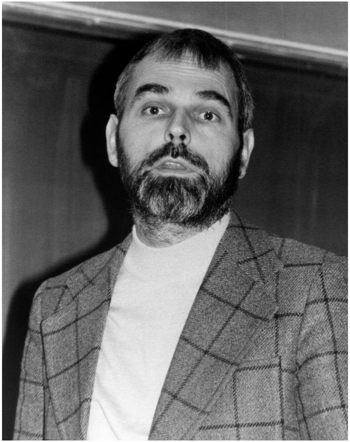
Strassen proved that the obvious $\mathcal{O}(n^3)$ algorithm for these (equivalent) problems is far from optimal

by designing a new one which takes only $\mathcal{O}(n^{2.8})$ operations

The possibility of obtaining even faster algorithms for these central problems set Strassen and many other computer scientists on a quest to obtain them, with the current record below $\mathcal{O}(n^{2.4})$



The quest to understand the matrix multiplication exponent ω is still raging on.

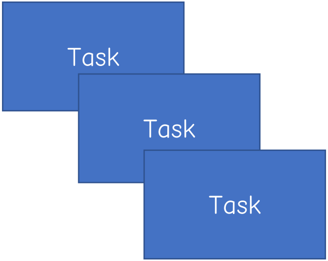


Decades later (1986–1991) Strassen developed his theory of

Asymptotic Spectra.

While motivated by trying to understand the complexity of matrix multiplication, this theory is far more general

leading to a broader framework that suits other problems and settings.



Central in this theory of asymptotic spectra:

What is the cost of a task if we have to perform it many times?

Arises in numerous parts of mathematics, physics, economics and computer science

- matrix multiplication
- circuit complexity (with Robert Robere)
- direct-sum problems
- Shannon capacity

Survey (with Avi Wigderson)
jeroenzuiddam.nl


GEOMETRIC

1. Shannon capacity
2. The asymptotic spectrum of graphs
3. The asymptotic spectrum duality theorem
4. Consequences and new directions

1. Shannon capacity

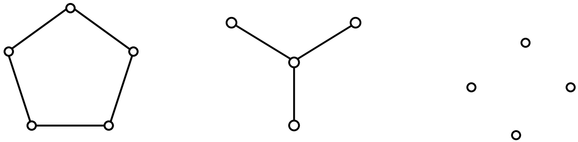
Measures amount of information that can be transmitted over a communication channel.

Understanding it has been an open problem in information theory and graph theory since its introduction by Shannon in 1956.

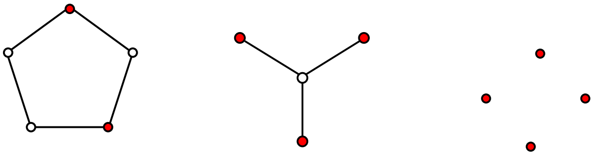
Translates to graph theoretical problem:

channel	graph
protocol	independent set
repeating	strong product

Graph



Independent set



Independence number

$\alpha(C_5) = 2$ $\alpha(S_3) = 3$ $\alpha(E_4) = 4$

Strong product

$G \boxtimes H$

$V(G \boxtimes H) = V(G) \times V(H)$

Adjacency matrix formulation:

The adjacency matrix of $G \boxtimes H$ is the tensor product of those of G and H

Independence number is super-multiplicative

$$\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$$

Example

$$\alpha(C_5) = 2$$

$$\alpha(C_5^{\boxtimes 2}) = 5$$

Shannon capacity

$$\Theta(G) = \sup_n \alpha(G^{\boxtimes n})^{1/n}$$

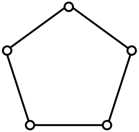
Example

$\Theta(C_5) = \sqrt{5}$ (Lovász)

$3.2578 \leq \Theta(C_7) \leq 3.3177$ (Schrijver–Polak)

How to upper bound α (and Θ)?

Matrix rank (Haemers bound)

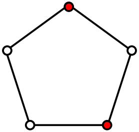


$$\begin{bmatrix} 1 & \ast & 0 & 0 & \ast \\ \ast & 1 & \ast & 0 & 0 \\ 0 & \ast & 1 & \ast & 0 \\ 0 & 0 & \ast & 1 & \ast \\ \ast & 0 & 0 & \ast & 1 \end{bmatrix}$$

1 on the diagonal

0 on the non-edges

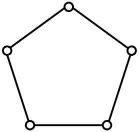
Every independent set gives an identity sub-matrix



$$\begin{bmatrix} 1 & \ast & 0 & 0 & \ast \\ \ast & 1 & \ast & 0 & 0 \\ 0 & \ast & 1 & \ast & 0 \\ 0 & 0 & \ast & 1 & \ast \\ \ast & 0 & 0 & \ast & 1 \end{bmatrix}$$

Independence number α is at most rank of any such matrix (and Θ too)

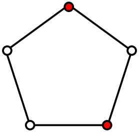
Largest eigenvalue (Lovász theta function)



$$\begin{bmatrix} 1 & * & 1 & 1 & * \\ * & 1 & * & 1 & 1 \\ 1 & * & 1 & * & 1 \\ 1 & 1 & * & 1 & * \\ * & 1 & 1 & * & 1 \end{bmatrix}$$

1 on the diagonal
1 on the non-edges

Every independent set gives an all-ones sub-matrix



$$\begin{bmatrix} 1 & * & 1 & 1 & * \\ * & 1 & * & 1 & 1 \\ 1 & * & 1 & * & 1 \\ 1 & 1 & * & 1 & * \\ * & 1 & 1 & * & 1 \end{bmatrix}$$

Independence number is at most largest eigenvalue of such matrix (and Θ too)

Q: How good are the Haemers and Lovász bounds?

2. The asymptotic spectrum of graphs

Models graphs as points in real space

Defined as the set X of all maps $F : \{\text{graphs}\} \to \mathbb{R}$ that are

1. additive under $\sqcup$
2. multiplicative under $\boxtimes$
3. monotone under cohomomorphism
4. normalized to 1 on the graph with one vertex E_1

Graphs as real points: $G \mapsto (F(G))_{F \in X}$

Examples of elements of X

- Lovász theta function ϑ
- fractional Haemers bound (Bukh–Cox)
- fractional clique cover number

3. Duality theorem

Recall that

- Shannon capacity is a maximization: $\Theta(G) = \sup_n \alpha(G^{\boxtimes n})^{1/n}$
- Lovász theta gives upper bound: $\Theta(G) \leq \vartheta(G)$

Lemma

Every $F \in X$ gives upper bound: $\Theta(G) \leq F(G)$

Q: Are the upper bounds from $F \in X$ powerful enough to reach Θ?

Duality Theorem ("yes", Zuiddam)

Shannon capacity is a minimization: $\Theta(G) = \min_{F \in X} F(G)$

Q: Is the duality theorem non-trivial?

Duality Theorem $\Theta(G) = \min_{F \in X} F(G)$

Conjecture (Shannon) $\Theta \in X$

Theorem (Haemers)
There are G, H for which $\Theta(G \boxtimes H) > \Theta(G)\Theta(H)$

Theorem (Alon)
There are G, H for which $\Theta(G \sqcup H) > \Theta(G) + \Theta(H)$

Corollary $\Theta \notin X$

Q: How is the duality theorem proven?

Duality Theorem $\Theta(G) = \min_{F \in X} F(G)$

More General Duality Theorem (Zuiddam)
$G^{\boxtimes n} \to H^{\boxtimes (n+o(n))}$ iff $F(G) \leq F(H)$ for all $F \in X$

Ideas:

- Real geometry, Positivstellensatz
- Kadison–Dubois representation theorem
- Extension of Linear Programming Duality

4. Consequences and new directions

Theorem ("Additivity if and only if multiplicativity", Holzman)
For any graphs G, H the following are equivalent:

(i) $\Theta(G \sqcup H) = \Theta(G) + \Theta(H)$
(ii) $\Theta(G \boxtimes H) = \Theta(G)\Theta(H)$
(iii) There is $F \in X$ such that $F(G) = \Theta(G)$ and $F(H) = \Theta(H)$

Proof (i) $\Rightarrow$ (iii)
Let $F \in X$ such that $\Theta(G \sqcup H) = F(G \sqcup H)$
Then $\Theta(G) + \Theta(H) = \Theta(G \sqcup H) = F(G \sqcup H) = F(G) + F(H)$
Always: $\Theta(G) \leq F(G)$ and $\Theta(H) \leq F(H)$
Therefore $\Theta(G) = F(G)$ and $\Theta(H) = F(H)$ $\square$

(iii) $\Rightarrow$ (i)

$$\Theta(G) + \Theta(H) \leq \Theta(G \sqcup H)$$
$$\leq F(G \sqcup H) = F(G) + F(H) = \Theta(H) + \Theta(H) \quad \square$$

Example ("Theorems of Haemers and Alon are equivalent")

$$\Theta(G \boxtimes H) > \Theta(G)\Theta(H) \quad \text{iff} \quad \Theta(G \sqcup H) > \Theta(G) + \Theta(H)$$

Example ("Shannon capacity is not attained at a finite power")

- $C_5 \boxtimes E_1 = C_5$
- $\Theta(C_5 \boxtimes E_1) = \Theta(C_5) = \Theta(C_5)\Theta(E_1)$
- $\Theta(C_5 \sqcup E_1) = \Theta(C_5) + \Theta(E_1) = \sqrt{5} + 1 \neq a^{1/n}$ for $a, n \in \mathbb{N}$

More general theorem

Let $G_1, \ldots, G_n$ be graphs. The following are equivalent:

(i) For every polynomial p we have $\Theta(p(G_1, \ldots, G_n)) = p(\Theta(G_1), \ldots, \Theta(G_n))$

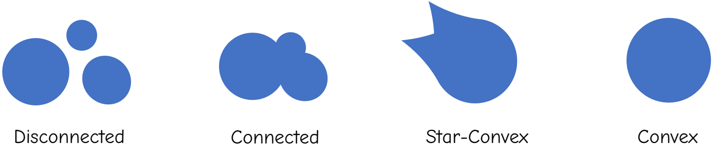
(ii) There exists a polynomial p (depending on all variables) such that $\Theta(p(G_1, \ldots, G_n)) = p(\Theta(G_1), \ldots, \Theta(G_n))$

(iii) There exists $F \in X$ such that $F(G_i) = \Theta(G_i)$ for all i

These we can also make quantitative, relating non-additivity and non-multiplicativity

New directions

- Topological structure of asymptotic spectra



Stronger topological structure $\Rightarrow$ new algorithmic methods
(with Avi Wigderson)

- New notion of graph limits

  distance $d(G, H) = \sup_{F \in X} |F(G) - F(H)|$

  (with David de Boer and Pjotr Buys)

- Hedetniemi properties of asymptotic spectrum

  (with Jim Wittebol)

- Direct-sum theorems in other areas (tensors)
  (with Visu Makam)

Problems

- What are the elements of the asymptotic spectrum of graphs?
- What other problems in math, CS and physics have asymptotic spectrum duality?
- Lovász theta function for hypergraphs?

Studying the computational complexity of
natural problems may both require and generate
deep and sophisticated mathematics