

Universal points in the asymptotic spectrum of tensors

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arXiv: 1709.07851

Asymptotic transformations between tensors

$$s \gtrsim t$$

The asymptotic spectrum of tensors

$$\Delta(\{\text{tensors}\})$$

Quantum functionals

$$F_\theta \in \Delta(\{\text{tensors}\})$$

Tensors

$$t = (t_{i_1 i_2 i_3})_{i_1 i_2 i_3} \in \mathbb{C}^{n_1 \times n_2 \times n_3} \cong \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$$
$$= \sum_{i_1 i_2 i_3} t_{i_1 i_2 i_3} |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle$$

Diagonal tensor

$$\langle n \rangle = (\delta(i_1 = i_2 = i_3))_{i_1 i_2 i_3} \in \mathbb{C}^{n \times n \times n} \cong \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$$
$$= \sum_{i \in [n]} |i\rangle \otimes |i\rangle \otimes |i\rangle$$

$\langle 2 \rangle$ = Greenberger–Horne–Zeilinger (GHZ) state

Restriction

$$t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$$

$$s \in \mathbb{C}^{m_1 \times m_2 \times m_3}$$

$t \geq s$ if there are matrices

$$A_1 \in \mathbb{C}^{m_1 \times n_1}$$

$$A_2 \in \mathbb{C}^{m_2 \times n_2}$$

$$A_3 \in \mathbb{C}^{m_3 \times n_3}$$

such that

$$(A_1 \otimes A_2 \otimes A_3) \cdot t = s$$

$$\begin{aligned} (I \otimes A_2 \otimes I) \cdot |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \\ = |i_1\rangle \otimes A_2 |i_2\rangle \otimes |i_3\rangle \end{aligned}$$

stochastic local operations and classical communication (slocc)

[Dür–Vidal–Cirac'00]

Direct sum and tensor product

$$t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$$

$$s \in \mathbb{C}^{m_1 \times m_2 \times m_3}$$

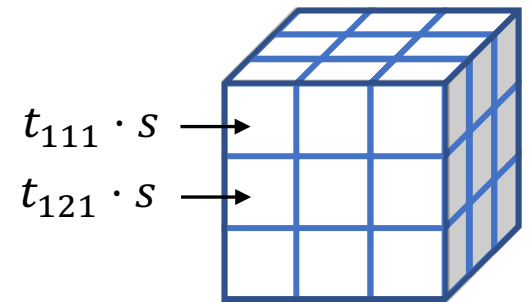
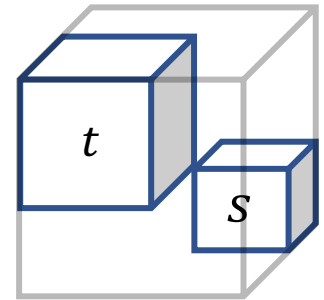
$$t \oplus s \in \mathbb{C}^{(n_1+m_1) \times (n_2+m_2) \times (n_3+m_3)}$$

$$\sum_{i_1 i_2 i_3} t_{i_1 i_2 i_3} |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle +$$

$$\sum_{j_1 j_2 j_3} s_{j_1 j_2 j_3} |n_1 + j_1\rangle \otimes |n_2 + j_2\rangle \otimes |n_3 + j_3\rangle$$

$$t \otimes s \in \mathbb{C}^{(n_1 \cdot m_1) \times (n_2 \cdot m_2) \times (n_3 \cdot m_3)}$$

$$\sum_{\substack{i_1 i_2 i_3 \\ j_1 j_2 j_3}} t_{i_1 i_2 i_3} \cdot s_{j_1 j_2 j_3} |i_1, j_1\rangle \otimes |i_2, j_2\rangle \otimes |i_3, j_3\rangle$$



Asymptotic restriction

$$t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$$

$$s \in \mathbb{C}^{m_1 \times m_2 \times m_3}$$

$t \succeq s$ if there are numbers

$$\epsilon_N \in \mathbb{N} \text{ with } \frac{\epsilon_N}{N} \rightarrow 0 \text{ when } N \rightarrow \infty$$

and

$$\forall N \in \mathbb{N} \quad t^{\otimes N + \epsilon_N} \geq s^{\otimes N}$$

asymptotic slocc transformation at rate 1

Rank and sub-rank

$$\langle m \rangle \leq t \leq \langle n \rangle ?$$

currency diagonals tensors $\langle n \rangle \in \mathbb{C}^{n \times n \times n}$

cost of t rank $R(t) = \min\{n \in \mathbb{N} : t \leq \langle n \rangle\} < \infty$

value of t sub-rank $Q(t) = \max\{m \in \mathbb{N} : \langle m \rangle \leq t\} \geq 0$

Computational complexity [Håstad90, Shitov16]

Deciding $R(t) \leq x$ is NP-hard

Asymptotic rank and asymptotic sub-rank

asymptotic rank $\underline{R}(t) = \lim_{N \rightarrow \infty} R(t^{\otimes N})^{1/N}$

asymptotic sub-rank $\underline{Q}(t) = \lim_{N \rightarrow \infty} Q(t^{\otimes N})^{1/N}$

$$Q(t) \leq \underline{Q}(t) \leq \underline{R}(t) \leq R(t)$$

Connections

$$\underset{\sim}{R}(\text{"matrix multiplication tensor"}) = 2^\omega \quad 2 \leq \omega \leq 2.3728639\dots$$

- complexity of matrix multiplication in **algebraic complexity theory** [Strassen, Coppersmith, Winograd, ...]
- upper bounds nondeterministic multiparty quantum **communication complexity** for pairwise equality in a triangle [Buhrman et al. '16]

$$\underset{\sim}{Q}(\text{special tensor})$$

- upper bound query complexity in **property testing** as in e.g. [Green'05, Fu–Kleinberg'13, ...]
- upper bound size of “cap sets” in **combinatorics** (requires finite field) [Ellenberg–Gijswijt'17, Tao, Tao–Sawin, Blasiak et al. '17, Christandl–Vrana–Zuiddam'17]

\approx

- asymptotic slocc transformation in **quantum information theory** [Yu et al. '10, Christandl–Vrana'13, Christandl–Vrana–Zuiddam'16, ...]

Problem: deciding asymptotic restriction

$$t \gtrsim s \quad ?$$

$$\text{i.e. } t^{\otimes N+o(N)} \geq s^{\otimes N} \quad ?$$

$$(\text{or } \underbrace{R(t)} = ? \quad \text{or} \quad \underbrace{Q(t)} = ?)$$

Two directions

- **Constructions:** matrices carrying out $t \gtrsim s$
- **Obstructions:** “certificates” that prohibit $t \gtrsim s$

The asymptotic spectrum of tensors (Strassen 1986)

$\Delta(\{\text{tensors}\})$ = set of maps $F : \{\text{tensors}\} \rightarrow \mathbb{R}_{\geq 0}$ such that

1. if $t \geq s$ then $F(t) \geq F(s)$ monotone
2. $F(s \oplus t) = F(s) + F(t)$ additive
3. $F(s \otimes t) = F(s)F(t)$ multiplicative
4. $F(\langle n \rangle) = n$ for $n \in \mathbb{N}$ normalized

$$t \succeq s \Rightarrow t^{\otimes N + \epsilon_N} \geq s^{\otimes N} \Rightarrow F(t)^{N + \epsilon_N} \geq F(s)^N \Rightarrow F(t) \geq F(s)$$

$F(t) < F(s) \Rightarrow t \not\succeq s$ F serves as an obstruction!

$$\underline{Q}(t) \leq F(t) \leq \underline{R}(t)$$

The spectral theorem (Strassen 1986)

$$\Delta(\{\text{tensors}\}) = \{ F : \{\text{tensors}\} \rightarrow \mathbb{R}_{\geq 0} : \text{monot, add, mult, norm} \}$$

i. $t \succeq s$ **iff** $\forall F \in \Delta(\{\text{tensors}\}) \quad F(t) \geq F(s)$

ii. $\underline{Q}(t) = \min_{F \in \Delta(\{\text{tensors}\})} F(t)$

$$\underline{R}(t) = \max_{F \in \Delta(\{\text{tensors}\})} F(t)$$

Remark

$\underline{Q}(t)$ and $\underline{R}(t)$ are **not** in $\Delta(\{\text{tensors}\})$!

Remark

Theorem still holds when $\Delta(\{\text{tensors}\})$ is replaced by $\Delta(S)$ where $S \subseteq \{\text{tensors}\}$ closed under \otimes, \oplus and $\langle 1 \rangle \in S$

Known: gauge points (Strassen 1986)

Transform tensor into matrix and take **matrix rank**

$$\zeta_1 : \{\text{tensors}\} \rightarrow \mathbb{R}_{\geq 0}$$

$$\left(t_{i_1 i_2 i_3} \right)_{i_1 i_2 i_3} \mapsto \text{rank} \left(t_{i_1(i_2, i_3)} \right)_{i_1(i_2, i_3)}$$

Theorem (observation)

$$\zeta_1, \zeta_2, \zeta_3 \in \Delta(\{\text{tensors}\})$$

$$= \{F : \{\text{tensors}\} \rightarrow \mathbb{R}_{\geq 0} : \text{monot, add, mult, norm}\}$$

Known: support functionals for “oblique tensors” (Strassen 1986)

Study **probability distributions on the support** of $t \in \mathbb{C}^{n_1 \times n_2 \times n_3}$
 $\text{supp } t = \{(i_1, i_2, i_3) : t_{i_1 i_2 i_3} \neq 0\} \subseteq [n_1] \times [n_2] \times [n_3]$

oblique tensor means $\text{supp } t$ is an antichain

$$\theta_1, \theta_2, \theta_3 \in \mathbb{R}_{\geq 0} \quad \theta_1 + \theta_2 + \theta_3 = 1$$

$$\zeta_\theta : \{\text{oblique tensors}\} \rightarrow \mathbb{R}_{\geq 0}$$
$$t \mapsto \max_{P \in \mathcal{P}(\text{supp } t)} 2^{\theta_1 H(P_1) + \theta_2 H(P_2) + \theta_3 H(P_3)}$$

Theorem

$$\zeta_\theta \in \Delta(\{\text{oblique tensors}\})$$
$$= \{F : \{\text{oblique tensors}\} \rightarrow \mathbb{R}_{\geq 0} : \text{monot, add, mult, norm}\}$$

New: quantum functionals

New: quantum functionals

Study marginal density matrices

- $s \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, $\|s\|_2 = 1$ pure tripartite quantum state
- $\rho = ss^\dagger \in \mathbb{C}^{(n_1 \times n_2 \times n_3) \times (n_1 \times n_2 \times n_3)}$ density matrix
- $\rho_1 = \text{Tr}_{2,3} \rho \in \mathbb{C}^{n_1 \times n_1}$ marginals
- $H(\rho_1)$ quantum entropy

$$\theta_1, \theta_2, \theta_3 \in \mathbb{R}_{\geq 0} \quad \theta_1 + \theta_2 + \theta_3 = 1$$

$$F_\theta : \{\text{tensors}\} \rightarrow \mathbb{R}_{\geq 0}$$

$$t \mapsto \sup\{2^{\theta_1 H(\rho_1) + \theta_2 H(\rho_2) + \theta_3 H(\rho_3)} : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$$

Main theorem (Christandl–Vrana–Zuiddam 2017)

$$F_\theta \in \Delta(\{\text{tensors}\}) = \{F: \{\text{tensors}\} \rightarrow \mathbb{R}_{\geq 0} : \text{monot, add, mult, norm}\}$$

Progress: Explicit points in the asymptotic spectrum of tensors

[S '86]	Gauge points	$\zeta_1, \zeta_2, \zeta_3 \in \Delta(\{\text{tensors}\})$
[S '86]	Support functionals	$\zeta_\theta \in \Delta(\{\text{oblique tensors}\})$
[CVZ '17]	Quantum functionals	$F_\theta \in \Delta(\{\text{tensors}\})$

$$\begin{aligned}\theta_1, \theta_2, \theta_3 &\in \mathbb{R}_{\geq 0} \\ \theta_1 + \theta_2 + \theta_3 &= 1\end{aligned}$$

Relations

1. $\zeta_1 = F_{(1,0,0)}$ $\zeta_2 = F_{(0,1,0)}$ $\zeta_3 = F_{(0,0,1)}$
2. $\zeta_\theta = F_\theta$ on oblique tensors

See our paper for more relations

Crucial connection: Entanglement polytopes

$$F_\theta(t) = \sup\{2^{\theta_1 H(\rho_1) + \theta_2 H(\rho_2) + \theta_3 H(\rho_3)} : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$$

$$\downarrow r_i = \text{spec}(\rho_i)$$

$$F_\theta(t) = \sup\{2^{\theta_1 H(r_1) + \theta_2 H(r_2) + \theta_3 H(r_3)} : r \in \mathbf{E}_t\}$$

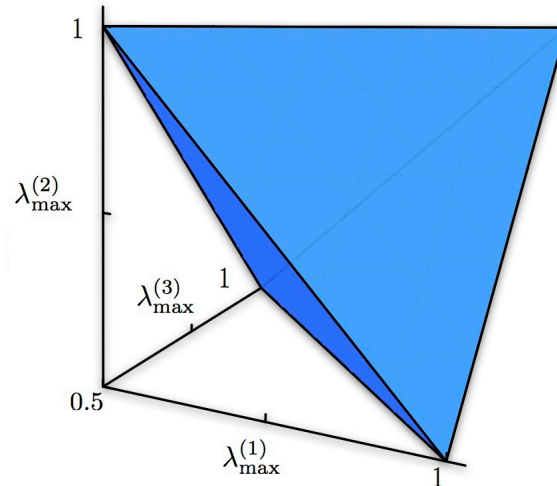
$$\mathbf{E}_t = \{(\text{spec}(\rho_1), \text{spec}(\rho_2), \text{spec}(\rho_3)) : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$$

Crucial connection: Entanglement polytopes

$$F_{\theta}(t) = \sup\{2^{\theta_1 H(r_1) + \theta_2 H(r_2) + \theta_3 H(r_3)} : r \in \mathbf{E}_t\}$$

$$\mathbf{E}_t = \{(\text{spec}(\rho_1), \text{spec}(\rho_2), \text{spec}(\rho_3)) : \rho = ss^\dagger; s \leq t; \|s\|_2 = 1\}$$

\mathbf{E}_t is the **entanglement polytope** of t



[Walter–Doran–Gross–Christandl‘13, Sawicki–Oszmaniec–Kuś‘14]
based on [Ness–Mumford‘84, Brion‘87]

Crucial connection: Entanglement polytopes

$$F_\theta(t) = \sup\{2^{\theta_1 H(r_1) + \theta_2 H(r_2) + \theta_3 H(r_3)} : r \in \mathbf{E}_t\}$$

Representation theory

$\lambda \vdash N$ is an integer partition of N

$P_\lambda^{\mathbb{C}^n} : (\mathbb{C}^n)^{\otimes N} \rightarrow$ “ GL_n -isotypic component of type λ ”

$\mathbf{E}_t =$ Euclidean closure of

$$\left\{ \left(\frac{\lambda}{N}, \frac{\mu}{N}, \frac{\nu}{N} \right) : \lambda, \mu, \nu \vdash N; \left(P_\lambda^{\mathbb{C}^{n_1}} \otimes P_\mu^{\mathbb{C}^{n_2}} \otimes P_\nu^{\mathbb{C}^{n_3}} \right) \cdot t^{\otimes N} \neq 0 \right\}$$

Connection leads to key ingredients for our proof:

entropy inequalities of Kronecker and Littlewood-Richardson coefficients

Maximal values of the quantum functional

$$F_\theta \leq n_1^{\theta_1} n_2^{\theta_2} n_3^{\theta_3} \text{ on } \mathbb{C}^{n_1 \times n_2 \times n_3}$$

Theorem [Christandl–Vrana–Zuiddam 2017]

$$\theta_1, \theta_2, \theta_3 > 0$$

The maximal value of F_θ equals $n_1^{\theta_1} n_2^{\theta_2} n_3^{\theta_3}$

iff

$\mathbb{C}^{n_1 \times n_2 \times n_3}$ contains a pure quantum state with completely mixed marginals

Bryan, Reichstein, and Van Raamsdonk characterized such formats (n_1, n_2, n_3) in Monday QIP talk!

Conclusion

- $\approx \underline{R}(t) \quad \underline{Q}(t)$
 - algebraic complexity theory
 - property testing
 - combinatorics
 - communication complexity
 - asymptotic slocc
- characterized by the **asymptotic spectrum of tensors** $\Delta(\{\text{tensors}\})$
- We construct a nontrivial family of points in $\Delta(\{\text{tensors}\})$ using quantum information methods: **quantum functionals** F_θ
- **Are these all elements of $\Delta(\{\text{tensors}\})$?**
(If yes, then matrix multiplication exponent ω equals 2.)