Part two

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Barriers for fast matrix multiplication

with
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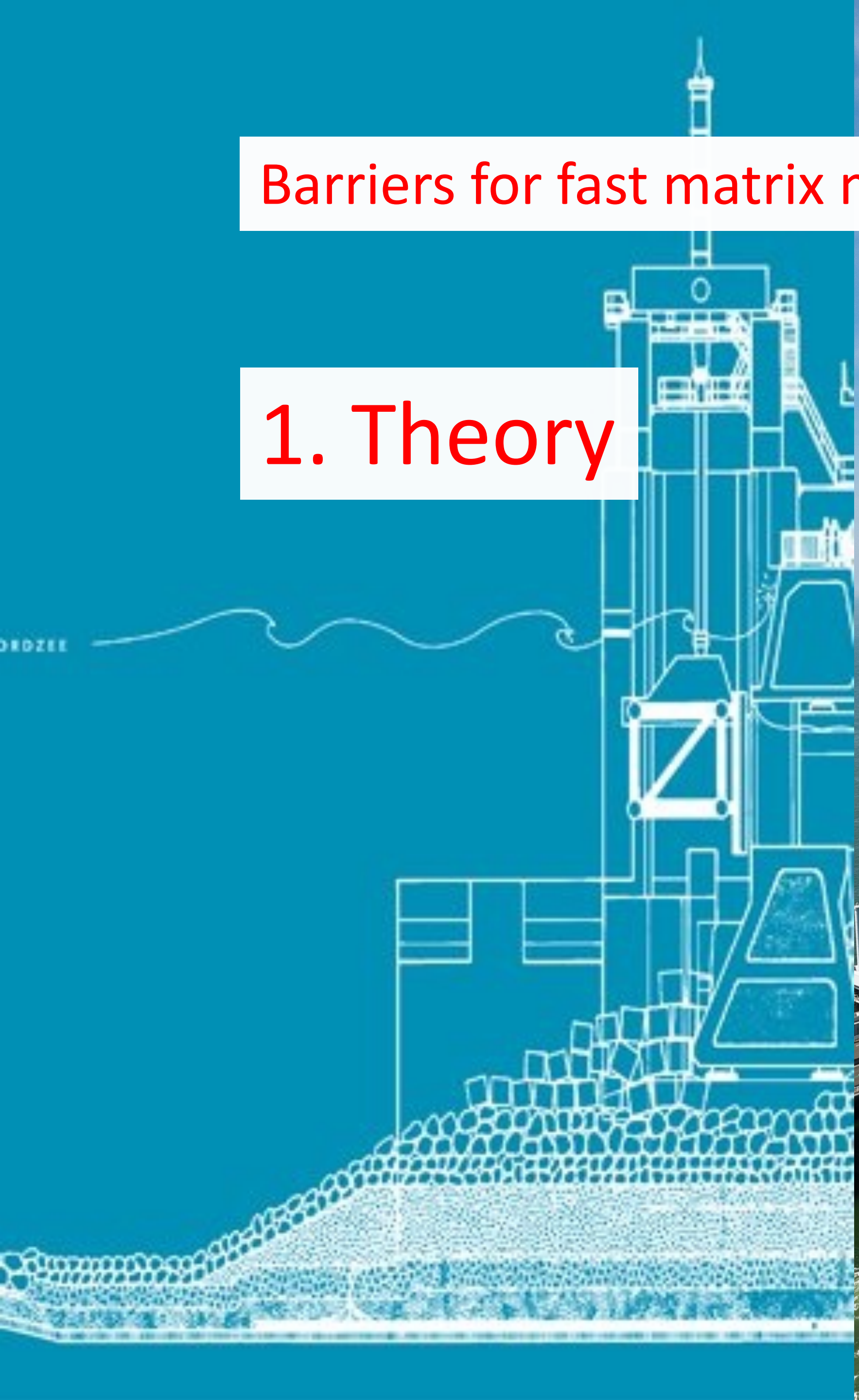
Barriers for fast matrix multiplication



1. Theory

2. Barrier

3. Tools

1.1 Matrix multiplication

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

ω

$O(n^{?})$ multiplications instead of $O(n^3)$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

7 multiplications instead of 8 Strassen 1969

block-wise multiplication

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

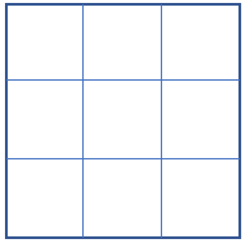
$O(n^{\log_2 7}) = O(n^{2.81})$ multiplications instead of n^3

approximation,
parallel computation of several matrix multiplications,
arithmetic progressions

$O(n^{2.372864})$ Coppersmith and Winograd 1990, Stothers 2010, V-Williams 2012, Le Gall 2014

$2 \leq \omega \leq 2.372864$ Is ω equal to 2?

Matrices



$M \leq N$ if $M = A \cdot N \cdot B$

matrix rank $R(M)$

min r $\quad M = \sum_{i=1}^{r} u_i \otimes v_i = A \cdot I_r \cdot B$

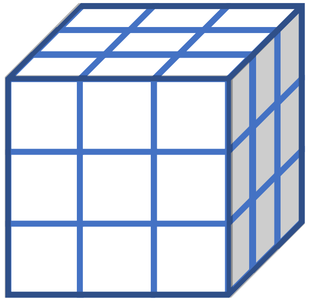
$M \leq I_r$

max r $\quad I_r = A \cdot M \cdot B$

$I_r \leq M$

Gaussian elimination!

Tensors



$S \leq T$ if $S = (A, B, C) \cdot T$

tensor rank $\mathrm{R}(S)$

min r $\quad S = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$

$S \leq \langle r \rangle$ $\quad r \times r \times r$ identity tensor

subrank $\mathrm{Q}(S)$ $\quad$ different notion!

max r $\quad \langle r \rangle \leq S$

1.2 Matrix multiplication tensor

collection of bilinear forms

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

r multiplications

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_{ij}} \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_{jk}} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_{ik}}$$

$$\sum_{i,j,k=1}^{n} e_{ij} \otimes e_{jk} \otimes e_{ik} =: \langle n, n, n \rangle$$

$$\in \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2}$$

$$\langle n, n, n \rangle = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

1.3 "Universal method"

$\mathrm{R}(\langle n, n, n\rangle) \le r$

i.e. $\langle n, n, n\rangle \le \langle r\rangle$ $\quad$ r small $\quad \Rightarrow \quad$ fast matrix multiplication

Universal method:

$\langle n, n, n\rangle \le T^{\otimes k} \le \langle r\rangle$ $\quad$ r small $\quad \Rightarrow \quad$ fast matrix multiplication

$T^{\otimes k}$ is the Kronecker product analogous to the Kronecker product $M^{\otimes k}$ for matrices

1.4 Popular and successful T

$\langle n,n,n\rangle \leq T^{\otimes k}$	T
$\omega \leq 2.8$ [Strassen 1969]	$\langle r\rangle = \sum_{i=1}^{r} e_i e_i e_i$
$\omega \leq 2.48$ [Strassen 1986]	$S_q = \sum_{i=1}^{q} e_i e_0 e_i + e_0 e_i e_i$
$\omega \leq 2.41$ [CW 1990]	$\mathrm{cw}_q = \sum_{i=1}^{q}(e_i e_0 e_i + e_0 e_i e_i + e_i e_i e_0)$
$\omega \leq 2.372864$ [CW 1990,…,Le Gall 2014]	$\mathrm{CW}_q = \sum_{i=1}^{q}(e_i e_0 e_i + e_0 e_i e_i + e_i e_i e_0)$ $+ e_{q+1} e_0 e_0 + e_0 e_{q+1} e_0 + e_0 e_0 e_{q+1}$

Barriers for fast matrix multiplication

2. Barrier

2.1 Barrier theorem

Universal method:

$\langle n, n, n \rangle \leq T^{\otimes m} \leq \langle r \rangle$ r small $\Rightarrow$ fast matrix multiplication

The *universal method* with $T = \mathrm{CW}_q$ can *at best* prove

$$\omega \leq 2.16$$

[Alman]
[Christandl, Vrana and Zuiddam]

Compare with: $\omega \leq 2.372864$

2.2 Source of barriers: subrank

Amazing and crucial subrank fact [Strassen]

$\mathrm{Q}(\langle n,n,n\rangle) = n^2$ i.e. $\langle n^2\rangle \leq \langle n,n,n\rangle$ roughly

Proof: Salem–Spencer set

Intuition of barrier:

- Clearly $\mathrm{Q}(\langle n^2\rangle) = \mathrm{R}(\langle n^2\rangle)$
- Imagine $\omega = 2$, then $\mathrm{Q}(\langle n,n,n\rangle) = \mathrm{R}(\langle n,n,n\rangle) = n^2$ roughly
- If $\mathrm{Q}\left(T^{\otimes m}\right) \ll \mathrm{R}\left(T^{\otimes m}\right)$, then T does not have enough *quality* to satisfy

$$\langle n,n,n\rangle \leq T^{\otimes m} \leq \langle n^2\rangle$$

2.3 General barrier theorem

[Christandl, Vrana and Zuiddam]
[Alman]

Universal method:

$\langle n,n,n\rangle \leq T^{\otimes m} \leq \langle r\rangle$ $\quad r$ small $\quad\Rightarrow\quad$ fast matrix multiplication

The *universal method* with T can *at best* prove*

$$\omega \leq 2 \cdot \frac{\log_2 \widetilde{\mathrm{R}}(T)}{\log_2 \widetilde{\mathrm{Q}}(T)}$$

$$\widetilde{\mathrm{R}}(T) \coloneqq \inf_n \mathrm{R}\left(T^{\otimes n}\right)^{1/n}$$

$$\widetilde{\mathrm{Q}}(T) \coloneqq \sup_n \mathrm{Q}\left(T^{\otimes n}\right)^{1/n}$$

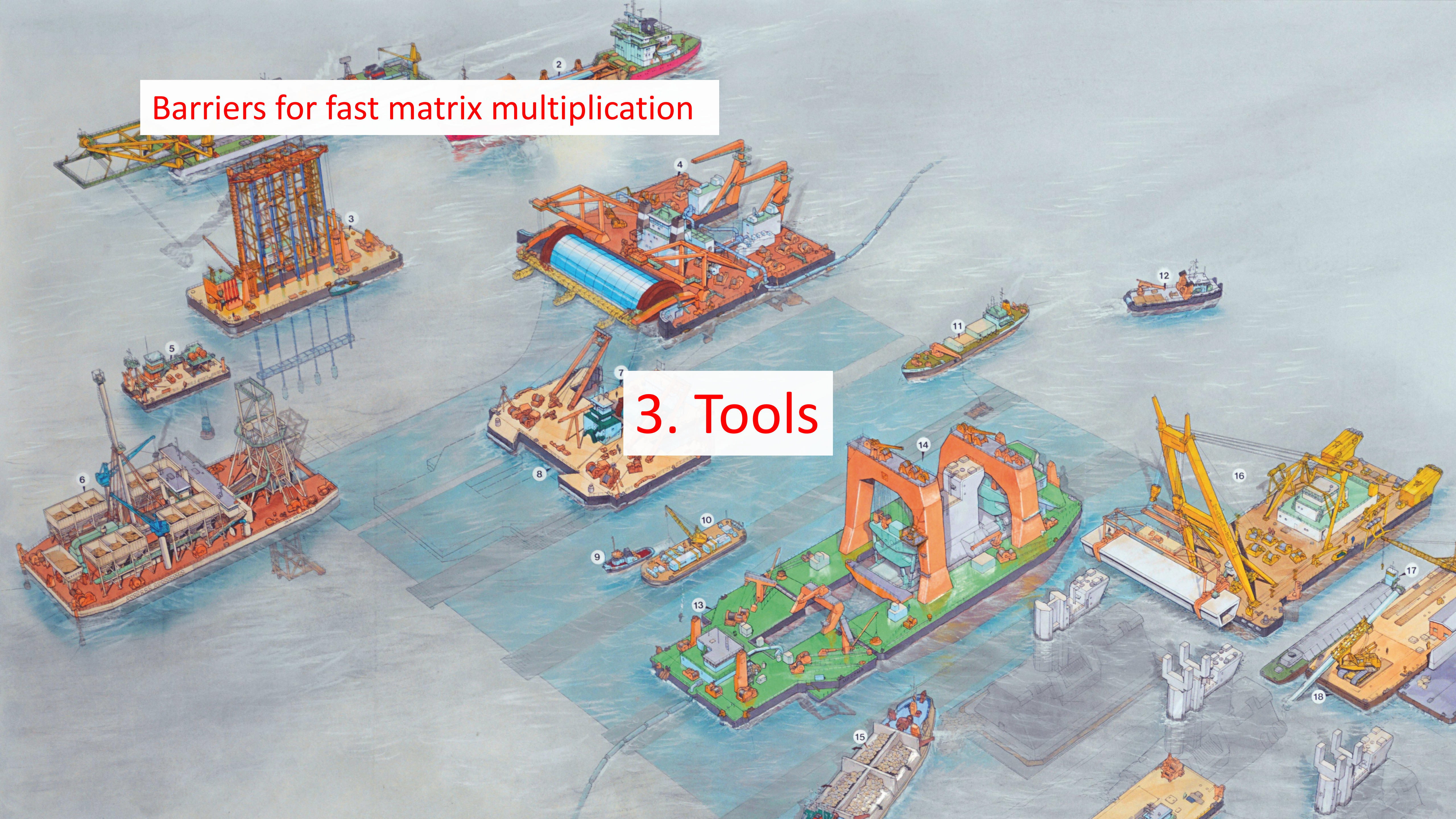
Proof sketch:

$\langle n^2\rangle \leq \langle n,n,n\rangle \leq T^{\otimes m}$

$\widetilde{\mathrm{Q}}(T)$ small $\Rightarrow$ $\mathrm{Q}(T^{\otimes m})$ small $\Rightarrow$ n small $\Rightarrow$ bad bound on ω

Barriers for fast matrix multiplication

3. Tools



3.1 Tools to compute barriers

$$\text{barrier}(T) := 2 \cdot \frac{\log_2 \widetilde{\mathrm{R}}(T)}{\log_2 \widetilde{\mathrm{Q}}(T)}$$

e.g. $T = \mathrm{CW}_q$

$$\widetilde{\mathrm{Q}}(T) \quad \leq \quad f(T) \qquad < \qquad g(T) \quad \leq \quad \widetilde{\mathrm{R}}(T)$$

- **support functionals** [Strassen 1986]
- **quantum functionals** [Christandl—Vrana—Zuiddam 2018]
- instability (from GIT) [Blasiak et al.]
- slice rank (from cap set problem) [Tao, Alman—V-Williams]

- flattening ranks

3.2 Tools for $\widetilde{\mathrm{Q}}(T) \le f(T)$

i.e. $k \to \infty$

Information-theoretic study of:

support functional $Z(T)$ — support of $T^{\otimes k}$ [Strassen]

quantum functional $F(T)$ / instability — representation-theoretic support of $T^{\otimes k}$

over $\mathbb{C}$, related to moment polytopes, scaling algorithms

[Christandl—Vrana—Zuiddam 2018] [Blasiak et al.]

$\widetilde{\mathrm{Q}}(T) \le Z(T)$

$\widetilde{\mathrm{Q}}(T) \le F(T) \le Z(T)$

no separations known

Best upper bound tools for $\widetilde{\mathrm{Q}}(T)$ that we know of

3.3 General theory of tools

Rich theory of asymptotic properties of tensors:

"Asymptotic spectrum of tensors"

[Strassen 1986]

Multiplicative, additive, normalized, ≤-monotone real functions F

$$\widetilde{\mathrm{Q}}(T) \leq F(T) \leq \widetilde{\mathrm{R}}(T)$$

Duality theorem

Analogous theory in study of Shannon capacity of graphs:

"Asymptotic spectrum of graphs"

[Zuiddam 2019]

e.g. Lovász theta number, fractional clique cover number, …

3.4 Example: barrier for big CW_q

$$\mathrm{CW}_q := e_0 e_0 e_{q+1} + e_0 e_{q+1} e_0 + e_{q+1} e_0 e_0 + \sum_{i=1}^{q} e_0 e_i e_i + e_i e_0 e_i + e_i e_i e_0$$

flattening

$$\widetilde{\mathrm{R}}(\mathrm{CW}_q) = q + 2$$

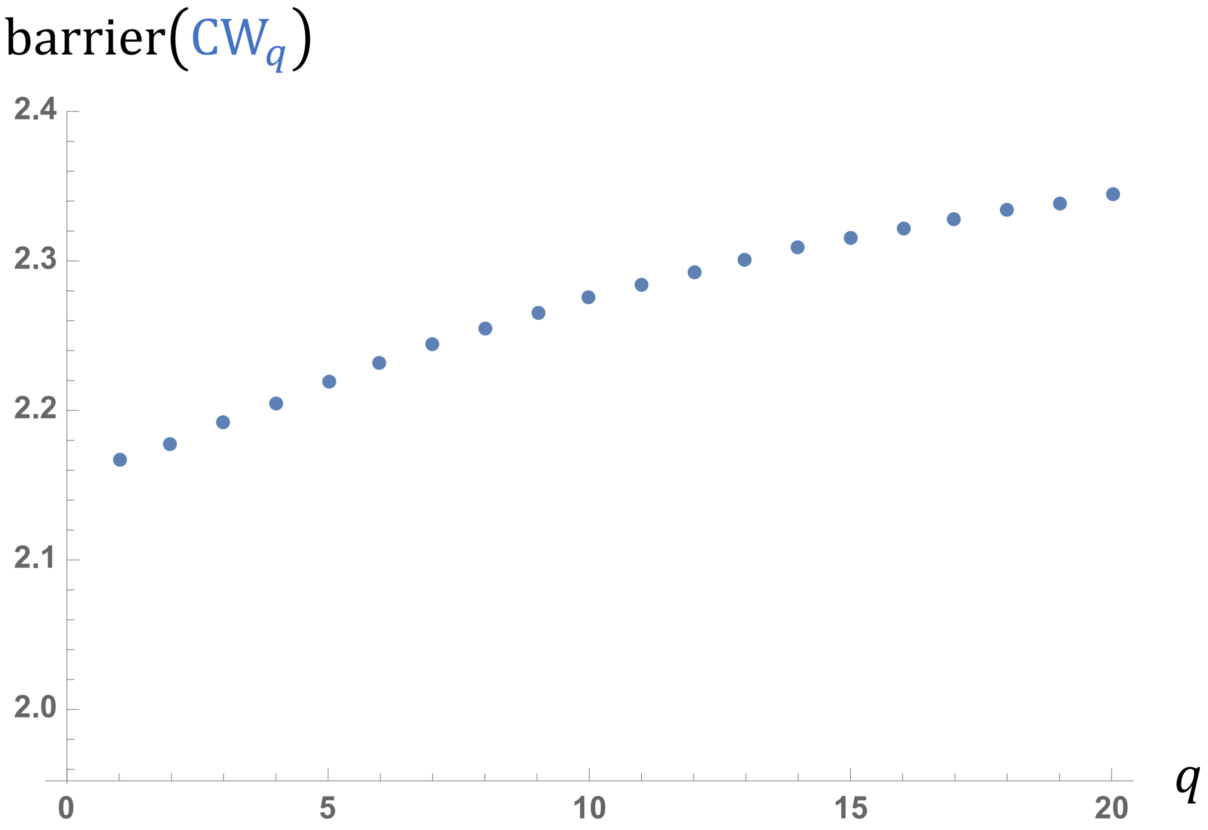
support functional

$$\widetilde{\mathrm{Q}}(\mathrm{CW}_q) \leq Z(\mathrm{CW}_q)$$

$$\mathrm{barrier}(\mathrm{CW}_q) \geq 2.16$$

minimum at $q = 2$

Josh proves inequality is tight
via Laser method



3.5 Example: barrier for small cw_q

$$\mathrm{cw}_q \coloneqq \sum_{i=1}^{q} e_0 e_i e_i + e_i e_0 e_i + e_i e_i e_0$$

flattening [CW90]

$$q + 1 \leq \widetilde{\mathrm{R}}(\mathrm{cw}_q) \leq q + 2$$

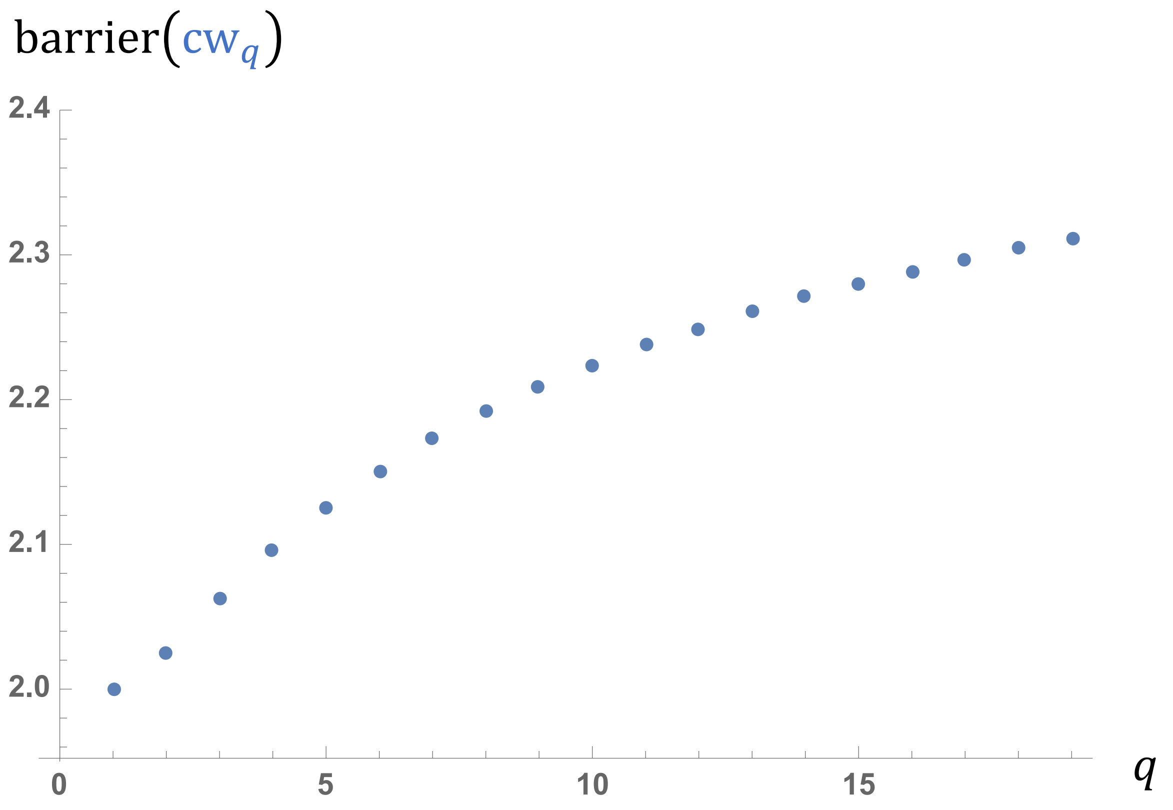
support functional

$$\widetilde{\mathrm{Q}}(\mathrm{cw}_q) \leq Z(\mathrm{cw}_q)$$

$$\mathrm{barrier}(\mathrm{cw}_q) \begin{cases} \geq 2.02 & q > 2 \\ = 2 & q = 2 \end{cases}$$

$$\widetilde{\mathrm{R}}(\mathrm{cw}_2) = 3 \Rightarrow \omega = 2$$

$$\widetilde{\mathrm{R}}(\mathrm{cw}_q) = q + 2 \Rightarrow \mathrm{barrier}(\mathrm{cw}_q) \geq 2.27$$



Conclusion: promising T to prove $\omega = 2$

T with $\widetilde{\mathrm{Q}}(T) = \widetilde{\mathrm{R}}(T)$

Example
$\widetilde{\mathrm{Q}}(\langle n \rangle) = \widetilde{\mathrm{R}}(\langle n \rangle) = n$

Example
$\omega = 2 \;\Rightarrow\; \widetilde{\mathrm{Q}}(\langle n,n,n \rangle) = \widetilde{\mathrm{R}}(\langle n,n,n \rangle)$
$= n^2$

Problem 1
other T with $\widetilde{\mathrm{Q}}(T) = \widetilde{\mathrm{R}}(T)$?

$T_1, T_2, \ldots$ with $\widetilde{\mathrm{R}}(T_i) \,/\, \widetilde{\mathrm{Q}}(T_i) \to 1$

Examples exist

Problem 2
use those T_i to upper bound ω

e.g. with group-theoretic method