Subrank

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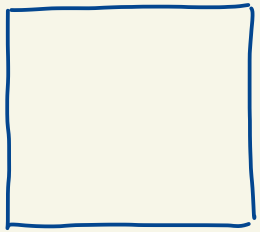
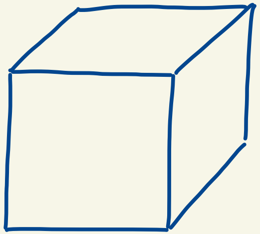
Tensor Ranks and
Tensor Invariants Seminar

matrix



matrix rank

tensor



tensor rank

slice rank

analytic rank

geometric rank

G-stable rank

→ Subrank

1. Subrank

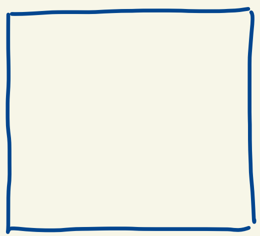
2. Generic subrank

3. Asymptotic subrank

4. Symmetric subrank

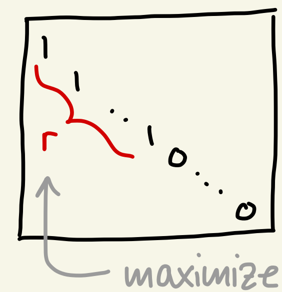
1. Subrank

matrix rank

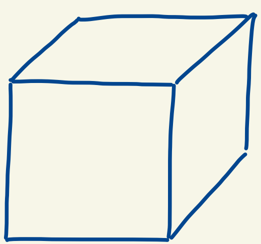


Gaussian elimination

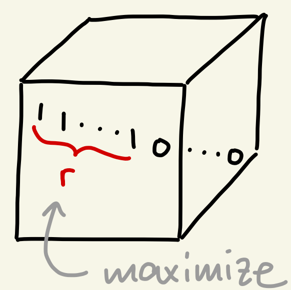
row and column operations



Tensor <u>subrank</u>



linear combinations of slices in all three directions



$T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$

Def. (Subrank, Strassen 1987)

$$Q(T) := \max\left\{ r : \exists\, A_i \in \mathrm{Mat}(r, n_i),\ (A_1 \otimes A_2 \otimes A_3)\cdot T = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \right\}$$

$$= \max\{ r : \langle r \rangle \leq T \}$$

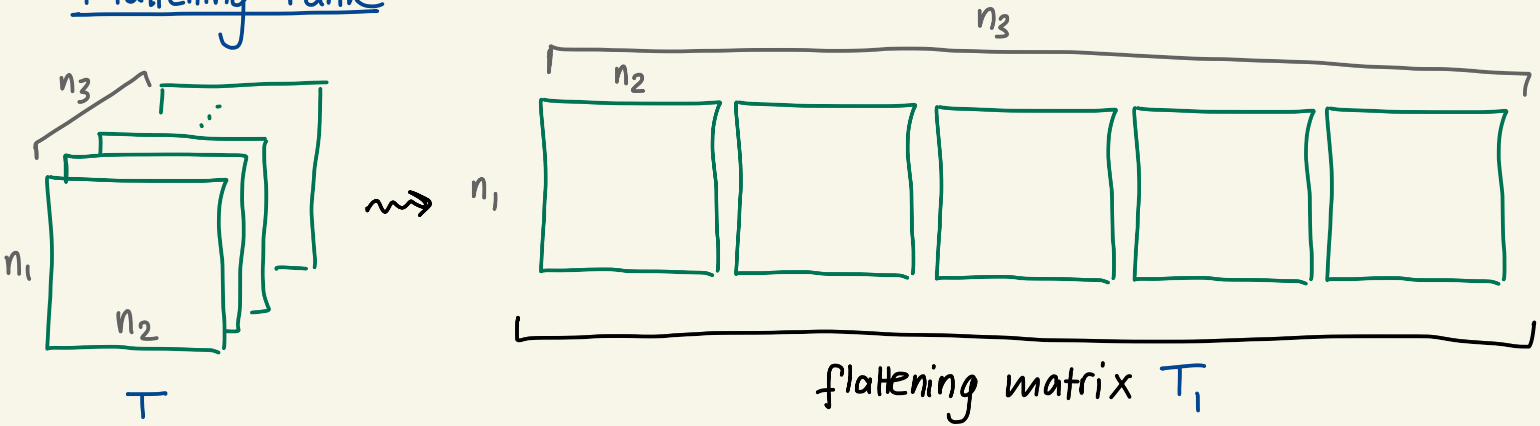
Ex. $Q(\langle r \rangle) = r$, $Q(W) = 1$, $Q(\langle n,n,n \rangle) = n^{2-o(1)}$

Applications:

- Matrix multiplication algorithms / Barriers
- Upper bounding hypergraph independence number
- Quantum entanglement

Open problem What is the computational complexity of subrank?

Flattening rank



Def Flattening rank $R_1(T) := \operatorname{rank} T_1$. Similarly define R_2 and R_3.

Lemma $Q(T) \leq R_i(T)$

Def If $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$ satisfies $R_i(T) = n_i$ for all i, then we call T concise. (Strassen: "The tensor is travelling in economy class.")

Lemma Any tensor is equivalent (under restriction) to a concise tensor.

Other parameters:

- Slicerank [Viktoriia Borovik, last week]
- Analytic rank $T: U \times V \times W \to \mathbb{F}_q$

$$-\log_q \frac{|Z|}{|U \times V|} \quad \text{where} \quad Z = \{(u,v) \in U \times V : T(u,v,\cdot) \equiv 0\}$$

- Geometric rank [Pierpaola Santarsiero, next week] codim of Z
- G-stable rank

Theorem The above parameters are (quasi-) linearly related (Moshkovitz-Zhu, Lampert, ...)

Theorem Subrank is at most the above parameters. (Tao, Lovett, Kopparty et al., Derksen)

Question How good are these upper bounds?

2. Generic subrank

$T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$

$$0 \leq \underset{Q}{\text{subrank}} \leq \underbrace{\begin{array}{c}\text{slicerank}\\ \text{analytic rank}\\ \text{geometric rank}\\ G\text{-stable rank}\end{array}}_{\approx n} \leq n \leq \underbrace{\underset{R}{\text{tensor rank}}}_{\approx n^2} \leq n^2$$

Theorem For almost all tensors:

Theorem (Derksen, Makam, Z) For almost all $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$, we have $Q(T) = \theta(\sqrt{n})$.

Remarks • "Almost all" = "random" = generic

- There is a non-empty Zariski-open $\mathcal{U} \subseteq \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ such that for all $T \in \mathcal{U}$ we have $Q(T) = \theta(\sqrt{n})$
- Very precise bounds: $\sqrt{3n-2} - 5 \leq Q(T) \leq \sqrt{3n-2}$

Upper bound proof

$Q(n) :=$ subrank of a generic tensor in $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$

To prove: $Q(n) \leq \sqrt{3n-2}$

$C_r := \{$ tensors in $\mathbb{F}^{n \times n \times n}$ with subrank $\geq r \}$

<u>Lemma 1</u> $Q(n) =$ largest r such that $\dim C_r = \underbrace{\dim \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n}_{n^3}$.

<u>Lemma 2</u> $\dim C_r \leq n^3 - r(r^2 - 3n + 2)$

Let $t = Q(n)$

Then $n^3 = \dim C_t \leq n^3 - t(t^2 - 3n + 2)$.

Then $t^2 - 3n + 2 \leq 0$

So $t \leq \sqrt{3n-2}$

$C_r := \{ \text{tensors in } \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n \text{ with subrank} \geq r \}$

<u>Lemma 2</u> $\dim C_r \leq n^3 - r(r^2 - 3n + 2)$

<u>Proof idea</u>

- Non-injective parametrization of C_r
- Compute dimension of parameter space $\quad 3n^2 + (n^3 - r^3 + r)$
- Subtract dimension of "over-count" (fiber dimension) $\quad 3(n(n-r) + r)$

$X_r = \{ \text{tensors in } \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n \text{ with } [r] \times [r] \times [r] \text{ subtensor arbitrary diag.} \}$

$$\Psi_r : GL_n \times GL_n \times GL_n \times X_r \to \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$$
$$(A, B, C, T) \mapsto (A \otimes B \otimes C) T \qquad \text{has image } C_r$$

$\square$

Application: Subrank is not additive under direct sum

Theorem There are tensors $S, T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ such that $Q(S), Q(T) \le \sqrt{3n-2}$ while $Q(S \oplus T) \geq n$.

Proof idea

- Let T be "random".
- Let $S = \langle n \rangle - T$. Then S is "random".
- Then $Q(S), Q(T) \le \sqrt{3n-2}$ by our theorem.
- On the other hand, $Q(S \oplus T) \geq Q(S+T) = Q(\langle n \rangle) = n$. $\square$

Related: Tensor rank additivity conjecture; counterexample (Shitov), much more complicated.

3. Asymptotic subrank

Def $\underset{\sim}{Q}(T) = \lim_{n\to\infty} Q(T^{\otimes n})^{1/n}$ ← sup

$\underset{\sim}{SR}(T) = \lim_{n\to\infty} SR(T^{\otimes n})^{1/n}$

Ex $\underset{\sim}{Q}(\langle r\rangle) = r$,

$\underset{\sim}{Q}(W) = 2^{h(1/3)} \approx 1.88$,

$\underset{\sim}{Q}(\langle n,n,n\rangle) = n^2$.

Theorem [Itai Leigh, earlier talk]

$$\underset{\sim}{Q}(T) = \min_{F\in X} F(T)$$

← asymptotic spectrum

Lemma $\underset{\sim}{Q}(T) \leq R_i(T)$

Theorem (Christandl, Vrana, Z)

$$\underset{\sim}{SR}(T) = \min_{F\in Y\subseteq X} F(T)$$

"quantum functionals" optimization over moment polytopes [Harold Nieuwboer, earlier talk]

Open problem $\underset{\sim}{Q} = \underset{\sim}{SR}$?

Question: What can we say about $V = \{ \underset{\sim}{Q}(T) \cdot T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3},\ n_i \in \mathbb{N} \}$?

Theorem (WZ) Closed under maps $\mathbb{N}[x]$.

?

Theorem (CGZ) $V = \{0, 1, 1.88\ldots, 2, 2.68\ldots\} \cup S$, $\quad S \subseteq [2.68\ldots, \infty)$

Theorem (BCLSZ) V is discrete for finite $\mathbb{F}$.

Remarks:

- discrete = has no accumulation points
  = any converging sequence must become constant
  = values are "gapped".
- similar result for other parameters and regimes (slice rank, tensor rank)

  $\{ \underset{\sim}{SR}(T) : T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3},\ n_1, n_2, n_3 \in \mathbb{N} \}$ is discrete

Theorem (BCLSZ) V is discrete for finite $\mathbb{F}$.

Proof ingredients:

- **Lemma 1** (Big tensors) If $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$ is concise, then $\underset{\sim}{Q}(T) \geq \min(n_1, n_2, n_3)^{1/3}$.

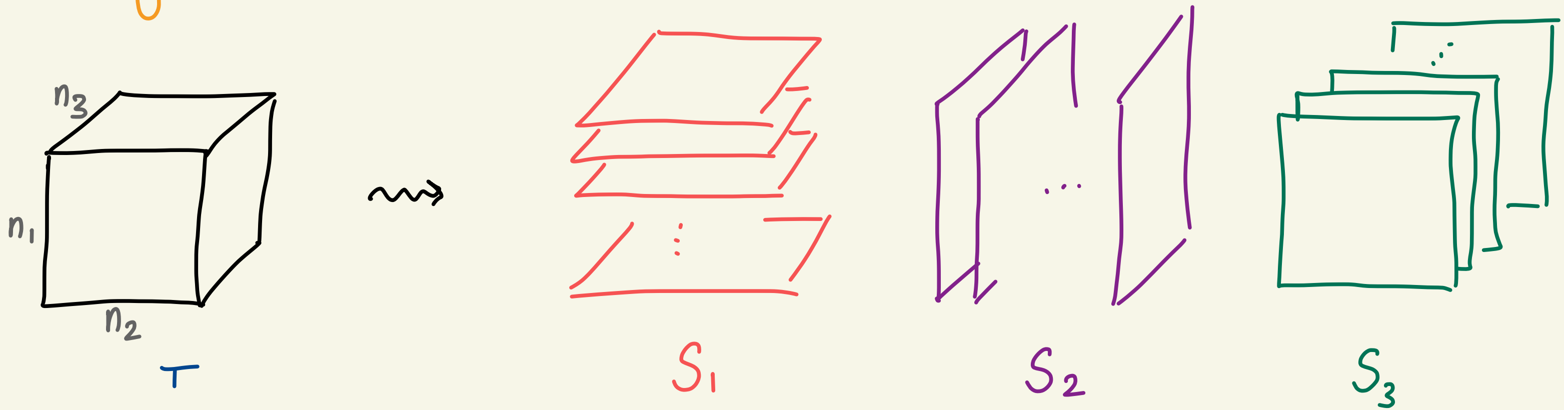
- **Lemma 2** (Thin tensors) For any c, there is an $N(c)$, for all $n_1 \geq N(c)$, for all concise $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{c}$, $\underset{\sim}{Q}(T) = c$.

Proof sketch of theorem:

Consider infinite sequence $\underset{\sim}{Q}(T_i)$ with $T_i \in \mathbb{F}^{a_i} \otimes \mathbb{F}^{b_i} \otimes \mathbb{F}^{c_i}$ concise.
There are finitely many tensors per choice of (a_i, b_i, c_i), so $\{(a_i, b_i, c_i)\}_i$ is infinite.

- If $\min(a_1, b_i, c_1) \to \infty$, then $\underset{\sim}{Q}(T_i) \to \infty$
- If $\max_i c_i = c$, then $a_i \to \infty$ so $\underset{\sim}{Q}(T_i)$ eventually constant $\square$

Proof ingredient for Lemma 1: max-rank



"Commutative rank"
[Leonie Kayser, later talk]

Def (max-rank) $Q_i(T) := \max \{ \operatorname{rank}(A) : A \in S_i \}$

Remark (flattening rank) $R_i(T) = \operatorname{rank}(T_i) = \dim S_i$

Lemma $Q_2(T)\, Q_3(T) \geq R_1(T)$.

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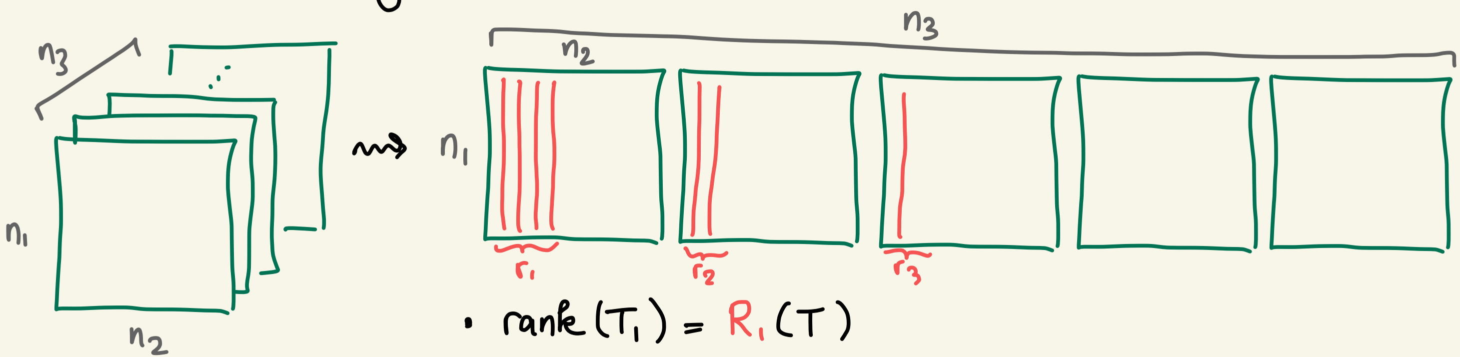
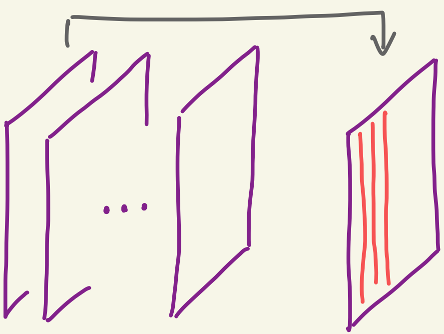
Lemma $Q_2(T)\,Q_3(T) \geq R_1(T)$.

Proof sketch: Apply random basis transformation to T. Consider flattening T_1:



- $\mathrm{rank}(T_1) = R_1(T)$
- Pick $R_1(T)$ lin. ind columns in T_1
- May assume they are the first $r_1, r_2, \dots, r_n$ columns.
- $\sum_i r_i = R_1(T)$, and $Q_3(T) \geq \max_i r_i$



- This slice has at least $\#\{i : r_i \neq 0\}$ many lin. independent columns, so $Q_2(T) \geq \#\{i : r_i \neq 0\}$.
- $Q_3(T) \cdot Q_2(T) \geq \max_i r_i \cdot \#\{i : r_i \neq 0\} \geq \sum_i r_i = R_1(T)$. $\square$

4. Symmetric subrank

Def (Symmetric rank) For a symmetric tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$,

$$R_s(T) = \min \{ r : \exists u_1, \ldots, u_r \in \mathbb{C}^n,\ T = \sum_{i=1}^{r} u_i \otimes u_i \otimes u_i \}$$

Comon's conjecture For symmetric tensors, $R = R_s$ Still open

Def (Symmetric subrank) $Q_s(T) = \max \{ r : \exists A,\ (A \otimes A \otimes A) \cdot T = \langle r \rangle \}$.

Ex. There is a tensor T with $Q_s(T) < Q(T)$, namely corresponding to the polynomial $f(x,y,z) = x^3 z^3 + y^6 + xyz^4$; $Q_s(f) \leq 1 < 2 \leq Q(f)$ (Shitov). Also over finite $\mathbb{F}$ (easier). Open problem: larger gaps

Theorem (CFTZ) For symmetric tensors, $\underset{\sim}{Q} = \underset{\sim}{Q}_s$. (alg. cl. field, char large)

$(\underset{\sim}{R} = \underset{\sim}{R}_s)$

Proof idea $S \geq T \Rightarrow S \boxtimes M \geq_s T$

$\Rightarrow S^{\boxtimes n + o(n)} \geq_s T^{\boxtimes n}$.

Thanks