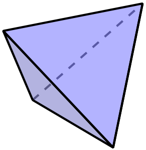
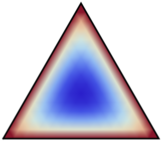
Universal points in the asymptotic spectrum of tensors

tensor $\mapsto$  $\mapsto$ 

Jeroen Zuiddam
CWI $\mapsto$ IAS

Péter Vrana BME Budapest
Matthias Christandl University of Copenhagen

constructions
e.g.
algorithms

obstructions
e.g.
“lower bounds”

Motivated by problems in

- algebraic complexity theory
- combinatorics

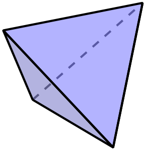
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

we study Strassen's "lower bound" method

asymptotic spectrum of tensors

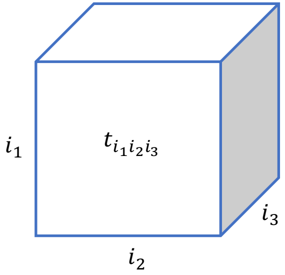
using

- moment polytopes / entanglement polytopes
- representation theory
- quantum information theory



3-tensor over $\mathbb{F}$

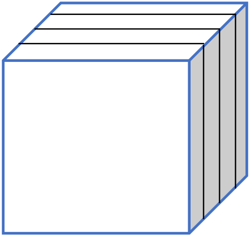
$$t = (t_{i_1 i_2 i_3})_{i_1 i_2 i_3} = \sum_{i_1 i_2 i_3} t_{i_1 i_2 i_3}\, e_i \otimes e_j \otimes e_k \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$$



restriction preorder

$$s \leq t \quad \text{iff} \quad s = (A_1 \otimes A_2 \otimes A_3) \cdot t \quad \text{for some linear maps } A_i$$

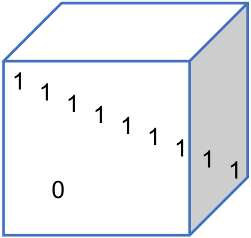
$$\begin{aligned}(A_1 \otimes I \otimes I) \cdot e_i \otimes e_j \otimes e_k &= (A_1 e_i) \otimes e_j \otimes e_k \\ &= \sum_{\ell} (A_1)_{\ell i}\, e_\ell \otimes e_j \otimes e_k\end{aligned}$$



t

diagonal tensors

$$\langle n \rangle := \sum_{i=1}^{n} e_i \otimes e_i \otimes e_i \;\in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n \quad \text{for } n \in \mathbb{N}$$



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rank and sub-rank

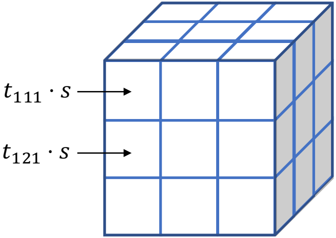
$$\mathrm{R}(t) := \min\{\, n \in \mathbb{N} \,:\, t \leq \langle n \rangle \,\} \qquad \text{cost}$$

$$\mathrm{Q}(t) := \max\{\, m \in \mathbb{N} \,:\, \langle m \rangle \leq t \,\} \qquad \text{value}$$

Kronecker product $\otimes$ on tensors

$$(\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}) \times (\mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2} \otimes \mathbb{F}^{m_3}) \to \mathbb{F}^{n_1 m_1} \otimes \mathbb{F}^{n_2 m_2} \otimes \mathbb{F}^{n_3 m_3}$$

$$(t, s) \mapsto t \otimes s := \sum_{\substack{i_1 i_2 i_3 \\ j_1 j_2 j_3}} t_{i_1 i_2 i_3} \cdot s_{j_1 j_2 j_3} \; e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3 j_3}$$



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asymptotic rank and asymptotic sub-rank

$$\underset{\sim}{\mathrm{R}}(t) := \lim_{N \to \infty} \mathrm{R}(t^{\otimes N})^{1/N} = \inf_N \mathrm{R}(t^{\otimes N})^{1/N}$$

$$\underset{\sim}{\mathrm{Q}}(t) := \lim_{N \to \infty} \mathrm{Q}(t^{\otimes N})^{1/N} = \sup_N \mathrm{Q}(t^{\otimes N})^{1/N}$$

asymptotic restriction

$$s \lesssim t \quad \text{iff} \quad \forall n\colon \; s^{\otimes n} \leq t^{\otimes n + a_n} \quad \tfrac{a_n}{n} \to 0, \quad n \to \infty$$

Example: Fast $n \times n$ matrix multiplication

$$\begin{pmatrix} 1 & 4 & 0 & 3 & 5 & 8 & 2 & 10 & 8 & 5 \\ 5 & 9 & 5 & 9 & 7 & 6 & 10 & 6 & 7 & 1 \\ 0 & 3 & 7 & 8 & 3 & 10 & 3 & 2 & 3 & 1 \\ 5 & 6 & 4 & 2 & 2 & 10 & 7 & 1 & 5 & 10 \\ 6 & 6 & 9 & 0 & 5 & 10 & 6 & 5 & 2 & 3 \\ 5 & 8 & 9 & 0 & 8 & 3 & 8 & 2 & 4 & 2 \\ 0 & 7 & 1 & 6 & 8 & 0 & 0 & 0 & 2 & 3 \\ 8 & 0 & 7 & 7 & 5 & 0 & 5 & 3 & 4 & 9 \\ 5 & 2 & 1 & 8 & 6 & 2 & 0 & 5 & 2 & 0 \\ 9 & 2 & 8 & 3 & 8 & 1 & 1 & 7 & 8 & 7 \end{pmatrix} \cdot \begin{pmatrix} 9 & 5 & 7 & 5 & 1 & 5 & 3 & 0 & 6 & 0 \\ 3 & 6 & 2 & 0 & 7 & 6 & 6 & 6 & 7 & 8 \\ 10 & 5 & 10 & 2 & 8 & 1 & 0 & 6 & 4 & 0 \\ 9 & 5 & 0 & 2 & 1 & 3 & 10 & 7 & 7 & 8 \\ 0 & 0 & 10 & 4 & 6 & 1 & 5 & 1 & 4 & 2 \\ 3 & 5 & 8 & 6 & 1 & 0 & 3 & 0 & 0 & 9 \\ 1 & 1 & 2 & 1 & 7 & 1 & 3 & 4 & 1 & 4 \\ 6 & 3 & 3 & 9 & 6 & 5 & 6 & 3 & 4 & 10 \\ 3 & 2 & 9 & 7 & 6 & 4 & 4 & 1 & 4 & 3 \\ 7 & 10 & 2 & 5 & 1 & 6 & 6 & 8 & 9 & 6 \end{pmatrix} = ?$$

by hand n^3

Strassen $n^{2.8}$

current best $n^{2.38}$

lower bound n^2

optimal n^ω all asymptotic in n

Lemma

$2^\omega = \underset{\sim}{\mathrm{R}}(\text{mamu tensor}), \quad \text{mamu tensor} \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$

Example: Cap set problem

$(\mathbb{Z}/3\mathbb{Z})^n$

• $u + 2v$

• $u + v$

• u

$(\mathbb{Z}/3\mathbb{Z})^3$

cap set

subset $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ without lines, except trivial lines (u, u, u)

Edel (2004)	$\exists A: \ |A| \geq 2.2^n$
Gijswijt–Ellenberg (2016)	$|A| \leq 2.755^n$

$2.755... = \underset{\sim}{Q}(\text{cap set tensor})$, cap set tensor $\in \mathbb{F}_3^3 \otimes \mathbb{F}_3^3 \otimes \mathbb{F}_3^3$ [Tao]

$$\text{constructions} \leq \underset{\sim}{\mathrm{Q}}(t) \leq \text{obstructions} \leq \underset{\sim}{\mathrm{R}}(t) \leq \text{constructions}$$

The asymptotic spectrum of tensors

Definition $X(\text{3-tensors over } \mathbb{F}) :=$ the set of all maps

$$F : \{\text{3-tensors over } \mathbb{F}\} \to \mathbb{R}_{\geq 0}$$

such that for all s, t

1. if $t \geq s$ then $F(t) \geq F(s)$ $\quad (A_1 \otimes A_2 \otimes A_3) \cdot t = s$
2. $F(s \oplus t) = F(s) + F(t)$
3. $F(s \otimes t) = F(s)F(t)$
4. $F(\langle n \rangle) = n$ for $n \in \mathbb{N}$ $\quad \langle n \rangle = \sum_{i=1}^{n} e_i \otimes e_i \otimes e_i$

Exercise $\underset{\sim}{Q}(t) \leq F(t) \leq \underset{\sim}{R}(t)$ for $F \in X(\text{3-tensors over } \mathbb{F})$

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Exercise $\underset{\sim}{\mathrm{Q}}(t) \leq F(t) \leq \underset{\sim}{\mathrm{R}}(t)$ for $F \in X$(3-tensors over $\mathbb{F}$)

Theorem [Strassen (1986)] completeness!

- $\underset{\sim}{\mathrm{Q}}(t) = \min\{F(t) : F \in X(\text{3-tensors over } \mathbb{F})\}$
- $\underset{\sim}{\mathrm{R}}(t) = \max\{F(t) : F \in X(\text{3-tensors over } \mathbb{F})\}$
- $s \lesssim t$ iff $\forall F \in X(\text{3-tensors over } \mathbb{F})$: $F(s) \leq F(t)$

Remark $\underset{\sim}{\mathrm{Q}}, \underset{\sim}{\mathrm{R}} \notin X$(3-tensors over $\mathbb{F}$)

$$F \in X \,(\text{3-tensors over } \mathbb{F})$$

$$\text{constructions} \leq \underset{\sim}{\mathrm{Q}}(t) \leq F(t) \leq \underset{\sim}{\mathrm{R}}(t) \leq \text{constructions}$$

- complete set of obstructions!
- but what do they look like?

Known: gauge points

"flatten" tensor into matrix and compute matrix rank

- $t = (t_{i_1 i_2 i_3})_{i_1 i_2 i_3} \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$
- flatten to matrix: $t^1 = (t_{i_1 (i_2, i_3)})_{i_1 (i_2, i_3)} \in \mathbb{F}^{n_1} \otimes (\mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3})$
- gauge point: $\zeta_{(1,0,0)} : t \mapsto \mathrm{rank}\,(t^1)$

Easy theorem

$\zeta_{(1,0,0)}, \zeta_{(0,1,0)}, \zeta_{(0,0,1)} \in X\,(\text{3-tensors over } \mathbb{F})$

Example

- $4 = \zeta_{(1,0,0)}(\text{mamu tensor}) \leq \underset{\sim}{\mathrm{R}}(\text{mamu tensor}) = 2^{\omega}$
- $2 \leq \omega$

Known: support functionals

$\{\text{oblique 3-tensor over } \mathbb{F}\} \subsetneq \{\text{3-tensor over } \mathbb{F}\}$

Theorem [Strassen (1986)]

$\zeta_{(\theta_1,\theta_2,\theta_3)} \in X(\text{oblique 3-tensors over } \mathbb{F})$

$\theta_i \in \mathbb{R}_{\geq 0}$, $\theta_1 + \theta_2 + \theta_3 = 1$

Example

- $\underset{\sim}{Q}(\text{cap set tensor}) \leq \zeta_{(1/3,\,1/3,\,1/3)}(\text{cap set tensor}) = 2.755\ldots$ [Strassen (1991)]
- reproves Gijswijt–Ellenberg result

New result

Known [Strassen]

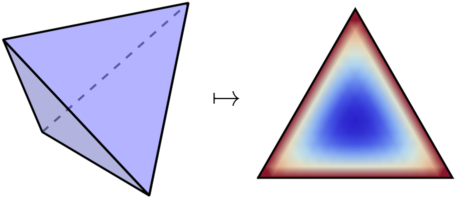
- three elements $\zeta_{(1,0,0)}, \zeta_{(0,1,0)}, \zeta_{(0,0,1)} \in X(\text{3-tensors over } \mathbb{F})$
- infinite family $\{\zeta_\theta : \theta\} \subseteq X(\underline{\text{oblique}} \text{ 3-tensors over } \mathbb{F})$

New result

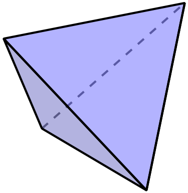
explicit infinite family $\{F_\theta : \theta\} \subseteq X(\text{3-tensors over } \mathbb{C})$ using

- moment polytopes
- Shannon entropy

called: quantum functionals

tensor $\mapsto$ 

Definition: moment polytope

$t \mapsto \Delta(t) =$ 

- Restriction $\leq$ has topologically closed version: degeneration $\trianglelefteq$
- Let $s \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, $s \trianglelefteq t$, $\|s\|_2 = 1$

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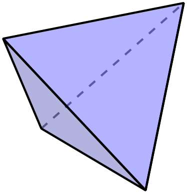
$t \mapsto \Delta(t) =$ 

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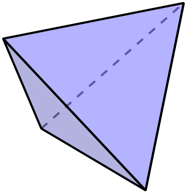
$t \mapsto \Delta(t) =$ 

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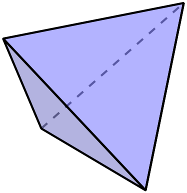
$t \mapsto \Delta(t) =$ 

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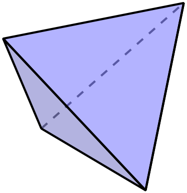
$t \mapsto \Delta(t) =$ 

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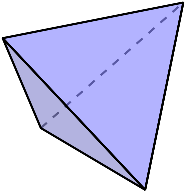
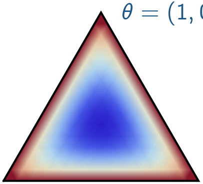
- probability vector: $\text{spec}(s^1) := (\sigma_1^2, \sigma_2^2, \cdots, \sigma_{n_1}^2)$
- moment polytope i.e. entanglement polytope:

  $\Delta(t) := \left\{(\text{spec}(s^1), \text{spec}(s^2), \text{spec}(s^3)) : s \trianglelefteq t, \|s\|_2 = 1\right\}$

Quantum functionals

$t \mapsto \Delta(t) =$  $\mapsto F_\theta(t) =$ 

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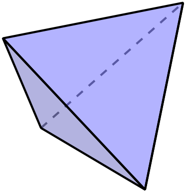
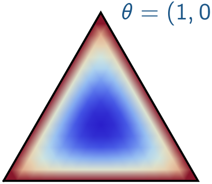
$t \mapsto \Delta(t) =$  $\mapsto F_\theta(t) =$ 

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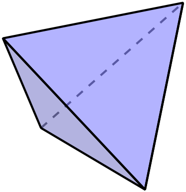

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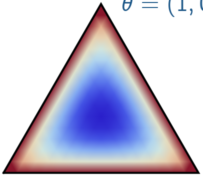

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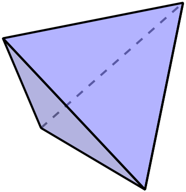
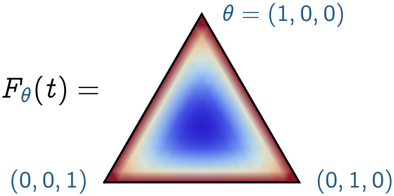
Quantum functionals

$t \mapsto \Delta(t) =$  $\mapsto F_\theta(t) =$ 

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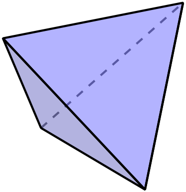

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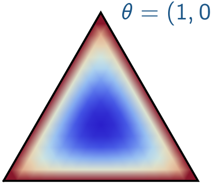

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- Shannon entropy: $H(x^1) := \sum_j x^1_j \log_2 \frac{1}{x^1_j}$
- θ-weighted average: $\sum_{i=1}^{3} \theta_i H(x^i)$

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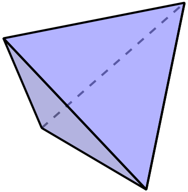

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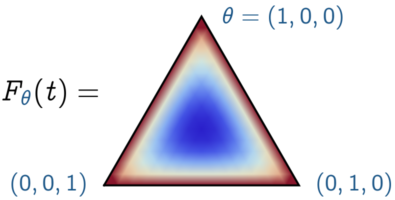

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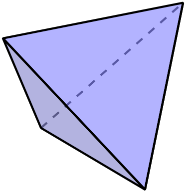
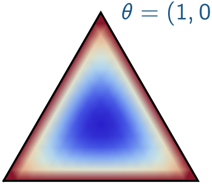
- $\Delta(t) := \{(\mathrm{spec}(s^1), \mathrm{spec}(s^2), \mathrm{spec}(s^3)) : s \trianglelefteq t, \|s\|_2 = 1\}$
- $x = (x^1, x^2, x^3) \in \Delta(t)$
- Shannon entropy: $H(x^1) := \sum_j x^1_j \log_2 \frac{1}{x^1_j}$
- θ-weighted average: $\sum_{i=1}^3 \theta_i H(x^i)$
- quantum functional: $F_\theta(t) := \max\{2^{\sum_{i=1}^3 \theta_i H(x^i)} : x \in \Delta(t)\}$

Quantum functionals

$t \mapsto \Delta(t) =$  $\mapsto F_\theta(t) =$ 

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Main theorem

$F_\theta \in X$(3-tensors over $\mathbb{C}$) $\qquad \theta_i \in \mathbb{R}_{\geq 0}, \ \theta_1 + \theta_2 + \theta_3 = 1$

Remark about the proof

$$F_\theta(t) \coloneqq \max\{2^{\sum_{i=1}^{3} \theta_i H(x^i)} : x \in \Delta(t)\}$$

Main theorem

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Remark about the proof

$$F_\theta(t) := \max\left\{2^{\sum_{i=1}^{3}\theta_i H(x^i)} : x \in \Delta(t)\right\}$$

Main theorem

$F_\theta \in X(\text{3-tensors over } \mathbb{C})$ $\theta_i \in \mathbb{R}_{\geq 0}$, $\theta_1 + \theta_2 + \theta_3 = 1$

To show:

F_θ is $\otimes$-multiplicative, $\oplus$-additive, $\leq$-monotone, $\langle n \rangle$-normalised

$$\Delta(t) := \left\{ \Big(\mathrm{spec}(s_1), \mathrm{spec}(s_2), \mathrm{spec}(s_3)\Big) : s \trianglelefteq t, \|s\|_2 = 1 \right\}$$

- super-multiplicativity
- super-additivity

Equivalent representation-theoretic description:

$$\Delta(t) \approx \left\{ \left(\tfrac{1}{n}\lambda_1, \tfrac{1}{n}\lambda_2, \tfrac{1}{n}\lambda_3\right) : \lambda_1, \lambda_2, \lambda_3 \vdash n, P_{\lambda_1} \otimes P_{\lambda_2} \otimes P_{\lambda_3} \cdot t^{\otimes n} \neq 0 \right\}$$

- sub-multiplicativity
- sub-additivity

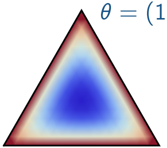
Relation to gauge points and support functionals

Known

- gauge points [Strassen 86]
  $\zeta_{(1,0,0)}, \zeta_{(0,1,0)}, \zeta_{(0,0,1)} \in X(\text{3-tensors over } \mathbb{F})$
- support functionals [Strassen 86]
  $\zeta_\theta \in X(\underline{\text{oblique}} \text{ 3-tensors over } \mathbb{F})$

New

- quantum functionals
  $F_\theta \in X(\text{3-tensors over } \mathbb{C})$

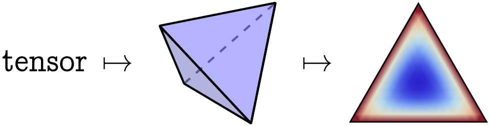


Relations

- over $\mathbb{C}$: $\zeta_{(1,0,0)} = F_{(1,0,0)}$, $\zeta_{(0,1,0)} = F_{(0,1,0)}$, $\zeta_{(0,0,1)} = F_{(0,0,1)}$
- on <u>oblique</u> tensors over $\mathbb{C}$: $\zeta_\theta = F_\theta$

Conclusion

- Various problems require computing $\underset{\sim}{\mathrm{Q}}$ and $\underset{\sim}{\mathrm{R}}$
- Knowing asymptotic spectrum X means knowing $\underset{\sim}{\mathrm{Q}}$ and $\underset{\sim}{\mathrm{R}}$
- Explicit infinite family $\{F_\theta : \theta\} \subseteq X$(3-tensors over $\mathbb{C}$) using moment/entanglement polytopes and (quantum) information ideas



- Quantum functionals F_θ generalise to all $k \geq 3$
- Are these all elements of X? We do not know! If all, then in particular $\omega = 2$.

Thank you