Asymptotic tensor rank is characterized by polynomials

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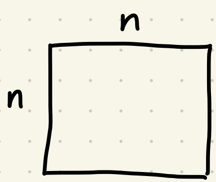
1. Tensors and ranks
2. Algebraic complexity theory
3. Polynomials
4. Consequences

1. Tensors and ranks

matrix



rank

$$M = \sum_{i=1}^{r} u_i \otimes v_i$$

minimal

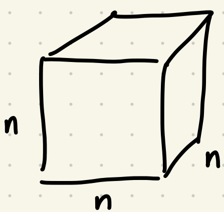
- easy to compute
- $R(M) \leq r$ iff all $(r+1) \times (r+1)$ submatrices have det $= 0$.

kronecker product

$$A \otimes B$$

- $R(A \otimes B) = R(A)R(B)$

tensor



tensor rank

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

minimal

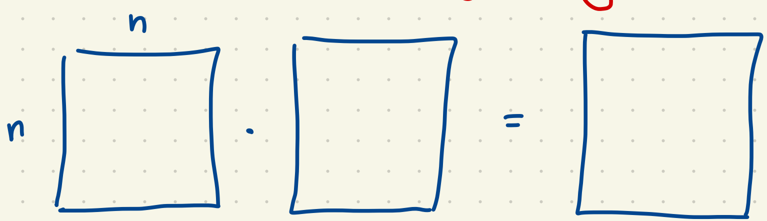
- NP-hard

$$T \otimes S$$

- $R(T \otimes S) \leq R(T)R(S)$

$\longrightarrow$ asymptotic tensor rank: $\underset{\sim}{R}(T) := \lim_{n \to \infty} R(T^{\otimes n})^{1/n}$.

2. Algebraic complexity theory



matrix mult. exponent

$O(n^{\omega})$ arithmetic operations

$2 \leq \omega \leq 2.3\ldots$

Conjecture: $\omega = 2$.

Th. $\underset{\sim}{R}(T) = 2^{\omega}$

matrix mult tensor $\in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$

Strassen's asymptotic rank conjecture: For any concise tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, $\underset{\sim}{R}(T) = m$.

What properties does $\underset{\sim}{R}$ have? Computable? Semicontinuous? Integer-valued?

3. Polynomials

$V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, $A \subseteq V$, $r \in \mathbb{R}$

Theorem 1 $\{T \in V : \underset{\sim}{R}(T) \leq r\}$ is Zariski-closed.

Def. $\underset{\sim}{R}[A] := \sup\{\underset{\sim}{R}(T) : T \in A\}$.

Theorem 2 $\underset{\sim}{R}[\bar{A}] = \underset{\sim}{R}[A]$

Theorem 2 $\Rightarrow$ Theorem 1. Let $A = \{T \in V : \underset{\sim}{R}(T) \leq r\}$. Then $\underset{\sim}{R}[A] \leq r$.

By Theorem 2, $\underset{\sim}{R}[\bar{A}] \leq r$. So for all $T \in \bar{A}$, $\underset{\sim}{R}(T) \leq r$, so $T \in A$.

Then $\bar{A} \subseteq A$ $\square$

Def. $A^{(n)} := \{ T^{\otimes n} : T \in A \}$.

Lemma 3 $(\bar{A})^{(n)} \subseteq \operatorname{span}(A^{(n)})$.

Proof. $\operatorname{span}(A^{(n)}) = \bigcap \{ \ker \ell : \ell \text{ linear form on } V^{\otimes n} \text{ vanishing on } A^{(n)} \}$

To prove: for any such ℓ, $(\bar{A})^{(n)} \subseteq \ker \ell$. Define $f : V \to \mathbb{F} : T \mapsto \ell(T^{\otimes n})$.

Then f is a polynomial function on V vanishing on A. Then f vanishes on $\bar{A}$.

Then ℓ vanishes on $(\bar{A})^{(n)}$ $\square$

Theorem 2 $\underset{\sim}{R}[\bar{A}] = \underset{\sim}{R}[A]$

Proof From $A \subseteq \bar{A}$, we have $\underset{\sim}{R}[A] \leq \underset{\sim}{R}[\bar{A}]$.

To prove: $\underset{\sim}{R}[\bar{A}] \leq \underset{\sim}{R}[A]$. Let $T \in \bar{A}$. By Lemma 3, $T^{\otimes n} \in \operatorname{span} A^{(n)}$.

So there are $S_1, \ldots, S_{p(n)} \in A$ and $\alpha_1, \ldots, \alpha_{p(n)} \in \mathbb{F}$ such that $T^{\otimes n} = \sum_{i=1}^{p(n)} \alpha_i S_i^{\otimes n}$

Note that $p(n)$ grows at most polynomially, because

$$p(n) \leq \dim \operatorname{span} A^{(n)} \leq \dim \operatorname{Sym}^n(V) = \binom{\dim(V)+n-1}{\dim(V)-1}.$$

Let $m \in \mathbb{N}$. Write

$$T^{\otimes nm} = \sum_{i_1, \ldots, i_m = 1}^{p(n)} \bigotimes_{j=1}^{m} \alpha_{i_j} S_{i_j}^{\otimes n}$$

Then (subadd.)

$$R(T^{\otimes nm}) \leq p(n)^m \max_{i_1, \ldots, i_m \in [p(n)]} R\Big(\bigotimes_{j=1}^{m} S_{i_j}^{\otimes n}\Big).$$

Rearranging,

$$R\Big(\bigotimes_{j=1}^{m} S_{i_j}^{\otimes n}\Big) = R\Big(\bigotimes_{i=1}^{p(n)} S_i^{\otimes m_i n}\Big) \quad \text{for some } m_i \text{ that sum to } m.$$

Then (subm.)

$$R\Big(\bigotimes_{i=1}^{p(n)} S_i^{\otimes m_i n}\Big) \leq \prod_{i=1}^{p(n)} R(S_i^{\otimes m_i n}). \quad \Big(\text{For large } \ell,\ R(S_i^{\otimes \ell})^{1/\ell} \leq \underset{\sim}{R}(S_i)+\varepsilon.\Big)$$

$$\leq \prod_{\substack{i:\\ m_i n \text{ large}}} (\underset{\sim}{R}(S_i)+\varepsilon)^{m_i n} \cdot \prod_{\substack{i:\\ m_i n \text{ small}}} B^{m_i n}$$

Then

$$R(T^{\otimes nm})^{1/nm} \leq p(n)^{1/n} \big(\underset{\sim}{R}[A]+\varepsilon\big)\ B^{p(n)\ c/nm}$$

Let $m \to \infty$,
$$\underset{\sim}{R}(T) \leq p(n)^{1/n}\big(\underset{\sim}{R}[A]+\varepsilon\big).$$ Let $\varepsilon \to 0$ and $n \to \infty$. $\square$

4. Consequences and more

$$\mathcal{R} := \{ \underset{\sim}{R}(T) . T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}, d_1, \ldots, d_k \in \mathbb{N} \}$$

Theorem $\mathcal{R}$ is well-ordered. (Every non-increasing sequence stabilizes.)

Noetherianity of $\mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ and $\underset{\sim}{R}(T) \geq \max_i d_i$ for concise T

Theorem Over $\mathbb{C}$, $\mathcal{R}$ is complete.

Baire property for affine varieties over $\mathbb{C}$.

Open problems

1. Is $\mathcal{R}$ discrete from below?

   $\Uparrow$

2. Is $\{ T \in V : \underset{\sim}{R}(T) \leq r \}$ an <u>irreducible</u> variety?

   $\Uparrow$

3. Use lower-semicontinuity of $\underset{\sim}{R}$ on concrete sequence of tensors.

Strassen's asymptotic rank conjecture