Asymptotic Spectrum Distance, graph limits, and the Shannon capacity

Jeroen Zuiddam

joint work with De Boer, Buys, Briët, Christandl, Leigh, Shpilka

Matrix multiplication

1969

$$\underset{\sim}{R}(\langle 2,2,2\rangle) = \underset{\sim}{R}(MM_2) = 2^{\omega}$$

Shannon capacity

1956

$$\Theta(G) = \lim_{n\to\infty} \alpha(G^{\boxtimes n})^{1/n}$$

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"Asymptotic spectrum distance"
Limits

$T_i \to T$

$G_i \to G$

- Shannon 1956 : G, H graphs

$G \boxtimes H$ strong product

$\alpha(G)$ independence number

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- Lovàsz 1979 : Lovász theta function ϑ, eigenvalues, SDP

  $\Theta(G) \leq \vartheta(G)$

  $\Theta(C_5) = \sqrt{5}$

  for perfect graphs: $\alpha(G) = \vartheta(G) = \bar{\chi}(G)$

- Haemers 1979, 1981 : Haemers bound, matrix rank
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:

- Polak-Schrijver 2019, Guruswami-Riazanov 2021, Google DeepMind 2024

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We propose a graph limit approach:

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- How to construct converging sequences?
- Where to look for "easy" graphs G_i?

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- What notion of convergence? Asymptotic spectrum distance
- How to construct converging sequences? "Induced subgraph covering"
- Where to look for "easy" graphs G_i? "Fraction graphs"

Main results:

(1) General construction of nontrivial sequences converging in asymptotic spectrum distance

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Infinite graphs as limit points (Borsuk-like, dynamical systems th.)

(3) All best-known lower bounds on Shannon capacity of small odd cycles can be obtained from a "finite version" of graph limit approach.

new bound for fifteen-cycle.

1. Shannon capacity and asymptotic spectrum distance
2. Converging sequences
3. Infinite graphs as limit points
4. Independent sets from orbits
5. Asymptotic spectrum distance of tensors, matrix multiplication

[Briët, Christandl, Leigh, Shpilka, Z. 24]

1. Shannon capacity and asymptotic spectrum distance

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<u>Def</u>. Asymptotic spectrum X = set of all functions f: graphs $\to \mathbb{R}_{\geq 0}$ that are $\boxtimes$-mult, $\sqcup$-add, K_1-norm and cohom – mon.

<u>Ex</u>. X contains Lovász theta, fract. Haemers, fract. clique cov. nr., ...

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Let $F_{G_i}, F_H \in X$ s.t. $F_{G_i}(G_i) = \Theta(G_i)$ and $F_H(H) = \Theta(H)$ (duality).

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Lemma. TFAE

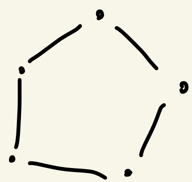
- $d(G,H) \leq \frac{a}{b}$
- $(E_b \boxtimes G)^{\boxtimes n} \leq ((E_b \boxtimes H) \sqcup E_a)^{\boxtimes (n + o(n))}$ and

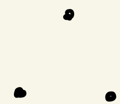
  $(E_b \boxtimes H)^{\boxtimes n} \leq ((E_b \boxtimes G) \sqcup E_a)^{\boxtimes (n + o(n))}$

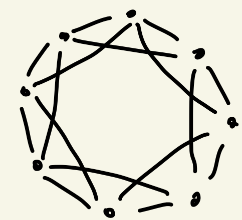
cohom →

2. Converging sequences

Def. Fraction graph $E_{a/b}$ has vertex set $\mathbb{Z}/a\mathbb{Z}$ and $u \sim v$ iff $-b < u-v \bmod a < b$.

$E_{\frac{5}{2}} =$ 

$E_{\frac{3}{1}} =$ 

$E_{\frac{8}{3}} =$ 

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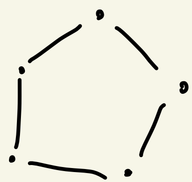
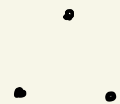
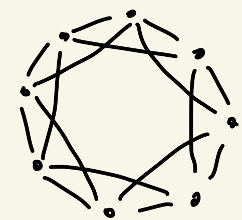
$E_{\frac{5}{2}} =$  $E_{\frac{3}{1}} =$  $E_{\frac{8}{3}} =$ 

Lemma. $E_{\frac{a}{b}} \leq E_{\frac{c}{d}}$ iff $\frac{a}{b} \leq \frac{c}{d}$ (in $\mathbb{Q}$).

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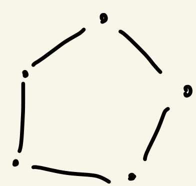
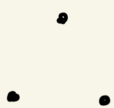
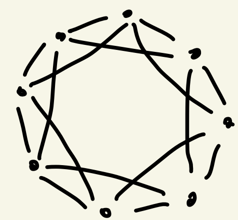
$E_{\frac{5}{2}} =$  $E_{\frac{3}{1}} =$  $E_{\frac{8}{3}} =$ 

Lemma. $E_{\frac{a}{b}} \leq E_{\frac{c}{d}}$ iff $\frac{a}{b} \leq \frac{c}{d}$ (in $\mathbb{Q}$).

Theorem A. For any $a/b \geq 2$, if $c_n/d_n \to a/b$ from above, then $E_{c_n/d_n} \to E_{a/b}$

Theorem B. For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy.

Ingredients

Lemma 1. G vertex transitive, $S \subseteq V(G)$, $\mathcal{F} \in X$, then

$$\mathcal{F}(G[S]) \leq \mathcal{F}(G) \leq \frac{|G|}{|S|} \cdot \mathcal{F}(G[S]).$$

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Proof sketch $G \le E_N \boxtimes G[S]$ with $N = \lceil |G| \cdot |S|^{-1} \cdot \log |G| \rceil$.

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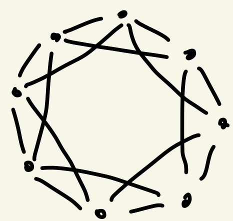
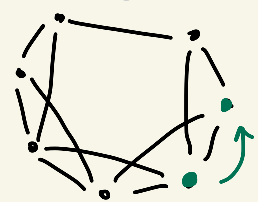
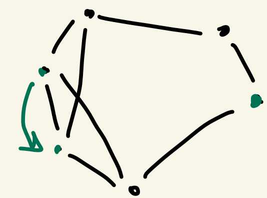
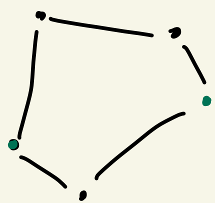
Lemma 2 $E_{\frac{8}{3}} =$  remove vertex $\longrightarrow$  $\le$ $\ge$ 

$\ge$ $\le$  $= E_{\frac{5}{2}}$

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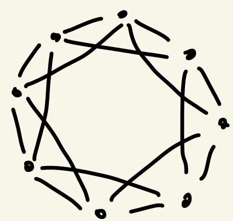
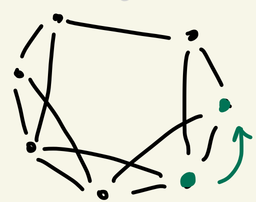
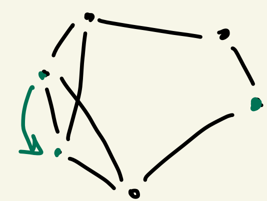


$\xrightarrow{\text{remove vertex}}$



$\le$
$\ge$



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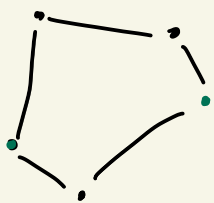
$\le$



$= E_{\frac{5}{2}}$

"Euclid's algorithm". $E_{p/q}$ - vertex is equivalent to $E_{p'/q'}$ for $p' < p$, $q' < q$ with $p \cdot q' - q \cdot p' = 1$.

Consequence: $F(E_{p'/q'}) \le F(E_{p/q}) \le \frac{p}{p-1} F(E_{p'/q'})$.

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For every n, $c_n \cdot b - d_n \cdot a = 1$. So

$$F(E_{a/b}) \leq F(E_{c_n/d_n}) \leq \frac{c_n}{c_n - 1} F(E_{a/b})$$

Let $n \to \infty$, then $c_n \to \infty$. $\square$

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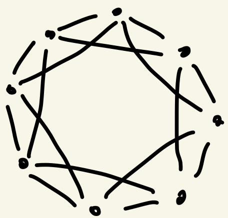
Proof sketch: continued fraction convergents $\frac{p_0}{q_0} < \frac{p_2}{q_2} \cdots < r < \cdots \frac{p_3}{q_3} < \frac{p_1}{q_1}$

$q_n p_{n-1} - p_n q_{n-1} = (-1)^n$. $\square$

3. Infinite graphs as limit points



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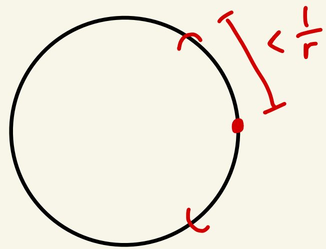
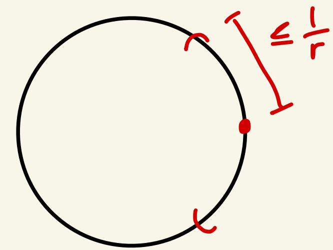


E_r^{o}



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Theorem. For any irrational $r \geq 2$, if $a_n/b_n \to r$, then $E_{a_n/b_n} \to E_r^{o}$

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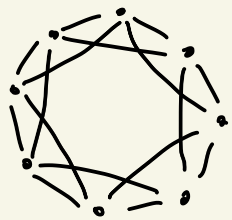

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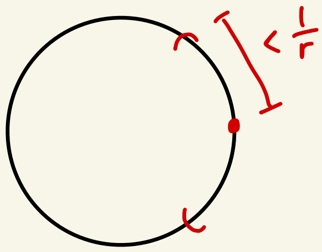

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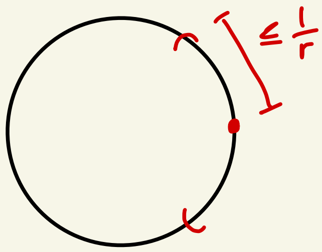

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Theorem For any irrational $r \geq 2$, if $a_n/b_n \to r$, then $E_{a_n/b_n} \to E_r^{o}$

Theorem TFAE: (i) E_r^c and E_r^o are asymptotically equivalent

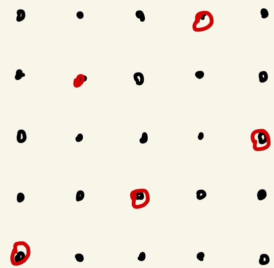
(ii) $a_n/b_n \to p/q$ from below $\Rightarrow$ $E_{a_n/b_n} \to E_{p/q}$.

Theorem E_r^c and E_r^o are not equivalent.

4. Independent sets from orbits

$$\Theta(C_5) = \sqrt{5}$$

$$\alpha(C_5^{\boxtimes 2}) = 5$$



$$\{ t \cdot (1,2) : t \in \mathbb{Z}_5 \}$$

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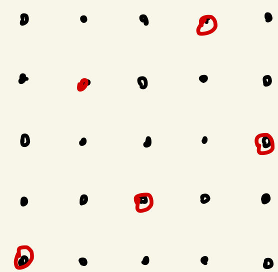

$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$

G	H	orbit independent set in $H^{\boxtimes k}$	reduction	$\leq \Theta(G)$
$E_{5/2}$	$E_{5/2}$	$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$	$H = G$	2.23 [Sha56]
$E_{7/2}$	$E_{382/108}$	$\{t \cdot (1,7,7^2,7^3,7^4) : t \in \mathbb{Z}_{382}\}$	$G \leq H$	3.25 [PS19]
$E_{9/2}$	$E_{9/2}$	$\{s \cdot (1,0,2) + t \cdot (0,1,4) : s,t \in \mathbb{Z}_9\}$	$H = G$	4.32 [BMR+71]
$E_{11/2}$	$E_{148/27}$	$\{t \cdot (1,11,11^2) : t \in \mathbb{Z}_{148}\}$	$H \leq G$	5.28 [BMR+71]
$E_{13/2}$	$E_{247/38}$	$\{t \cdot (1,19,117) : t \in \mathbb{Z}_{247}\}$	$H \leq G$	6.27 [BMR+71][18]
$E_{15/2}$	$E_{2873/381}$	$\{t \cdot (1,15,1073,1125) : t \in \mathbb{Z}_{2873}\}$	$G \leq H$	7.30 (Section 6.2)

5 Asymptotic spectrum distance of tensors, matrix multiplication

$$d(S,T) = \max_{F \in X} |F(S) - F(T)|$$

$$T_i \to \langle 2,2,2 \rangle \Rightarrow \underset{\sim}{R}(T_i) \to 2^{\omega}$$

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Theorem same for $\underset{\sim}{Q}$, $\underset{\sim}{SR}$ Much harder proof!

Theorem Over $\mathbb{C}$, $\underset{\sim}{SR}$ is discrete. Relies on rep. theory, moment pol., quantum functionals

Questions:

- Is asymptotic rank discrete over $\mathbb{C}$?
- other notions of convergence for tensors?