Discreteness of Asymptotic Tensor Ranks

Jeroen Zuiddam

Joint work with Briët, Christandl, Leigh, and Shpilka

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- Play central role in algebraic complexity theory (fast matrix multiplication), quantum information (entanglement cost and distillation) and combinatorics (cap sets, sunflower-free sets).

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- Raises the question (for a given F): What values can $\underset{\sim}{F}(T)$ take when varying T over all tensors of order three? Are there gaps? accumulation points? Is it discrete?
- Our result: We prove for several parameters and regimes that the set of possible values is discrete.

1. $\underbrace{\text{Subrank}}_{F}$ and $\underbrace{\text{asymptotic subrank}}_{\underset{\sim}{F}}$

2. Discreteness theorem

3. Proof ingredients

4. General result

1. Subrank and asymptotic subrank

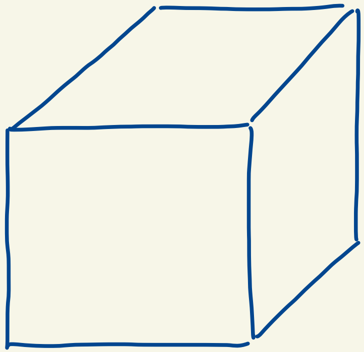
Matrix rank

$$M = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \quad \overset{\text{linear combinations of rows and columns}}{\rightsquigarrow} \quad \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

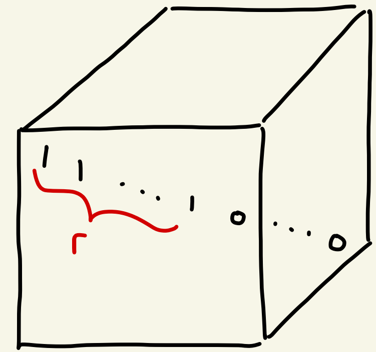
r

max r

Subrank

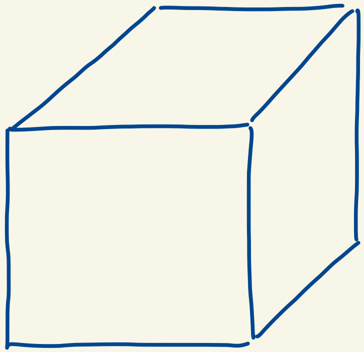
$T =$ 

linear combinations of slices in all three directions



$Q(T) = \max r$

Subrank

$T =$ 

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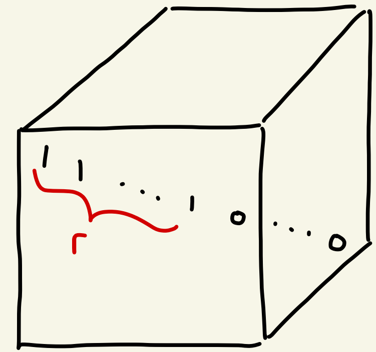

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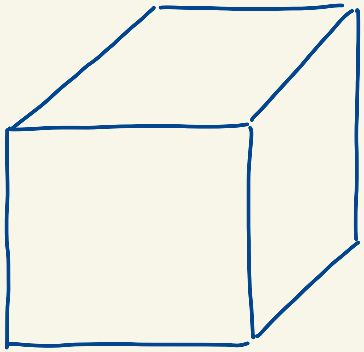
Asymptotic subrank

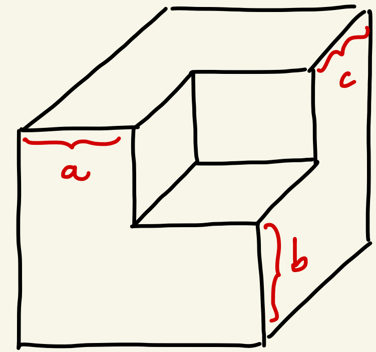
$$\underset{\sim}{Q}(T) = \lim_{n \to \infty} Q(T^{\boxtimes n})^{1/n}$$

Slice rank

$T =$ 

invertible
linear combinations
of slices in all
three directions

$\rightsquigarrow$

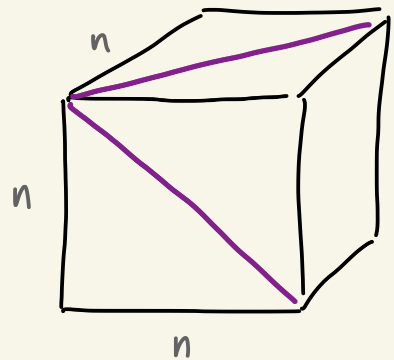


$$SR(T) = \min\ a + b + c$$

Asymptotic slice rank

$$\underset{\sim}{SR}(T) = \lim_{n \to \infty} SR(T^{\boxtimes n})^{1/n}$$

Strassen's null algebra

$T =$ 

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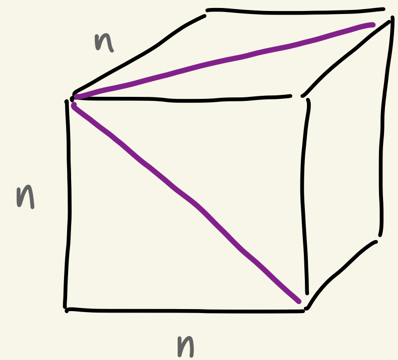
$T =$ 

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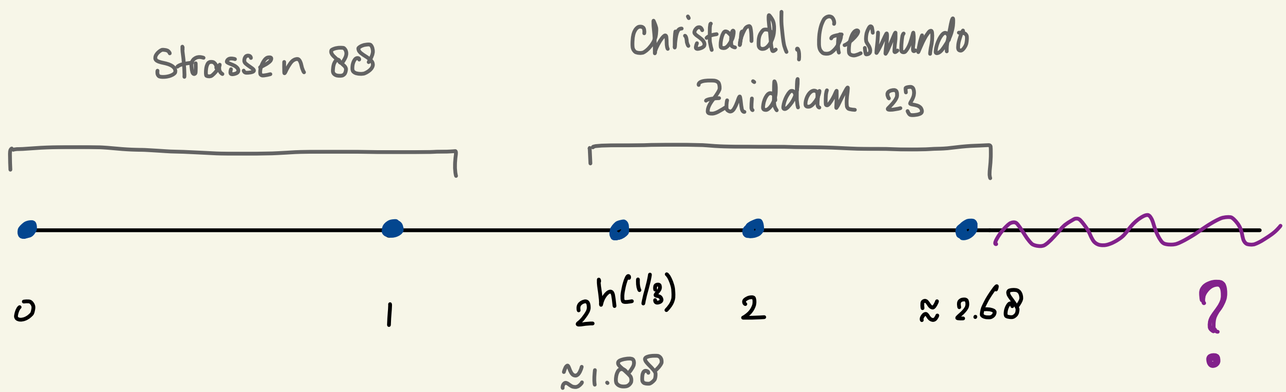
Question: Is asymptotic subrank discrete?

Values of $\underset{\sim}{Q}$



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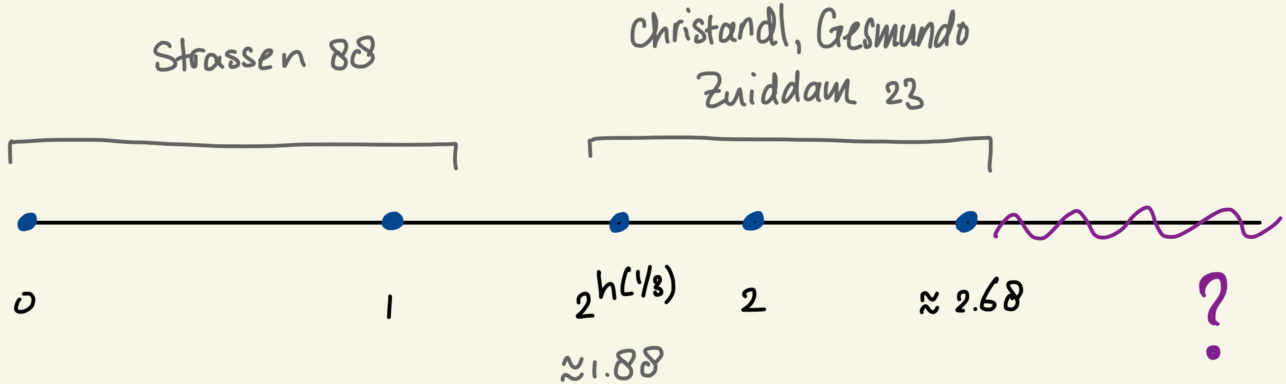

- $\underset{\sim}{Q}$ takes countably many values over $\mathbb{C}$
  [Blatter - Draisma - Rupniewski 22a]

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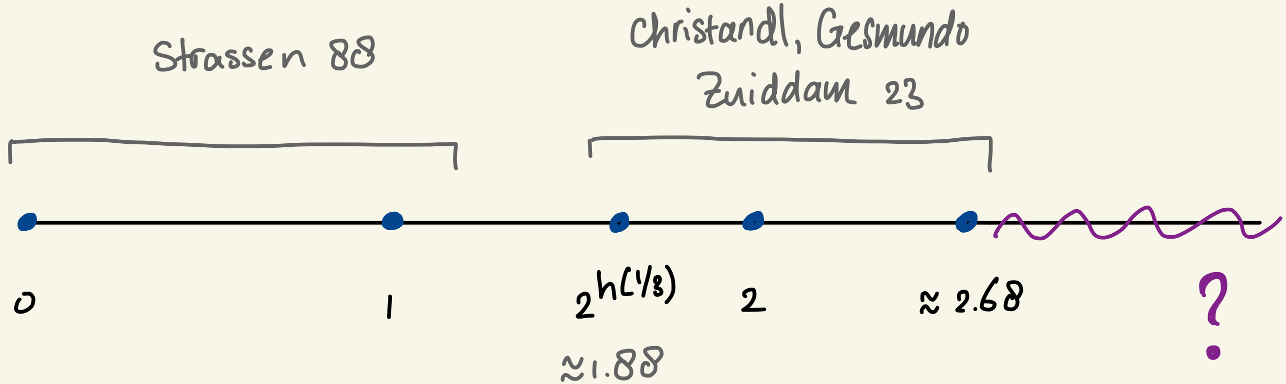

- $\underset{\sim}{Q}$ takes countably many values over $\mathbb{C}$
  [Blatter - Draisma - Rupniewski 22a]

- $\underset{\sim}{Q}$ is well-ordered over finite fields (no accumulation points from above)
  [Blatter - Draisma - Rupniewski 22b]

2. Discreteness theorem (simplest to explain version)

Theorem Over any finite set of coefficients $S \subseteq \mathbb{F}$, the set

$$\left\{ \underset{\sim}{Q}(T) : T \in S^{n_1} \otimes S^{n_2} \otimes S^{n_3},\ n_1, n_2, n_3 \in \mathbb{N} \right\}$$

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$$\left\{ \underset{\sim}{SR}(T) : T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3},\ n_1, n_2, n_3 \in \mathbb{N} \right\}$$

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- null algebra: gap between nth and (n+1)th value at most $O(1/\sqrt{n})$.
- similar result for other parameters and regimes (slice rank, complex numbers)

3. Proof ingredients

Lemma 1 (Big tensors)

Lemma 2 (Thin tensors)

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Lemma 2 (Thin tensors)

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Lemma 2 (Thin tensors) If $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{c}$ is concise and $n_1 \geq N(c)$, then $\underset{\sim}{Q}(T) = c$.

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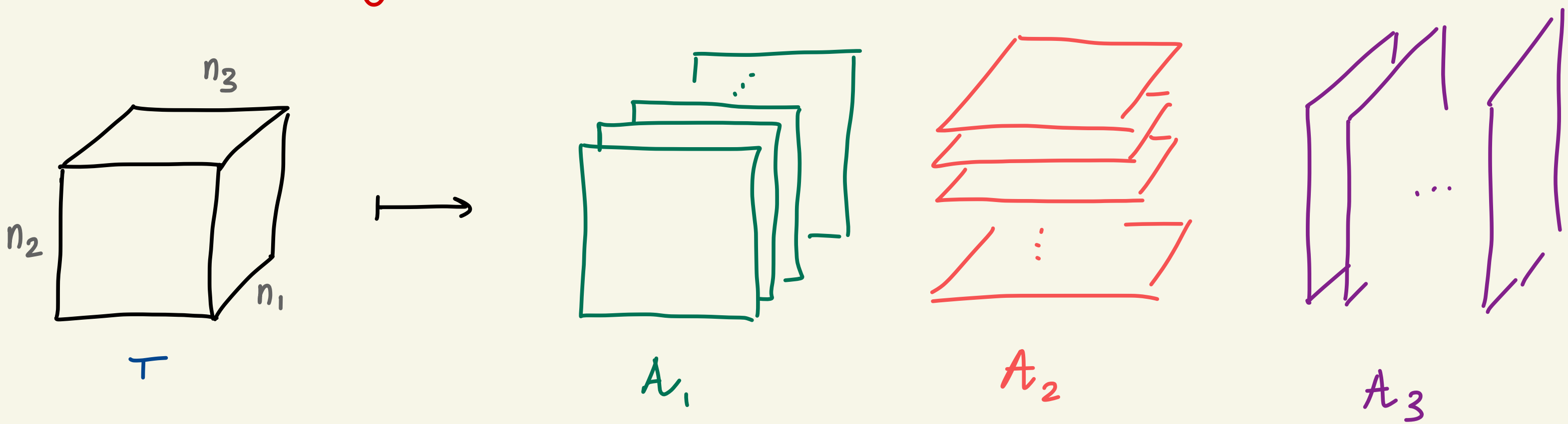
Proof sketch of main result:

Consider infinite sequence $\underset{\sim}{Q}(T_i)$ with $T_i \in \mathbb{F}^{a_i} \otimes \mathbb{F}^{b_i} \otimes \mathbb{F}^{c_i}$ concise.

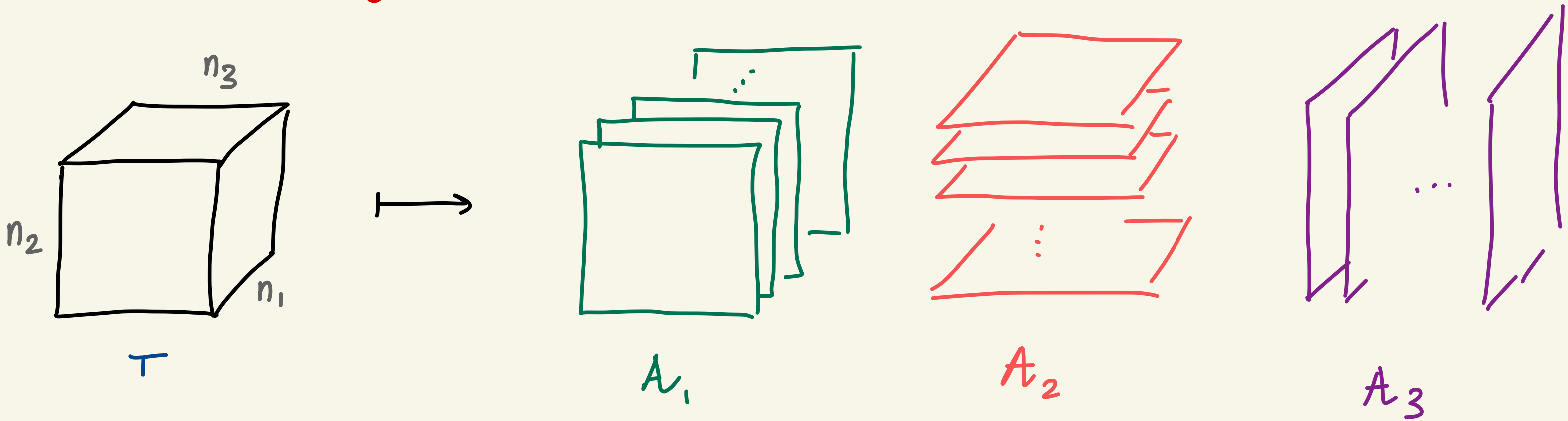
- If $\min(a_i, b_i, c_i) \to \infty$, then $\underset{\sim}{Q}(T_i) \to \infty$
- If $\max_i c_i = c$, then $a_i \to \infty$ so $\underset{\sim}{Q}(T_i)$ eventually constant $\square$

Lemma 1 Proof ingredient



- $Q_i(T) = \max\{\operatorname{rank}(A) : A \in \mathcal{A}_i\}$

Lemma: Proof ingredient



- $Q_i(T) = \max\{\operatorname{rank}(A) : A \in \mathcal{A}_i\}$

Lemma For concise $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$, and any distinct $i, j, k \in [3]$,

$$Q_i(T)\, Q_j(T) \geq n_k .$$

Lemma 2 Proof ingredient

- $\mathrm{minrank}(\mathcal{A}_i) = \min\{\mathrm{rank}(A) : 0 \neq A \in \mathcal{A}_i\}$
- relation between minrank and subrank
- tensor power trick

4. General result

Theorem. We have discreteness when

- finite $S \subseteq \mathbb{F}$
  - asymptotic subrank (this talk)
  - asymptotic slice rank
- $\mathbb{F} = \mathbb{C}$ for asymptotic slice rank (uses entanglement polytopes, quantum functionals)
- $\mathbb{F}$ arbitrary
  - asymptotic subrank and asymptotic slice rank for "tight" tensors
  - asymptotic slice rank for "oblique" tensors.

Open problems

1. Is $\underset{\sim}{Q}(T) \geq n^{1/3}$ optimal for concise $n \times n \times n$ tensors?

   For symmetric T, we have a better lower bound $n^{1/2}$.

2. Values of $\underset{\sim}{Q}(T)$

3. Higher order tensors