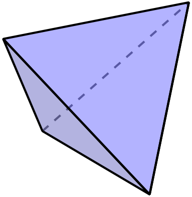
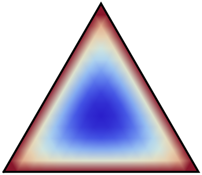
# The asymptotic spectrum of tensors

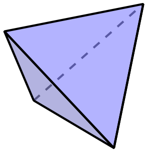
tensor $\mapsto$  $\mapsto$ 

Jeroen Zuiddam
(QuSoft & CWI)

joint work with
Péter Vrana (BME Budapest)
Matthias Christandl (University of Copenhagen)

We use

- moment polytopes
- representation theory
- quantum information theory



to study asymptotic properties of tensors

motivated by problems in

- computational complexity theory
- additive combinatorics

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

# 1. Tensors

## tensor

$$t = (t_{i_1 i_2 i_3})_{i_1 i_2 i_3} \qquad \in \mathbb{F}^{n_1 \times n_2 \times n_3}$$

$$t = \sum_{i_1 i_2 i_3} t_{i_1 i_2 i_3} \, e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \qquad \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$$

## “restriction” of tensors

$t \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$
$s \in \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2} \otimes \mathbb{F}^{m_3}$

We say $t$ *restricts to* $s$ and write

$$t \geq s$$

if there are linear maps $A_i : \mathbb{F}^{n_i} \to \mathbb{F}^{m_i}$ such that

$$(A_1 \otimes A_2 \otimes A_3) \cdot t = s$$

## restriction

$$s \leq t \quad \text{iff} \quad s = (A_1 \otimes A_2 \otimes A_3) \cdot t \quad \text{for some linear } A_i$$

## "diagonal" tensor

ones on the main diagonal, zeros elsewhere

$$\langle n \rangle = \sum_{i=1}^{n} e_i \otimes e_i \otimes e_i \qquad \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$$

## rank and sub-rank

$$\mathrm{R}(t) = \min\{\, n \in \mathbb{N} \,:\, t \leq \langle n \rangle \,\}$$

$$\mathrm{Q}(t) = \max\{\, m \in \mathbb{N} \,:\, \langle m \rangle \leq t \,\}$$

## 2. Asymptotic properties of tensors

$$\text{property}(t^{\otimes n}) \quad n \to \infty$$

## tensor product ⊗ on tensors

$$(\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}) \times (\mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2} \otimes \mathbb{F}^{m_3}) \to \mathbb{F}^{n_1 m_1} \otimes \mathbb{F}^{n_2 m_2} \otimes \mathbb{F}^{n_3 m_3}$$

$$(t, s) \mapsto t \otimes s = \sum_{\substack{i_1 i_2 i_3 \\ j_1 j_2 j_3}} t_{i_1 i_2 i_3} \cdot s_{j_1 j_2 j_3} \; e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3 j_3}$$

## asymptotic rank and asymptotic sub-rank

$$\underset{\sim}{\mathrm{R}}(t) = \lim_{N \to \infty} \mathrm{R}(t^{\otimes N})^{1/N}$$

$$\underset{\sim}{\mathrm{Q}}(t) = \lim_{N \to \infty} \mathrm{Q}(t^{\otimes N})^{1/N}$$

## asymptotic restriction

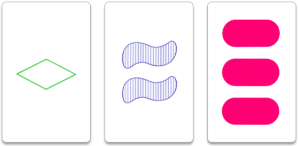
$$s \lesssim t \quad \text{iff} \quad \forall n \quad s^{\otimes n} \leq t^{\otimes n + a_n} \quad \frac{a_n}{n} \to 0, \quad n \to \infty$$

## 3. Illustrations of $\underset{\sim}{\mathrm{Q}}(t)$ and $\underset{\sim}{\mathrm{R}}(t)$

### fast matrix multiplication

$$\begin{pmatrix} 1 & 4 & 0 & 3 & 5 & 8 & 2 & 10 & 8 & 5 \\ 5 & 9 & 5 & 9 & 7 & 6 & 10 & 6 & 7 & 1 \\ 0 & 3 & 7 & 8 & 3 & 10 & 3 & 2 & 3 & 1 \\ 5 & 6 & 4 & 2 & 2 & 10 & 7 & 1 & 5 & 10 \\ 6 & 6 & 9 & 0 & 5 & 10 & 6 & 5 & 2 & 3 \\ 5 & 8 & 9 & 0 & 8 & 3 & 8 & 2 & 4 & 2 \\ 0 & 7 & 1 & 6 & 8 & 0 & 0 & 0 & 2 & 3 \\ 8 & 0 & 7 & 7 & 5 & 0 & 5 & 3 & 4 & 9 \\ 5 & 2 & 1 & 8 & 6 & 2 & 0 & 5 & 2 & 0 \\ 9 & 2 & 8 & 3 & 8 & 1 & 1 & 7 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 9 & 5 & 7 & 5 & 1 & 5 & 3 & 0 & 6 & 0 \\ 3 & 6 & 2 & 0 & 7 & 6 & 6 & 6 & 7 & 8 \\ 10 & 5 & 10 & 2 & 8 & 1 & 0 & 6 & 4 & 0 \\ 9 & 5 & 0 & 2 & 1 & 3 & 10 & 7 & 7 & 8 \\ 0 & 0 & 10 & 4 & 6 & 1 & 5 & 1 & 4 & 2 \\ 3 & 5 & 8 & 6 & 1 & 0 & 3 & 0 & 0 & 9 \\ 1 & 1 & 2 & 1 & 7 & 1 & 3 & 4 & 1 & 4 \\ 6 & 3 & 3 & 9 & 6 & 5 & 6 & 3 & 4 & 10 \\ 3 & 2 & 9 & 7 & 6 & 4 & 4 & 1 & 4 & 3 \\ 7 & 10 & 2 & 5 & 1 & 6 & 6 & 8 & 9 & 6 \end{pmatrix}$$

### *cap set* problem



# Fast matrix multiplication

$$
\begin{pmatrix}
1 & 4 & 0 & 3 & 5 & 8 & 2 & 10 & 8 & 5 \\
5 & 9 & 5 & 9 & 7 & 6 & 10 & 6 & 7 & 1 \\
0 & 3 & 7 & 8 & 3 & 10 & 3 & 2 & 3 & 1 \\
5 & 6 & 4 & 2 & 2 & 10 & 7 & 1 & 5 & 10 \\
6 & 6 & 9 & 0 & 5 & 10 & 6 & 5 & 2 & 3 \\
5 & 8 & 9 & 0 & 8 & 3 & 8 & 2 & 4 & 2 \\
0 & 7 & 1 & 6 & 8 & 0 & 0 & 0 & 2 & 3 \\
8 & 0 & 7 & 7 & 5 & 0 & 5 & 3 & 4 & 9 \\
5 & 2 & 1 & 8 & 6 & 2 & 0 & 5 & 2 & 0 \\
9 & 2 & 8 & 3 & 8 & 1 & 1 & 7 & 8 & 7
\end{pmatrix}
\cdot
\begin{pmatrix}
9 & 5 & 7 & 5 & 1 & 5 & 3 & 0 & 6 & 0 \\
3 & 6 & 2 & 0 & 7 & 6 & 6 & 6 & 7 & 8 \\
10 & 5 & 10 & 2 & 8 & 1 & 0 & 6 & 4 & 0 \\
9 & 5 & 0 & 2 & 1 & 3 & 10 & 7 & 7 & 8 \\
0 & 0 & 10 & 4 & 6 & 1 & 5 & 1 & 4 & 2 \\
3 & 5 & 8 & 6 & 1 & 0 & 3 & 0 & 0 & 9 \\
1 & 1 & 2 & 1 & 7 & 1 & 3 & 4 & 1 & 4 \\
6 & 3 & 3 & 9 & 6 & 5 & 6 & 3 & 4 & 10 \\
3 & 2 & 9 & 7 & 6 & 4 & 4 & 1 & 4 & 3 \\
7 & 10 & 2 & 5 & 1 & 6 & 6 & 8 & 9 & 6
\end{pmatrix}
= ?
$$

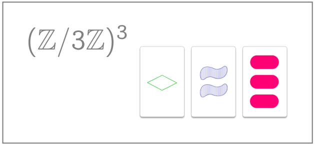
| | |
|---|---|
| by hand | $2 \cdot n^3$ |
| Strassen | $4.7 \cdot n^{2.8}$ |
| current best | $C \cdot n^{2.32}$ |
| lower bound | $n^2$ |
| optimal | $C \cdot n^{\omega}$ |

$$2^{\omega} = \underset{\sim}{\mathrm{R}}(\text{mamu tensor}), \quad \text{mamu tensor} \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$$

# Cap set problem

$(\mathbb{Z}/3\mathbb{Z})^n$

- $u + 2v$
- $u + v$
- $u$




cap set

subset $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ without lines, except trivial lines $(u, u, u)$

trivial upper bound $\quad |A| \leq 3^n$

Gijswijt–Ellenberg (2016) $\quad |A| \leq C \cdot 2.755^n$

$2.755\ldots = \underset{\sim}{Q}(\text{cap set tensor})$, cap set tensor $\in \mathbb{F}_3^3 \otimes \mathbb{F}_3^3 \otimes \mathbb{F}_3^3$

# 4. Asymptotic spectrum of tensors

[Strassen 1986]

## The asymptotic spectrum of tensors

$\mathcal{T}$ = 3-tensors over $\mathbb{F} = \bigcup_{n_1,n_2,n_3} \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$

$X(\mathcal{T})$ = set of maps $F : \mathcal{T} \to \mathbb{R}_{\geq 0}$ such that

1. if $t \geq s$ then $F(t) \geq F(s)$ — monotone
2. $F(s \oplus t) = F(s) + F(t)$ — additive
3. $F(s \otimes t) = F(s)F(t)$ — multiplicative
4. $F(\langle n \rangle) = n$ for $n \in \mathbb{N}$ — normalised

## The asymptotic spectrum of tensors

$\mathcal{T}$ = 3-tensors over $\mathbb{F} = \bigcup_{n_1,n_2,n_3} \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$

$X(\mathcal{T})$ = set of maps $F : \mathcal{T} \to \mathbb{R}_{\geq 0}$ such that

1. if $t \geq s$ then $F(t) \geq F(s)$ — monotone
2. $F(s \oplus t) = F(s) + F(t)$ — additive
3. $F(s \otimes t) = F(s)F(t)$ — multiplicative
4. $F(\langle n \rangle) = n$ for $n \in \mathbb{N}$ — normalised

- Let $F \in X(\mathcal{T})$
- $t^{\otimes n} \leq \langle \mathrm{R}(t^{\otimes n}) \rangle$
- $F(t)^n = F(t^{\otimes n}) \leq F(\langle \mathrm{R}(t^{\otimes n}) \rangle) = \mathrm{R}(t^{\otimes n})$
- $F(t) \leq \lim_{n\to\infty} R(t^{\otimes n})^{1/n} = \underset{\sim}{\mathrm{R}}(t)$

## The asymptotic spectrum of tensors

$\mathcal{T}$ = 3-tensors over $\mathbb{F} = \bigcup_{n_1,n_2,n_3} \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$

$X(\mathcal{T})$ = set of maps $F : \mathcal{T} \to \mathbb{R}_{\geq 0}$ such that

1. if $t \geq s$ then $F(t) \geq F(s)$ — monotone
2. $F(s \oplus t) = F(s) + F(t)$ — additive
3. $F(s \otimes t) = F(s)F(t)$ — multiplicative
4. $F(\langle n \rangle) = n$ for $n \in \mathbb{N}$ — normalised

- Let $F \in X(\mathcal{T})$
- $t^{\otimes n} \leq \langle \mathrm{R}(t^{\otimes n}) \rangle$
- $F(t)^n = F(t^{\otimes n}) \leq F(\langle \mathrm{R}(t^{\otimes n}) \rangle) = \mathrm{R}(t^{\otimes n})$
- $F(t) \leq \lim_{n\to\infty} R(t^{\otimes n})^{1/n} = \underset{\sim}{\mathrm{R}}(t)$

### Observation

- $\underset{\sim}{\mathrm{Q}}(t) \leq F(t) \leq \underset{\sim}{\mathrm{R}}(t)$
- $s \lesssim t$ implies $\forall F \in X(\mathcal{T}) \quad F(s) \leq F(t)$

Observation

- $\underset{\sim}{Q}(t) \leq F(t) \leq \underset{\sim}{R}(t)$
- $s \lesssim t$ implies $\forall F \in X(\mathcal{T}) \quad F(s) \leq F(t)$

Theorem

- $\underset{\sim}{Q}(t) = \min_{F \in X(\mathcal{T})} F(t)$
- $\underset{\sim}{R}(t) = \max_{F \in X(\mathcal{T})} F(t)$
- $s \lesssim t \quad \text{iff} \quad \forall F \in X(\mathcal{T}) \quad F(s) \leq F(t)$

Remark

$\underset{\sim}{Q}(t), \underset{\sim}{R}(t) \not\in X(\mathcal{T})$

Goal Describe $X(\mathcal{T})$ explicitly

# Known: gauge points

Transform tensor into matrix and compute matrix rank

$$\begin{aligned} \zeta_1 : \mathcal{T} &\to \mathbb{R}_{\geq 0} \\ (t_{i_1 i_2 i_3})_{i_1 i_2 i_3} &\mapsto \mathtt{rank}(t_{i_1 (i_2, i_3)})_{i_1 (i_2, i_3)} \end{aligned}$$

# Known: gauge points

Transform tensor into matrix and compute matrix rank

$$\zeta_1 : \mathcal{T} \to \mathbb{R}_{\geq 0}$$
$$(t_{i_1 i_2 i_3})_{i_1 i_2 i_3} \mapsto \mathrm{rank}(t_{i_1 (i_2, i_3)})_{i_1 (i_2, i_3)}$$

**Theorem (observation)** [Strassen (1986)]
The three gauge points are in the asymptotic spectrum

$$\zeta_1, \zeta_2, \zeta_3 \in X(\mathcal{T})$$

# Known: gauge points

Transform tensor into matrix and compute matrix rank

$$\zeta_1 : \mathcal{T} \to \mathbb{R}_{\geq 0}$$
$$(t_{i_1 i_2 i_3})_{i_1 i_2 i_3} \mapsto \mathrm{rank}(t_{i_1 (i_2, i_3)})_{i_1 (i_2, i_3)}$$

**Theorem (observation)** [Strassen (1986)]
The three gauge points are in the asymptotic spectrum

$$\zeta_1, \zeta_2, \zeta_3 \in X(\mathcal{T})$$

**Example**

- mamu tensor $= \sum_{ijk \in [2]} e_{ij} \otimes e_{jk} \otimes e_{ki} \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$
- $\zeta_1$(mamu tensor) $= \mathrm{rank}(\sum_{ijk \in [2]} e_{ij} \otimes e_{jki}) = 4$
- $4 \leq \underset{\sim}{\mathrm{R}}$(mamu tensor) $= 2^\omega$

# Known: support functionals

Study probability distributions on support of $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$

$$\operatorname{supp} t = \{(i_1, i_2, i_3) : t_{i_1 i_2 i_3} \neq 0\} \subseteq [n_1] \times [n_2] \times [n_3]$$

Oblique tensor: tensor for which $\operatorname{supp} t$ is antichain in some basis

$$\zeta_\theta : \{\text{oblique tensors}\} \to \mathbb{R}_{\geq 0}$$
$$t \mapsto \max_{P \in \operatorname{prob}(\operatorname{supp} t)} 2^{\theta_1 H(P_1) + \theta_2 H(P_2) + \theta_3 H(P_3)}$$

Theorem [Strassen (1986)]
For every weighting $\theta$

$$\zeta_\theta \in X(\{\text{oblique tensors}\})$$

# Known: support functionals

Study probability distributions on support of $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$

$$\operatorname{supp} t = \{(i_1, i_2, i_3) : t_{i_1 i_2 i_3} \neq 0\} \subseteq [n_1] \times [n_2] \times [n_3]$$

Oblique tensor: tensor for which $\operatorname{supp} t$ is antichain in some basis

$$\zeta_\theta : \{\text{oblique tensors}\} \to \mathbb{R}_{\geq 0}$$
$$t \mapsto \max_{P \in \operatorname{prob}(\operatorname{supp} t)} 2^{\theta_1 H(P_1) + \theta_2 H(P_2) + \theta_3 H(P_3)}$$

Theorem [Strassen (1986)]
For every weighting $\theta$

$$\zeta_\theta \in X(\{\text{oblique tensors}\})$$

Example

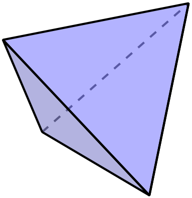
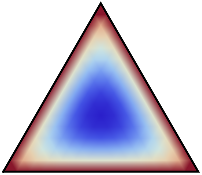
- $\zeta_{(1/3,1/3,1/3)}(\text{cap set tensor}) = 2.755\ldots$
- $\underset{\sim}{Q}(\text{cap set tensor}) \leq 2.755$ in fact, equality holds

## Summary of what was known

- *three* elements in $X(\mathcal{T})$
- infinite family in $X(\mathcal{S})$ for certain sub-semirings $\mathcal{S} \subseteq \mathcal{T}$

# 5. New: quantum functionals

infinite family of elements in $X(\mathcal{T})$ when $\mathbb{F} = \mathbb{C}$
via moment polytopes

tensor $\mapsto$  $\mapsto$ 

## Moment polytope
## Representation theoretic description

- $t \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$

## Moment polytope

### Representation theoretic description

- $t \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- $t^{\otimes n} \in (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} \cong (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$

## Moment polytope
## Representation theoretic description

- $t \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- $t^{\otimes n} \in (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} \cong (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- $S_n \times (\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \times \mathrm{GL}_{d_3}) \circlearrowright (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$

## Moment polytope
### Representation theoretic description

- $t \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- $t^{\otimes n} \in (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} \cong (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- $S_n \times (\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \times \mathrm{GL}_{d_3}) \ \circlearrowright \ (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- isotypical components are labeled by triples of partitions $\lambda_1, \lambda_2, \lambda_3 \vdash n$

## Moment polytope
## Representation theoretic description

- $t \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- $t^{\otimes n} \in (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} \cong (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- $S_n \times (\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \times \mathrm{GL}_{d_3}) \circlearrowright (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- isotypical components are labeled by triples of partitions $\lambda_1, \lambda_2, \lambda_3 \vdash n$
- $(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n} = \bigoplus_{\lambda_1, \lambda_2, \lambda_3 \vdash n} P_\lambda \, (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$

## Moment polytope
### Representation theoretic description

- $t \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$
- $t^{\otimes n} \in (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} \cong (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- $S_n \times (\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \times \mathrm{GL}_{d_3}) \ \circlearrowright \ (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$
- isotypical components are labeled by triples of partitions $\lambda_1, \lambda_2, \lambda_3 \vdash n$
- $(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n} = \bigoplus_{\lambda_1,\lambda_2,\lambda_3 \vdash n} P_\lambda \, (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})^{\otimes n}$

$$\Delta_t = \left\{ \left( \tfrac{1}{n}\lambda_1, \tfrac{1}{n}\lambda_2, \tfrac{1}{n}\lambda_3 \right) : n \in \mathbb{N},\ \lambda_1, \lambda_2, \lambda_3 \vdash n,\ P_\lambda \cdot t^{\otimes n} \neq 0 \right\}$$

## Moment polytope

## Marginal spectrum description

(natural point of view for quantum information theory)

- $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $\|s\|_2 = 1$ quantum state

## Moment polytope
## Marginal spectrum description

(natural point of view for quantum information theory)

- $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $\|s\|_2 = 1$ — quantum state
- $\rho^s = ss^* : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \to \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ — density matrix

## Moment polytope
## Marginal spectrum description

(natural point of view for quantum information theory)

- $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $\|s\|_2 = 1$ — quantum state
- $\rho^s = ss^* : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \to \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ — density matrix
- $\rho_1^s = \mathrm{Tr}_{23}\, \rho^s : \mathbb{C}^{n_1} \to \mathbb{C}^{n_1}$ — reduced density matrix

## Moment polytope
## Marginal spectrum description

(natural point of view for quantum information theory)

- $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $\|s\|_2 = 1$ — quantum state
- $\rho^s = ss^* : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \to \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ — density matrix
- $\rho_1^s = \mathrm{Tr}_{23}\, \rho^s : \mathbb{C}^{n_1} \to \mathbb{C}^{n_1}$ — reduced density matrix
- $\mathrm{spec}(\rho_1^s) = (p_1 \geq p_2 \geq \cdots)$ — ordered spectrum

## Moment polytope

## Marginal spectrum description

(natural point of view for quantum information theory)

- $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $\|s\|_2 = 1$ — quantum state
- $\rho^s = ss^* : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \to \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ — density matrix
- $\rho_1^s = \mathrm{Tr}_{23}\, \rho^s : \mathbb{C}^{n_1} \to \mathbb{C}^{n_1}$ — reduced density matrix
- $\mathrm{spec}(\rho_1^s) = (p_1 \geq p_2 \geq \cdots)$ — ordered spectrum

$$\Delta_t = \left\{ (\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \trianglelefteq t, \|s\|_2 = 1 \right\}$$

## Moment polytope

## Marginal spectrum description

(natural point of view for quantum information theory)

- $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $\|s\|_2 = 1$ quantum state
- $\rho^s = ss^* : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3} \to \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ density matrix
- $\rho_1^s = \mathrm{Tr}_{23}\, \rho^s : \mathbb{C}^{n_1} \to \mathbb{C}^{n_1}$ reduced density matrix
- $\mathrm{spec}(\rho_1^s) = (p_1 \geq p_2 \geq \cdots)$ ordered spectrum

$$\Delta_t = \left\{ (\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \trianglelefteq t, \|s\|_2 = 1 \right\}$$

## Theorem

- the two descriptions of $\Delta_t$ indeed coincide
- $\Delta_t$ is a convex polytope

[Guillemin–Sternberg, Kempf, Ness, Mumford, Brion, Kirwan, Walter–Doran–Gross–Christandl]

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$
- $\Delta_t = \{(\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq t, \|s\|_2 = 1\}$

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$
- $\Delta_t = \{(\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq t, \|s\|_2 = 1\}$
- $x = (x^1, x^2, x^3) \in \Delta_t$

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$
- $\Delta_t = \{(\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq t, \|s\|_2 = 1\}$
- $x = (x^1, x^2, x^3) \in \Delta_t$
- $H(x^1) = \sum_i x_i^1 \log_2 1/x_i^1$ Shannon entropy

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$
- $\Delta_t = \{(\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq t, \|s\|_2 = 1\}$
- $x = (x^1, x^2, x^3) \in \Delta_t$
- $H(x^1) = \sum_i x_i^1 \log_2 1/x_i^1$ Shannon entropy
- $\sum_{i=1}^{3} \theta_i H(x^i)$ $\theta$-weighted average

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$
- $\Delta_t = \{(\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq t, \|s\|_2 = 1\}$
- $x = (x^1, x^2, x^3) \in \Delta_t$
- $H(x^1) = \sum_i x_i^1 \log_2 1/x_i^1$ Shannon entropy
- $\displaystyle\sum_{i=1}^{3} \theta_i H(x^i)$ $\theta$-weighted average
- $F_\theta(t) = \sup\{2^{\sum_{i=1}^{3} \theta_i H(x^i)} : x \in \Delta_t\}$ supr. over moment polyt.

## Construction of the quantum functional

- $t \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$
- $\Delta_t = \{(\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq t, \|s\|_2 = 1\}$
- $x = (x^1, x^2, x^3) \in \Delta_t$
- $H(x^1) = \sum_i x_i^1 \log_2 1/x_i^1$ Shannon entropy
- $\displaystyle\sum_{i=1}^{3} \theta_i H(x^i)$ $\theta$-weighted average
- $F_\theta(t) = \sup\{2^{\sum_{i=1}^{3} \theta_i H(x^i)} : x \in \Delta_t\}$ supr. over moment polyt.

**Theorem** [Christandl–Vrana–Zuiddam 2017]
For every weighting $\theta$

$$F_\theta \in X(\mathcal{T})$$

$$\underset{\sim}{\mathrm{Q}}(t) \leq F_\theta(t) \leq \underset{\sim}{\mathrm{R}}(t)$$

# Example

- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

# Example

- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
- $\Delta_w = \left\{ (\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq w, \|s\| = 1 \right\} \subseteq \mathbb{R}^{2+2+2}$

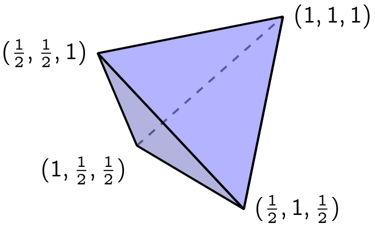
# Example

- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
- $\Delta_w = \left\{ (\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq w, \|s\| = 1 \right\} \subseteq \mathbb{R}^{2+2+2}$
- $\mathrm{spec}(\rho_i^s) = (p_i, q_i) \mapsto p_i$

# Example

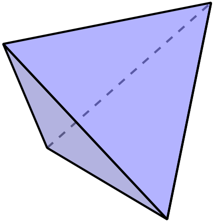
- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
- $\Delta_w = \left\{ (\mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s)) : s \leq w, \|s\| = 1 \right\} \subseteq \mathbb{R}^{2+2+2}$
- $\mathrm{spec}(\rho_i^s) = (p_i, q_i) \mapsto p_i$

$\Delta_w =$

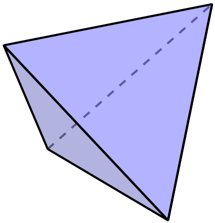


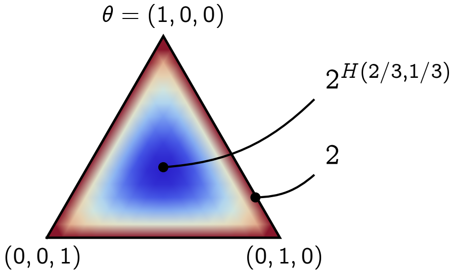

[Walter–Doran–Gross–Christandl]

- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$
- $F_\theta(w) = \max_{(p_1,p_2,p_3)\in\Delta_w} 2^{\sum_{i=1}^{3} \theta_i H(p_i, 1-p_i)}$

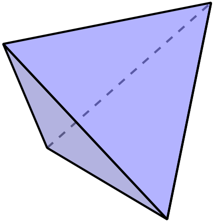
$\Delta_w =$ 

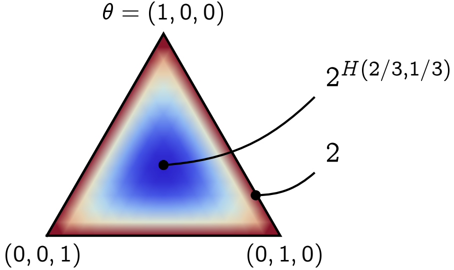
- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$
- $F_\theta(w) = \max_{(p_1,p_2,p_3) \in \Delta_w} 2^{\sum_{i=1}^{3} \theta_i H(p_i, 1-p_i)}$

$\Delta_w =$ 



- $w = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$
- $F_\theta(w) = \max_{(p_1, p_2, p_3) \in \Delta_w} 2^{\sum_{i=1}^{3} \theta_i H(p_i, 1-p_i)}$

$\Delta_w =$ 




- $\underset{\sim}{\mathrm{Q}}(w) \leq 2^{H(2/3,1/3)}$ and $2 \leq \underset{\sim}{\mathrm{R}}(w)$
- In fact these are equalities.

# Some remarks about the proof

$$F_\theta(t) = \sup\{2^{\sum_{i=1}^3 \theta_i H(x^i)} : x \in \Delta_t\}$$

To show:
$F_\theta$ is multiplicative, additive, $\leq$-monotone, $\langle n \rangle$-normalised

$$\Delta_t = \left\{ \left( \tfrac{1}{n}\lambda_1, \tfrac{1}{n}\lambda_2, \tfrac{1}{n}\lambda_3 \right) : \ n \in \mathbb{N}, \ \lambda_1, \lambda_2, \lambda_3 \vdash n, \ P_\lambda \cdot t^{\otimes n} \neq 0 \right\}$$

- sub-multiplicativity
- sub-additivity

$$\Delta_t = \left\{ \left( \mathrm{spec}(\rho_1^s), \mathrm{spec}(\rho_2^s), \mathrm{spec}(\rho_3^s) \right) : s \trianglelefteq t, \|s\|_2 = 1 \right\}$$

- super-multiplicativity
- super-additivity

# Proof ingredient for sub-multiplicativity

### Schur-Weyl duality

$$S_n \times \mathrm{GL}_d \circlearrowright (\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash_d n} [\lambda] \otimes \mathbb{S}_\lambda(\mathbb{C}^d)$$

### Kronecker coefficient

$$S_n \circlearrowright [\lambda] \otimes [\mu] \cong \bigoplus_{\nu \vdash n} [\nu]^{\oplus g_{\lambda\mu\nu}}$$

### Semigroup property

If $g_{\lambda\mu\nu} \neq 0$ and $g_{\alpha\beta\gamma} \neq 0$, then $g_{\lambda+\alpha,\mu+\beta,\nu+\gamma} \neq 0$

### Semigroup property + dimension bounds $\rightarrow$ entropy inequality

If $g_{\lambda\mu\nu} \neq 0$, then $H(\frac{1}{n}\lambda) \leq H(\frac{1}{n}\mu) + H(\frac{1}{n}\nu)$

# Relation to gauge points and support functionals

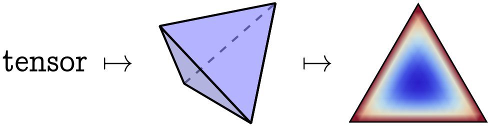
- gauge points [Strassen 86] $\zeta_1, \zeta_2, \zeta_3 \in X(\mathcal{T})$
- support functionals [Strassen 86] $\zeta_\theta \in X(\{\text{oblique tensors}\})$
- quantum functionals [CVZ 17] $F_\theta \in X(\mathcal{T})$

Relations

1. $\zeta_1 = F_{(1,0,0)}$, $\zeta_2 = F_{(0,1,0)}$, $\zeta_3 = F_{(0,0,1)}$
2. $\zeta_\theta = F_\theta$ on oblique tensors

# 6. Conclusion

- Knowing the asymptotic spectrum means knowing $\underset{\sim}{\mathrm{Q}}$ and $\underset{\sim}{\mathrm{R}}$.
- We construct an infinite family of elements in the asymptotic spectrum $X(\mathcal{T})$ of tensors over $\mathbb{C}$ via quantum information ideas and moment polytopes.




- Are these all? We do not know for 3-tensors. For 4-tensors: there are more.
- We do not improve the bounds on $2^\omega = \underset{\sim}{\mathrm{R}}(\text{mamu tensor})$.

Thank you