Discreteness of Asymptotic Tensor Ranks

Briët, Christandl, Leigh, Shpilka and Zuiddam

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- These are of the form $\underset{\sim}{F}(T) = \lim_{n\to\infty} F(T^{\otimes n})^{1/n}$

- Play central role in algebraic complexity theory (fast matrix multiplication), quantum information (entanglement cost and distillation) and combinatorics (cap sets, sunflower-free sets).

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- Our result: We prove for several parameters and regimes that the set of possible values is discrete.

1. Asymptotic ranks, applications and context

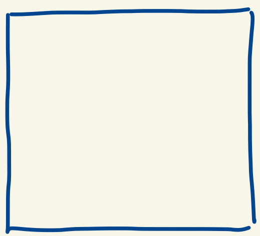
2. Discreteness theorem

3. Proof ingredients

4. General result

1. Asymptotic ranks and applications

Warm-up: Matrix rank

$M =$ 

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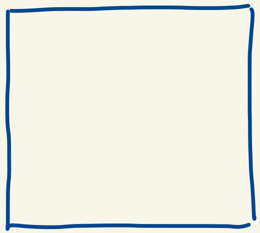
$M =$ 

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minimize

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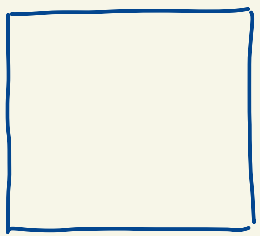
$M =$ 

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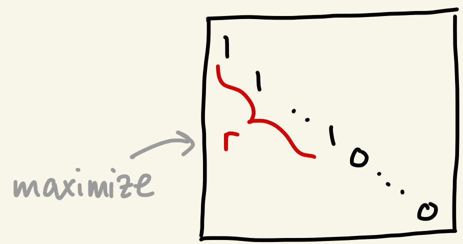
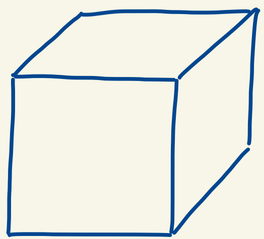


Tensor ranks

$T =$ 

Tensor ranks

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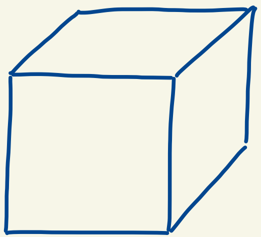
$T =$ 

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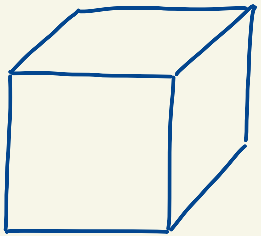
$T =$ 

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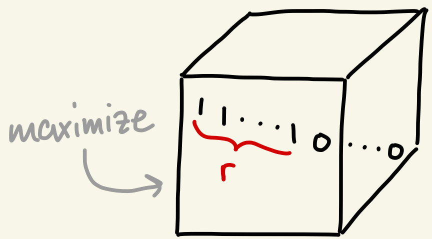

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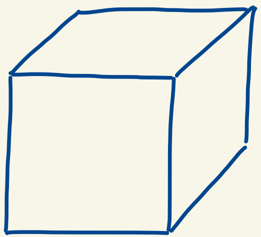
$T =$ 

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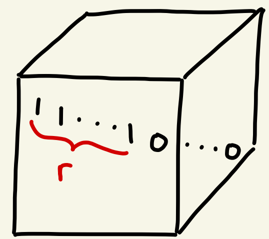
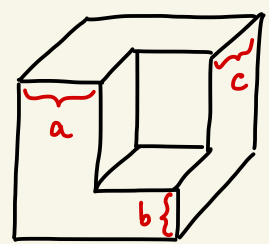


(3) slicerank



$$\underbrace{a+b+c}_{\text{minimize}}$$

Asymptotic ranks

"Rank" $\rightsquigarrow$ "Asymptotic rank"

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Tensor rank R $\qquad$ Asymptotic tensor rank $\underset{\sim}{R}$

Subrank Q $\qquad$ Asymptotic subrank $\underset{\sim}{Q}$

Slice rank SR $\qquad$ Asymptotic slice rank $\underset{\sim}{SR}$

Applications and context

<u>Asymptotic tensor rank</u> $\underset{\sim}{R}$

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Known: Closed under applying any (univariate) polynomial with non-negative integer coefficients [Wigderson–Zuiddam 23]

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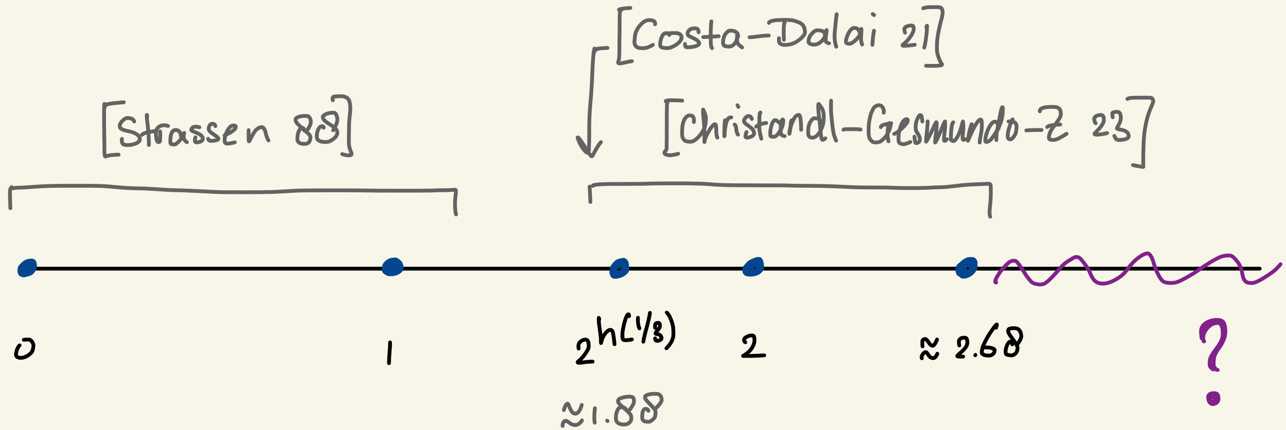
Gives information on the power of the slice rank method

Known: Closed under polynomials, as before [Wigderson-Zuiddam 23]

Known: Values of $\underset{\sim}{Q}$ and $\underset{\sim}{SR}$



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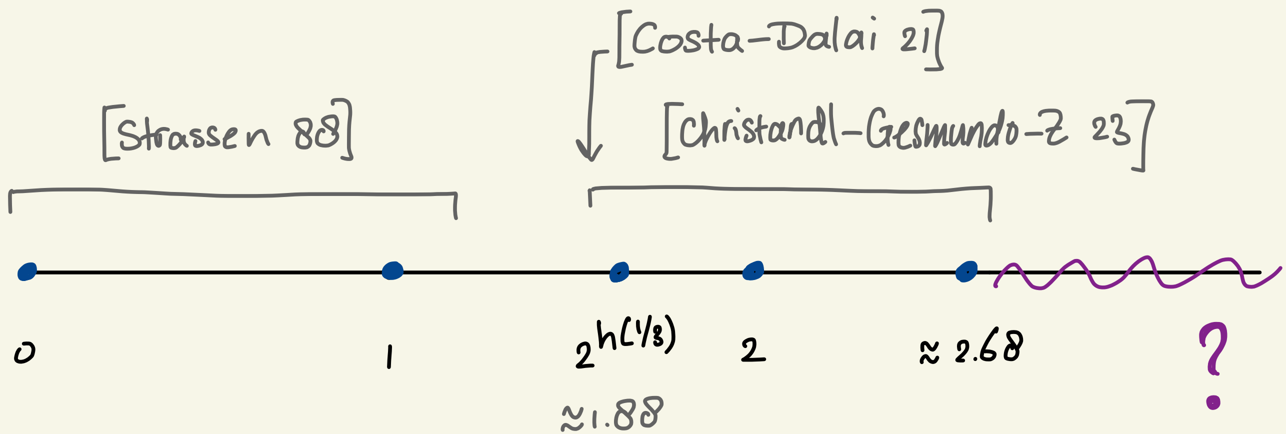

- Countably many values over $\mathbb{C}$
  [Blatter – Draisma – Rupniewski 22a]
- Well-ordered over finite fields (no accumulation points from above)
  [Blatter – Draisma – Rupniewski 22b]

2. Discreteness theorem (simple to explain version)

Theorem Over any finite set of coefficients $S \subseteq \mathbb{F}$, the set

$$\left\{ \underset{\sim}{Q}(T) : T \in S^{n_1} \otimes S^{n_2} \otimes S^{n_3},\ n_1, n_2, n_3 \in \mathbb{N} \right\}$$

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Lemma 2 (Thin tensors)

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Lemma 2 (Thin tensors) If $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{c}$ is concise and $n_1 \geq N(c)$, then $\underset{\sim}{Q}(T) = c$.

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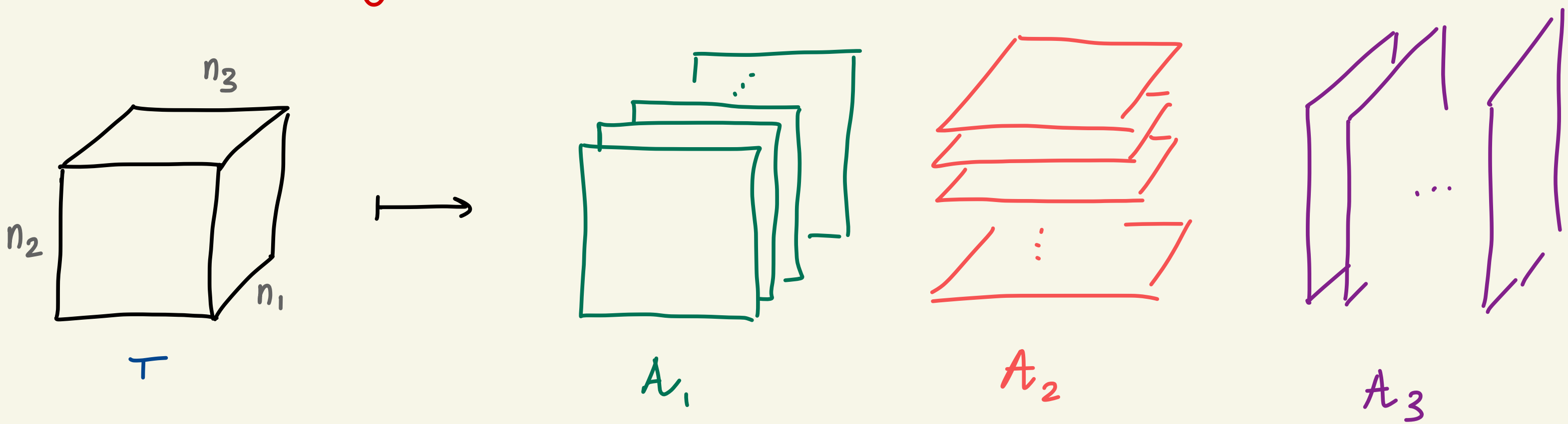
Proof sketch of main result:

Consider infinite sequence $\underset{\sim}{Q}(T_i)$ with $T_i \in \mathbb{F}^{a_i} \otimes \mathbb{F}^{b_i} \otimes \mathbb{F}^{c_i}$ concise.

- If $\min(a_1, b_i, c_1) \to \infty$, then $\underset{\sim}{Q}(T_i) \to \infty$
- If $\max_i c_i = c$, then $a_i \to \infty$ so $\underset{\sim}{Q}(T_i)$ eventually constant $\square$

Lemma 1 Proof ingredient



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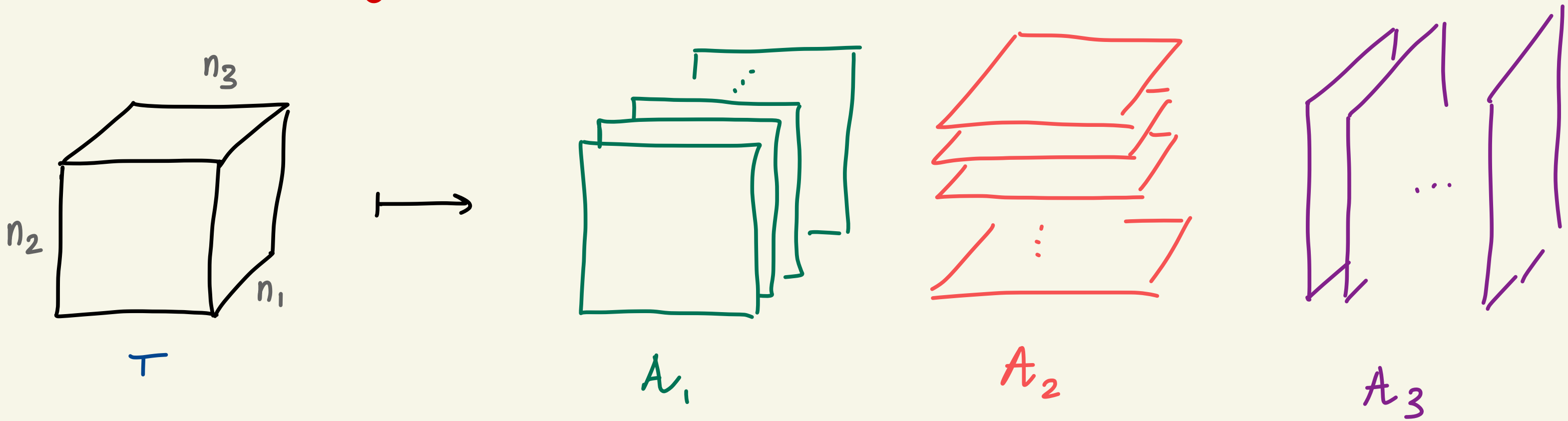
Lemma: Proof ingredient



$$Q_i(T) = \max\{\operatorname{rank}(A) : A \in \mathcal{A}_i\}$$

Lemma For concise $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$, and any distinct $i, j, k \in [3]$,

$$Q_i(T)\, Q_j(T) \geq n_k .$$

Lemma 2 Proof ingredient

- $\operatorname{minrank}(\mathcal{A}_i) = \min\{\operatorname{rank}(A) : 0 \neq A \in \mathcal{A}_i\}$
- relation between minrank and subrank
- tensor power tricks

4. General result

Theorem. We have discreteness when

- finite $S \subseteq \mathbb{F}$
  - asymptotic subrank
  - asymptotic slice rank
  - asymptotic tensor rank (simple proof)
- $\mathbb{F} = \mathbb{C}$ for asymptotic slice rank (uses entanglement polytopes, quantum functionals)
- $\mathbb{F}$ arbitrary
  - asymptotic subrank and asymptotic slice rank for "tight" tensors
  - asymptotic slice rank for "oblique" tensors.

Open problems

1. Is $\underset{\sim}{Q}(T) \geq n^{1/3}$ optimal for concise $n \times n \times n$ tensors?

   For symmetric T, we have a better lower bound $n^{1/2}$.

2. Values of $\underset{\sim}{Q}(T)$

3. Higher order tensors

4. Arbitrary fields