Tensors and their applications: foundations and recent advances

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TA: Maxim van den Berg

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[Strassen, 1969]

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- Currently: $\leq O(n^{2.37})$, best exponent still open

What is the cost of a task
if we have to perform it many times?

Direct-sum problems/
Economies of scale/
Savings

$$\lim_{n\to\infty} F(T^n)^{1/n}$$

Asymptotic spectrum duality

[Strassen, 1987 - 1991]

Applications

- matrix multiplication
- combinatorics
- entanglement

...

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Applications $\rightsquigarrow$ Tensor ranks

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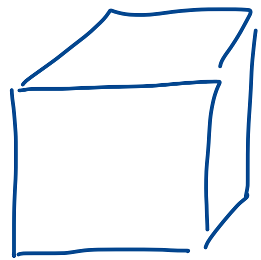

Tensor

Asymptotic spectrum duality

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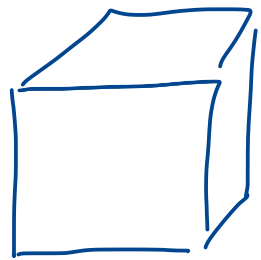

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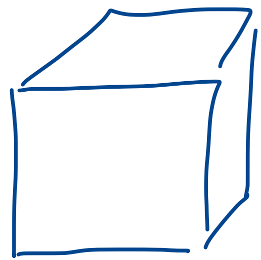

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$$X$$

$\longleftarrow$

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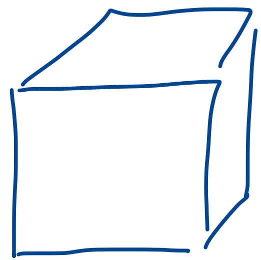

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- convexity
- geometry
- information theory

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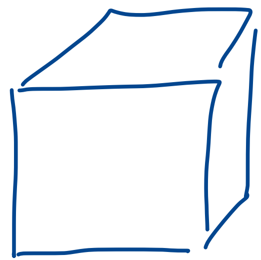

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Asymptotic spectrum duality

Recent developments: [2018 – ...]

- quantum information
- representation theory

Master class

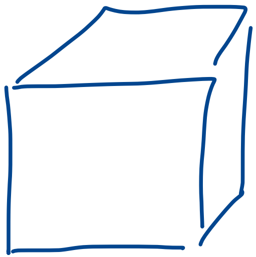
1. Tensors, ranks, applications
2. Matrix multiplication and asymptotic rank
3. Cap set problem and slice rank
4. Asymptotic spectrum duality
5. Quantum information and quantum functionals (Maxim)

PhD School

6. Asymptotic spectrum duality
7. Schur-Weyl and moment polytopes (Maxim)
8. Quantum functionals
9. Convexity
10. Semicontinuity

We will see many

Tensors



$\langle n \rangle$ $\langle a,b,c \rangle$

W $\mathbb{C}[x]/(x^2)$

$\dots$

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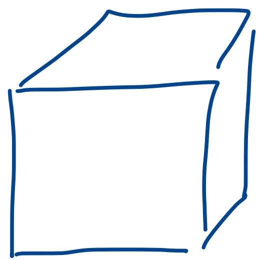

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Ranks

R $\underline{R}$ $\underset{\sim}{R}$

Q $\underline{Q}$ $\underset{\sim}{Q}$

SR GR AR

R_i

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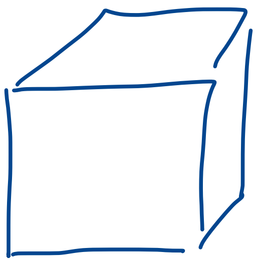

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Orders

$\leq \quad \trianglelefteq \quad \underset{\sim}{\leq}$

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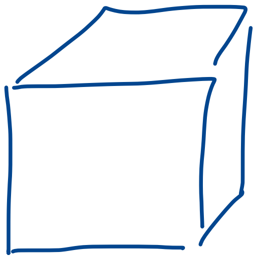

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Special functions

support functionals,
quantum functionals

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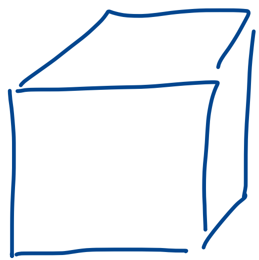

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Special functions

support functionals,
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Special classes of tensors

tight $\subseteq$ oblique $\subseteq$ free,
generic

Lecture 1 Tensors, ranks, applications

1. Applications in computer science, mathematics and physics
2. Matrices
3. Tensors: Rank, unit tensor, restriction, subrank
4. Border rank
5. Degeneration
6. Generic rank, explicit tensors of high rank

1. Applications

- Computer science

  algebraic complexity theory

  → matrix multiplication

- Mathematics

  additive combinatorics

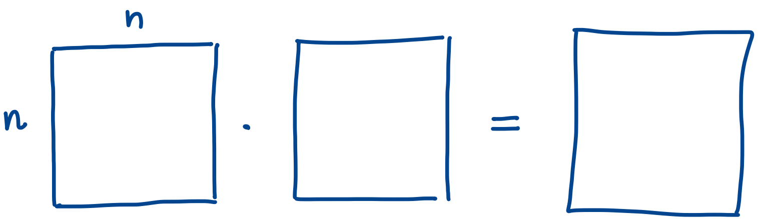
  → cap set problem

- Physics

  quantum information

  → entanglement

1.1 Matrix multiplication



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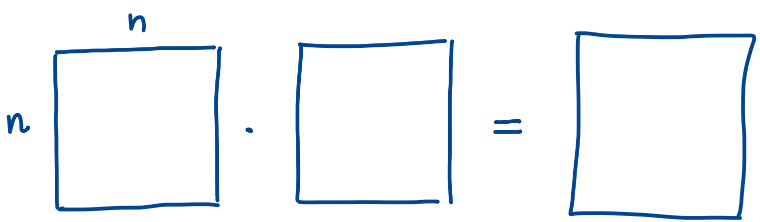

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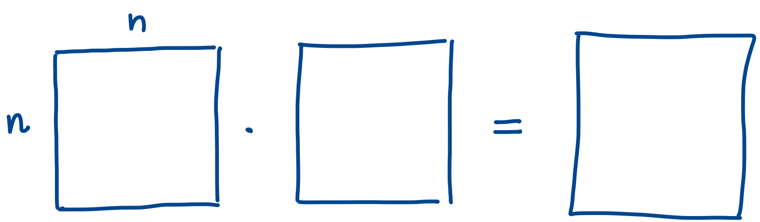

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Optimal: $\mathcal{O}(n^{\omega})$ matrix mult. exponent

$2 \leq \omega \leq 2.3\ldots$

<u>Conjecture</u>: $\omega = 2$

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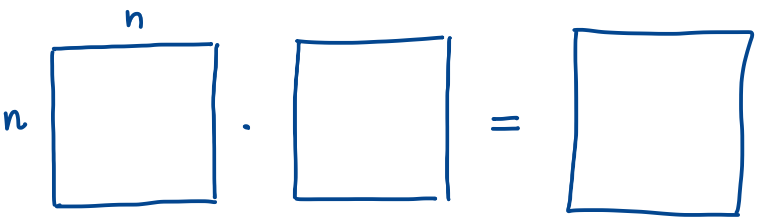

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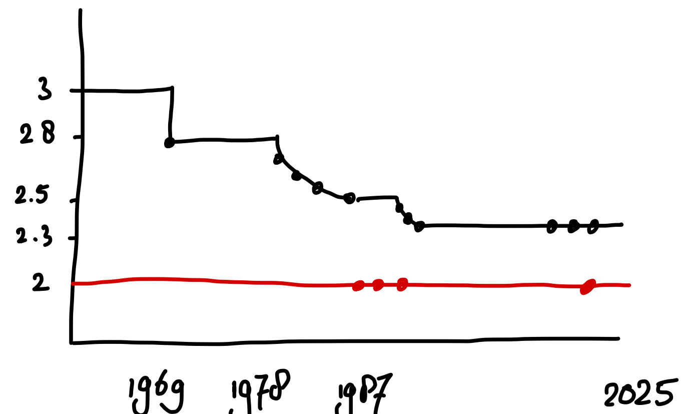

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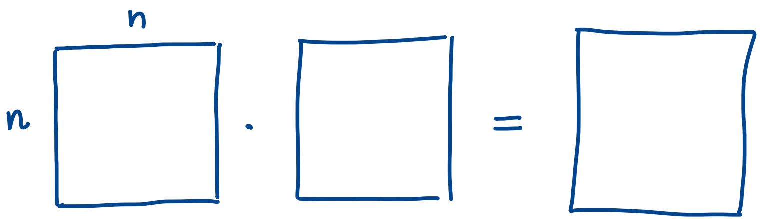

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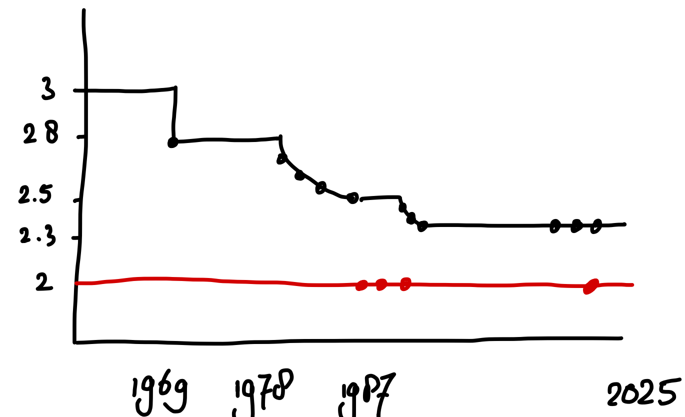

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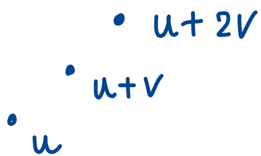
$\boxed{\text{Characterized by tensor rank!}}$

1.2 Cap set problem

- $x, y, z \in (\mathbb{Z}/3\mathbb{Z})^n$ form a line if $(x,y,z) = (u, u+v, u+2v)$ for some $u, v \in (\mathbb{Z}/3\mathbb{Z})^n$, $v \neq 0$.



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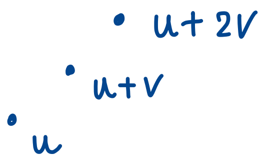

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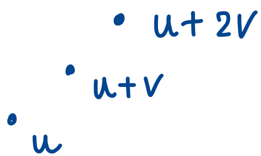

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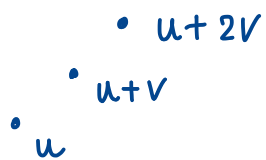

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[Edel, Tyrell, DeepMind]

construction

$\boxed{\text{Tensor obstruction! (subrank)}}$

[Ellenberg–Gijswijt, Tao]

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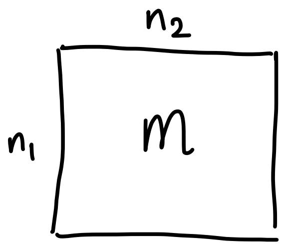
- They now manipulate their local system in their lab, and discuss by classical communication, to get state $S \in V$ (SLOCC)
- What states S can they obtain?
- What if they have many copies of T? (Tensor restriction!)

  Savings?

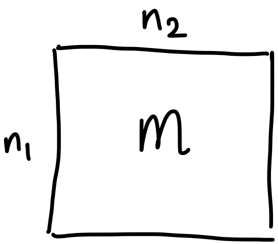
2. Matrices

$\mathbb{F}^{n_1 \times n_2}$



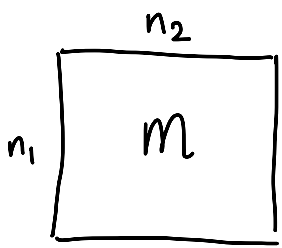
2. Matrices

$\mathbb{F}^{n_1 \times n_2}$ $\qquad$ $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$



$$M = \sum_{i,j} M_{ij} \, e_i \otimes e_j$$

2. Matrices

		lin. map	bil. map
$\mathbb{F}^{n_1 \times n_2}$	$\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$	$\mathbb{F}^{n_2} \to \mathbb{F}^{n_1}$	$\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \to \mathbb{F}$
	$M = \sum_{i,j} M_{ij}\, e_i \otimes e_j$	$v \mapsto Mv$	$(u,v) \mapsto u^T M v$

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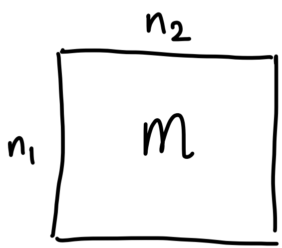
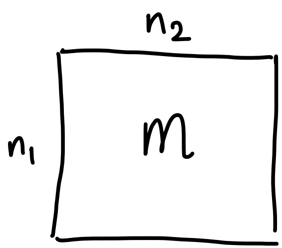
	$M = \sum_{i,j} M_{ij}\, e_i \otimes e_j$	$v \mapsto Mv$	$(u,v) \mapsto u^T M v$

matrix rank $\quad R(M)$ is smallest r s.t. $M = \sum_{i=1}^{r} u_i \otimes v_i$ for $u_i, v_i \in \mathbb{F}^n$

2. Matrices

$\mathbb{F}^{n_1 \times n_2}$ | $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$ | lin. map $\mathbb{F}^{n_2} \to \mathbb{F}^{n_1}$ | bil. map $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \to \mathbb{F}$



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restriction For $M_1 \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$, $M_2 \in \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2}$ define

$$M_1 \leq M_2 \;:\Leftrightarrow\; \exists A_i \in \mathbb{F}^{n_i \times m_i},\ M_1 = A_1 M_2 A_2^T$$

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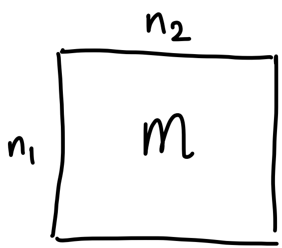	$M = \sum_{i,j} M_{i,j}\, e_i \otimes e_j$	$v \mapsto Mv$	$(u,v) \mapsto u^T M v$

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Theorem

- $R(M) = \min\{r : M \leq I_r\} = \max\{r : I_r \leq M\}$.
- $M_1 \leq M_2 \iff R(M_1) \leq R(M_2)$

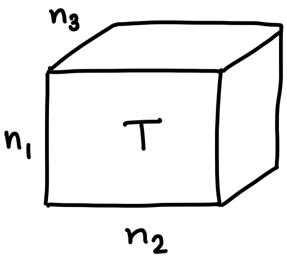
matrix rank:

- smallest r s.t. $M = \sum_{i=1}^{r} u_i \otimes v_i$ for $u_i \in \mathbb{F}^{n_1}, v_i \in \mathbb{F}^{n_2}$
- $\min\{r : M \leq I_r\}$
- $\max\{r : I_r \leq M\}$
- $\dim \operatorname{Im} M$
- $\operatorname{codim} \operatorname{Ker} M$
- smallest r s.t. all $(r+1)\times(r+1)$ minors are zero

...

3. Tensors

$$\mathbb{F}^{n_1 \times n_2 \times n_3}$$



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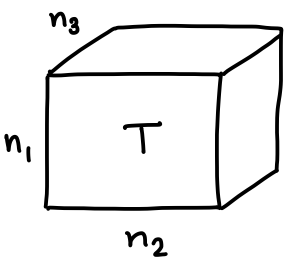

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$$T = \sum_{i,j} T_{i,j,k}\, e_i \otimes e_j \otimes e_k$$

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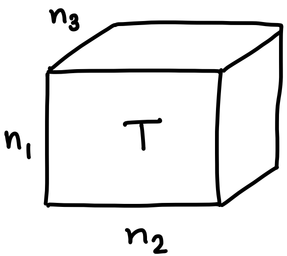

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bil. map $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \to \mathbb{F}^{n_3}$

tril. map $\mathbb{F}^{n_1} \times \mathbb{F}^{n_2} \times \mathbb{F}^{n_3} \to \mathbb{F}$

3. Tensors

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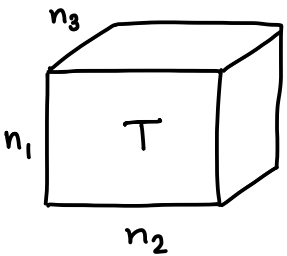

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- every matrix in $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$ gives a tensor in $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{1}$.

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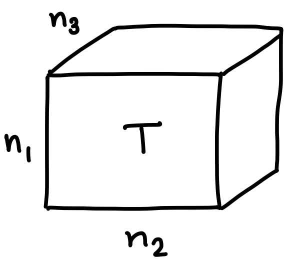

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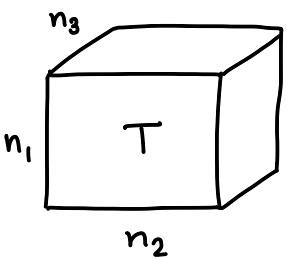

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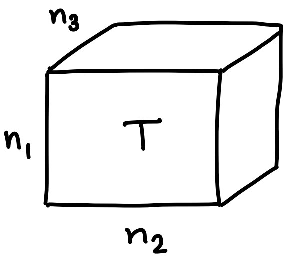

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- every algebra has a bilinear multiplication map, which corresponds to a tensor

  e.g. polynomials $\mathbb{C}[x]/(x^2)$ or matrix algebra $\mathbb{C}^{n \times n}$

Acting on tensors. For $T \in \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2} \otimes \mathbb{F}^{m_3}$, $A_i \in \mathbb{F}^{n_i \times m_i}$

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Equivalence $S \sim T \; :\Leftrightarrow \; S \leq T$ and $T \leq S$.

Isomorphism $S \cong T \; :\Leftrightarrow \; \exists A_i \in GL_{n_i},\; S = (A_1 \otimes A_2 \otimes A_3) \cdot T$

General linear group $GL_n \subseteq \mathbb{F}^{n \times n}$ invertible matrices

Rank $R(T) := \min\{ r : T \leq \langle r \rangle \}$

Lemma $R(T) =$ smallest r s.t. $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$ (exercise)

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- for matrices, $Q = R$
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Dictionary

unit tensor	$\langle r \rangle$	currency
rank	R	cost
subrank	Q	value
restriction	$\leq$	transformation/reduction

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For tensor rank the analogous statement is false!

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What gaps are there? There are $n \times n \times n$ tensors T_n with $R(T_n) \geq 3 \cdot \underline{R}(T_n) - o(n)$

We will see a reverse inequality next lecture. [Z 2017]

5 Degeneration, border subrank

Approximative relaxation of restriction.

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We will see applications of these.

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Theorem $\trianglelefteq^E, \trianglelefteq^Z, \trianglelefteq^A$ coincide (over appropriate fields) (Proof not simple.)

We will write $\trianglelefteq$ for this, and $\underline{R}, \underline{Q}$ for corresponding border rank and subrank.

6. Generic rank, explicit tensors of high rank

Recall: $Q \leq \underline{Q} \leq \underline{R} \leq R$

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Theorem Let $T \in \mathbb{C}^{n \times n \times n}$ random. Then (whp)

- $Q(T) = \sqrt{3n-2}$ [Derksen–Makam–Z 2024, Pielasa–Safranek–Shatsila 24]
- $\underline{Q}(T) \in [\lfloor \sqrt{4n} \rfloor - 3, C \cdot \sqrt{n}]$ [Biaggi–Chang–Draisma–Rupniewski 24]

$$\left.\begin{array}{l} \bullet\ \underline{R}(T) = \left\lceil \frac{n^3}{3n-2} \right\rceil = \Theta(n^2) \\ \bullet\ R(T) = \quad \text{''} \end{array}\right\} \text{(except for } n=3\text{)}$$

[Lickteig 1985][Strassen]

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Open problem Construct explicit tensors of high rank

This would lead to breakthroughs in alg. compl. th. [Bläser, "Explicit tensors"]
[Strassen 1973, Raz 2010]