Shannon capacity, graph limits, and asymptotic spectrum distance

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Strassen, in his seminal 1969 paper

"Gaussian Elimination is Not Optimal"

sent a clear message to the scientific community:

Natural, obvious and centuries-old methods for solving important computational problems may be far from the fastest.

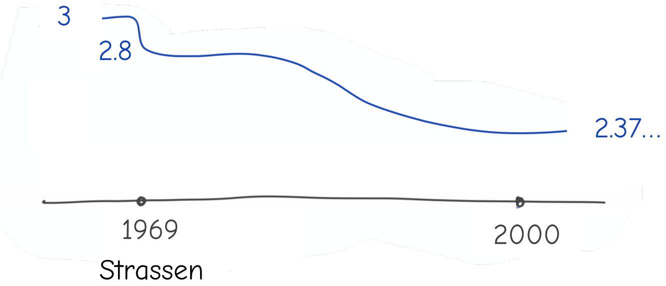
"Gaussian elimination is not optimal"

- multiplying $n \times n$ matrices
- inverting $n \times n$ matrices
- solving a system of n linear equations in n unknowns
- computing the determinant of an $n \times n$ matrix

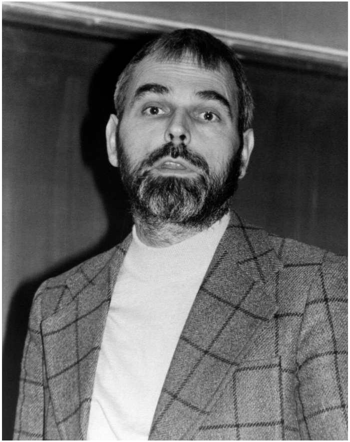
Strassen proved that the obvious $\mathcal{O}(n^3)$ algorithm for these (equivalent) problems is far from optimal

by designing a new one which takes only $\mathcal{O}(n^{2.8})$ operations

The possibility of obtaining even faster algorithms for these central problems set Strassen and many other computer scientists on a quest to obtain them, with the current record below $\mathcal{O}(n^{2.4})$



The quest to understand the matrix multiplication exponent ω is still raging on.

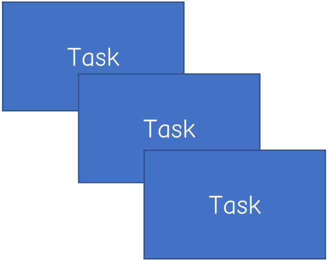


Decades later (1986–1991) Strassen developed his theory of

Asymptotic Spectra

While motivated by trying to understand the complexity of matrix multiplication, this theory is far more general

leading to a broader framework that suits other problems and settings.



Central in this theory of asymptotic spectra:

What is the cost of a task if we have to perform it many times?

Arises in numerous parts of mathematics, physics, economics and computer science

- matrix multiplication
- circuit complexity
- direct-sum problems
- Shannon capacity

Survey with Avi Wigderson (Bull. AMS)
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David de Boer

Pjotr Buys

1. Shannon capacity
2. The asymptotic spectrum of graphs
3. Duality theorem
4. A graph limit approach
5. Infinite graphs as limits
6. New bound on Shannon capacity

1. Shannon capacity

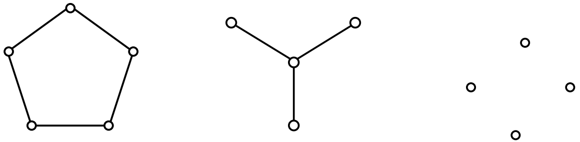
Measures amount of information that can be transmitted over a communication channel.

Understanding it has been an open problem in information theory and graph theory since its introduction by Shannon in 1956.

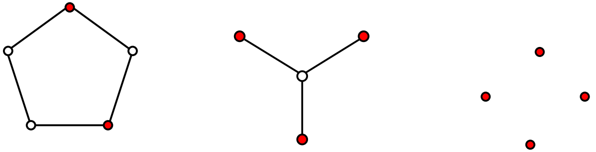
Translates to graph theoretical problem:

channel	graph
protocol	independent set
repeating	strong product

Graph



Independent set



Independence number

$\alpha(C_5) = 2$ $\alpha(S_3) = 3$ $\alpha(E_4) = 4$

Strong product

$G \boxtimes H$

$V(G \boxtimes H) = V(G) \times V(H)$

Adjacency matrix formulation:

The adjacency matrix of $G \boxtimes H$ is the tensor product of those of G and H

Independence number is super-multiplicative

$$\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$$

Example

$$\alpha(C_5) = 2$$

$$\alpha(C_5^{\boxtimes 2}) = 5$$

Shannon capacity

$$\Theta(G) = \sup_n \alpha(G^{\boxtimes n})^{1/n}$$

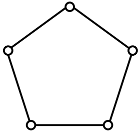
Example

$$\Theta(C_5) = \sqrt{5} \quad \text{(Lovász)}$$

$$3.2578 \leq \Theta(C_7) \leq 3.3177 \quad \text{(Schrijver–Polak)}$$

How to upper bound α (and Θ)?

Matrix rank (Haemers bound)

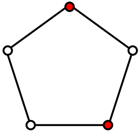


$$\begin{bmatrix} 1 & * & 0 & 0 & * \\ * & 1 & * & 0 & 0 \\ 0 & * & 1 & * & 0 \\ 0 & 0 & * & 1 & * \\ * & 0 & 0 & * & 1 \end{bmatrix}$$

1 on the diagonal

0 on the non-edges

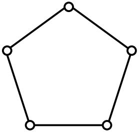
Every independent set gives an identity sub-matrix



$$\begin{bmatrix} 1 & * & 0 & 0 & * \\ * & 1 & * & 0 & 0 \\ 0 & * & 1 & * & 0 \\ 0 & 0 & * & 1 & * \\ * & 0 & 0 & * & 1 \end{bmatrix}$$

Independence number α is at most rank of any such matrix (and Θ too)

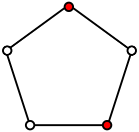
Largest eigenvalue (Lovász theta function)



$$\begin{bmatrix} 1 & * & 1 & 1 & * \\ * & 1 & * & 1 & 1 \\ 1 & * & 1 & * & 1 \\ 1 & 1 & * & 1 & * \\ * & 1 & 1 & * & 1 \end{bmatrix}$$

1 on the diagonal
1 on the non-edges

Every independent set gives an all-ones sub-matrix



$$\begin{bmatrix} 1 & * & 1 & 1 & * \\ * & 1 & * & 1 & 1 \\ 1 & * & 1 & * & 1 \\ 1 & 1 & * & 1 & * \\ * & 1 & 1 & * & 1 \end{bmatrix}$$

Independence number is at most largest eigenvalue of such matrix (and Θ too)

Q: How good are the Haemers and Lovász bounds?

2. The asymptotic spectrum of graphs

Models graphs as points in real space

Defined as the set X of all maps $F : \{\text{graphs}\} \to \mathbb{R}$ that are

1. additive under $\sqcup$
2. multiplicative under $\boxtimes$
3. monotone under cohomomorphism: $G \leq H \Rightarrow F(G) \leq F(H)$
4. normalized to 1 on the graph with one vertex E_1

Graphs as real points: $G \mapsto (F(G))_{F \in X}$

Examples of elements of X

- Lovász theta function ϑ
- fractional Haemers bound (Bukh–Cox)
- fractional clique cover number

3. Duality theorem

Recall that

- Shannon capacity is a maximization: $\Theta(G) = \sup_n \alpha(G^{\boxtimes n})^{1/n}$
- Lovász theta gives upper bound: $\Theta(G) \leq \vartheta(G)$

Lemma

Every $F \in X$ gives upper bound: $\Theta(G) \leq F(G)$

Q: Are the upper bounds from $F \in X$ powerful enough to reach Θ?

Duality Theorem ("yes", Zuiddam, 2018)

Shannon capacity is a minimization: $\Theta(G) = \min_{F \in X} F(G)$

Q: Is the duality theorem non-trivial?

Duality Theorem $\Theta(G) = \min_{F \in X} F(G)$

Conjecture (Shannon) $\Theta \in X$

Theorem (Haemers)
There are G, H for which $\Theta(G \boxtimes H) > \Theta(G)\Theta(H)$

Theorem (Alon)
There are G, H for which $\Theta(G \sqcup H) > \Theta(G) + \Theta(H)$

Corollary $\Theta \notin X$

Q: How is the duality theorem proven?

Duality Theorem $\Theta(G) = \min_{F \in X} F(G)$

More General Duality Theorem (Zuiddam, 2018)
$G^{\boxtimes n} \leq H^{\boxtimes (n+o(n))}$ iff $F(G) \leq F(H)$ for all $F \in X$

Ideas:

- Real geometry, Positivstellensatz
- Kadison–Dubois representation theorem
- Extension of Linear Programming Duality

Consequences

Theorem ("Additivity if and only if multiplicativity", Holzman)
For any graphs G, H the following are equivalent:

(i) $\Theta(G \sqcup H) = \Theta(G) + \Theta(H)$
(ii) $\Theta(G \boxtimes H) = \Theta(G)\Theta(H)$
(iii) There is $F \in X$ such that $F(G) = \Theta(G)$ and $F(H) = \Theta(H)$

Example ("Theorems of Haemers and Alon are equivalent")

$$\Theta(G \boxtimes H) > \Theta(G)\Theta(H) \quad \text{iff} \quad \Theta(G \sqcup H) > \Theta(G) + \Theta(H)$$

Example ("Shannon capacity may not be attained at a finite power")

- $C_5 \boxtimes E_1 = C_5$
- $\Theta(C_5 \boxtimes E_1) = \Theta(C_5) = \Theta(C_5)\Theta(E_1)$
- $\Theta(C_5 \sqcup E_1) = \Theta(C_5) + \Theta(E_1) = \sqrt{5} + 1 \neq a^{1/n}$ for $a, n \in \mathbb{N}$

4. A graph limit approach

Despite tremendous effort, even small instances of the Shannon capacity problem have remained open:

$$\Theta(C_7) = ?$$

We propose a graph limit approach:

$$G_i \text{ “}\!\to\!\text{” } C_7 \quad \Longrightarrow \quad \Theta(G_i) \to \Theta(C_7)$$

- What notion of convergence? Asymptotic spectrum distance
- How to construct converging sequences?
- Where to look for “easy” graphs G_i?

Mains results (de Boer, Buys, Zuiddam, 2024):

(1) General construction of nontrivial sequences converging in asymptotic spectrum distance

(2) Cauchy sequences of finite graphs that do not converge to any finite graph

(3) All best-known lower bounds on Shannon capacity of small odd-cycles can be obtained from a finite version of the graph limit approach

(4) New bound for fifteen-cycle C_{15}!

Asymptotic spectrum distance

$$d(G,H) = \max_{F\in X} |F(G) - F(H)|$$

Lemma

$G_i \to H \Rightarrow \Theta(G_i) \to \Theta(H)$

Proof Let $\varepsilon > 0$. There is an N such that for all $i > N$ and all $F \in X$,

$$|F(G_i) - F(H)| < \varepsilon.$$

Let $F_{G_i}, F_H \in X$ such that $F_{G_i}(G_i) = \Theta(G_i)$ and $F_H(H) = \Theta(H)$ (duality). Then

$$\Theta(H) = F_H(H) > F_H(G_i) - \varepsilon \geq \Theta(G_i) - \varepsilon$$

and similarly with H and G_i swapped. □

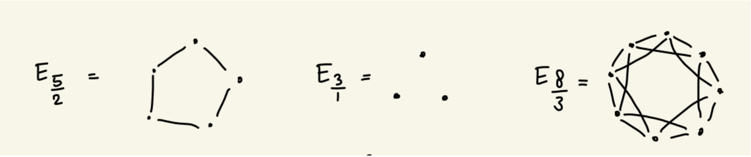
Lemma TFAE

- $d(G,H) \leq a/b$
- $(E_b \boxtimes G)^{\boxtimes n} \leq ((E_b \boxtimes H) \sqcup E_a)^{\boxtimes(n+o(n))}$ and with G, H swapped

Converging sequences

Fraction graph $E_{a/b}$

vertex set $\mathbb{Z}/a\mathbb{Z}$ and $u \sim v$ iff $-b < u - v \pmod a < b$.



Lemma $E_{a/b} \leq E_{c/d}$ iff $a/b \leq c/d$ (in $\mathbb{Q}$)

Theorem A For any $a/b \geq 2$, if $c_n/d_n \to a/b$ from above, then $E_{c_n/d_n} \to E_{a/b}$

Theorem B For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy

Ingredients

Lemma 1 Let G vertex transitive, $S \subseteq V(G)$, $F \in X$. Then

$$F(G[S]) \leq F(G) \leq \frac{|G|}{|S|} \cdot F(G[S])$$

Proof "Cover" G with N copies of $G[S]$ with $N = \lceil |G| \cdot |S|^{-1} \cdot \log |G| \rceil$.

Lemma 2 ("Euclid's algorithm") $E_{a/b}$ minus any vertex is equivalent to $E_{a'/b'}$ for $a' < a$, $b' < b$ with $a \cdot b' - b \cdot a' = 1$.

Consequence: $F(E_{a'/b'}) \leq F(E_{a/b}) \leq \frac{a}{a-1} \cdot F(E_{a'/b'})$

Theorem A For any $a/b \geq 2$, if $c_n/d_n \to a/b$ from above, then $E_{c_n/d_n} \to E_{a/b}$

Proof sketch Let a, b coprime. There are x, y with $x \cdot b - y \cdot a = 1$. Let $c_n = x + a \cdot n$ and $d_n = y + b \cdot n$. For every n, $c_n \cdot b - d_n \cdot a = 1$. So

$$F(E_{a/b}) \leq F(E_{c_n/d_n}) \leq \frac{c_n}{c_n - 1} F(E_{a/b}).$$

Let $n \to \infty$, then $c_n \to \infty$. $\square$

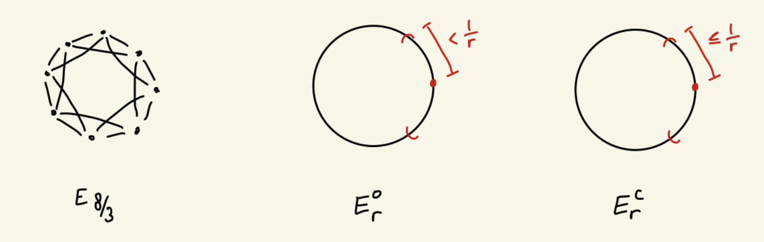
Theorem B For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy

Proof sketch Continued fraction convergents:

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < r < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

Property: $q_n \cdot p_{n-1} - p_n \cdot q_{n-1} = (-1)^n$. $\square$

5. Infinite graphs as limits



Theorem For any irrational $r \geq 2$, if $a_n/b_n \to r$, then $E_{a_n/b_n} \to E_r^o$

Theorem TFAE

(i) E_r^c and E_r^o are equivalent under asymptotic cohomomorphism

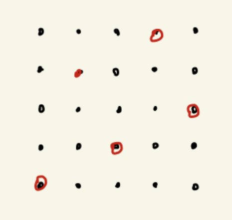
(ii) $a_n/b_n \to p/q$ from below $\Rightarrow$ $E_{a_n/b_n} \to E_{p/q}$

Theorem E_r^c and E_r^o are not equivalent under cohomomorphism

6. New bounds on Shannon capacity

$\Theta(C_5) = \sqrt{5}$

$\alpha(C_5^{\boxtimes 2}) = 5$



$\{g \cdot (1,2) : g \in \mathbb{Z}/5\mathbb{Z}\}$

G	H	orbit independent set in $H^{\boxtimes k}$	reduction	$\leq \Theta(G)$
$E_{5/2}$	$E_{5/2}$	$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$	$H = G$	2.23 [Sha56]
$E_{7/2}$	$E_{382/108}$	$\{t \cdot (1,7,7^2,7^3,7^4) : t \in \mathbb{Z}_{382}\}$	$G \leq H$	3.25 [PS19]
$E_{9/2}$	$E_{9/2}$	$\{s \cdot (1,0,2) + t \cdot (0,1,4) : s,t \in \mathbb{Z}_9\}$	$H = G$	4.32 [BMR+71]
$E_{11/2}$	$E_{148/27}$	$\{t \cdot (1,11,11^2) : t \in \mathbb{Z}_{148}\}$	$H \leq G$	5.28 [BMR+71]
$E_{13/2}$	$E_{247/38}$	$\{t \cdot (1,19,117) : t \in \mathbb{Z}_{247}\}$	$H \leq G$	6.27 [BMR+71][18]
$E_{15/2}$	$E_{2873/381}$	$\{t \cdot (1,15,1073,1125) : t \in \mathbb{Z}_{2873}\}$	$G \leq H$	7.30 (Section 6.2)

arXiv:2404.16763

Problems

- What are the elements of the asymptotic spectrum of graphs?
- How effective are "orbit constructions"?
- Prove more new bounds on Shannon capacity using converging sequences of graphs
- What other problems in math, CS and physics have asymptotic spectrum duality?