Lecture 10 Semicontinuity of asymptotic rank

1. Asymptotic rank
2. Semicontinuity
3. Proof
4. Consequences
5. Discreteness
6. Asymptotic spectrum distance

1. Asymptotic rank

$$\underset{\sim}{R}(\langle 2,2,2\rangle) = 2^{\omega}$$

Central problems:

(1) Determine whether $\omega = 2$ or $\omega > 2$? $\underset{\sim}{R}(\langle 2,2,2\rangle) = 4$ or >4?

(2) Is there any tensor $T \in \mathbb{F}^{n} \otimes \mathbb{F}^{n} \otimes \mathbb{F}^{n}$ with $\underset{\sim}{R}(T) > n$?

(3) What is the structure (geometric, topological, algebraic, ...) of
$$\{ \underset{\sim}{R}(T) : T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3},\ n_i \in \mathbb{N} \} \ ?$$

(4) What properties does $\underset{\sim}{R}$ have? Computable? Semicontinuous?

2. Semicontinuity $\quad V = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$

Theorem $\{T \in V : \underset{\sim}{R}(T) \leq r\}$ is Zariski-closed.

Remarks:

(1) Meaning: $\forall d, r$, $\exists$ polynomials $p_1, \ldots, p_\ell$, $\forall T \in V$.

$$\underset{\sim}{R}(T) \leq r \iff p_1(T) = \cdots = p_\ell(T) = 0.$$

So there is an algorithm to decide upper bounds.

(2) Consequence:

$$T_1, T_2, \ldots \to T \text{ and } \forall i,\ \underset{\sim}{R}(T_i) \leq r \implies \underset{\sim}{R}(T) \leq r$$

(Euclidean distance)

Theorem $\forall F \in X$, $\{T \in V : F(T) \leq r\}$ is Zariski-closed.

Theorem $\forall T \in V$, $\{S \in V : S \underset{\sim}{\leq} T\}$ is Zariski-closed.

3. Proof

Theorem 1 $\{T \in V : \underset{\sim}{R}(T) \le r\}$ is Zariski-closed.

Def $\underset{\sim}{R}[A] := \sup\{\underset{\sim}{R}(T) : T \in A\}$ for $A \subseteq V$

Theorem 2 $\underset{\sim}{R}[\bar{A}] = \underset{\sim}{R}[A]$

Theorem 2 $\Rightarrow$ Theorem 1.

- Let $A = \{T \in V : \underset{\sim}{R}(T) \le r\}$. Then $\underset{\sim}{R}[A] \le r$.
- By Theorem 2, $\underset{\sim}{R}[\bar{A}] \le r$.
- So for all $T \in \bar{A}$, $\underset{\sim}{R}(T) \le r$, so $T \in A$.
- Then $\bar{A} \subseteq A$ $\square$

Two ingredients:

Lemma $\underset{\sim}{R}(S \oplus T) \leq \underset{\sim}{R}(S) + \underset{\sim}{R}(T)$

Proof Recall: $\exists F \in X$, $\underset{\sim}{R}(S \oplus T) = F(S \oplus T) = F(S) + F(T) \leq \underset{\sim}{R}(S) + \underset{\sim}{R}(T)$. $\square$

Lemma $\forall A \subseteq \mathbb{C}^d, \forall n, \ \{ v^{\otimes n} : v \in \overline{A} \} \subseteq \text{span} \{ v^{\otimes n} : v \in A \}$.

Example
$$\overline{A} = \bigcap_{\substack{p : p(A) = 0 \\ \text{poly}}} \{ p = 0 \} \subseteq \text{span}\, A = \bigcap_{\substack{\ell : \ell(A) = 0 \\ \text{lin. form}}} \{ \ell = 0 \}.$$

Proof
- Let ℓ be a linear form vanishing on the RHS
- $f : T \mapsto \ell(T^{\otimes n})$ is a polynomial vanishing on A
- So f vanishes on $\overline{A}$
- Then ℓ vanishes on the LHS $\square$

Theorem 2 $\underset{\sim}{R}[\bar{A}] = \underset{\sim}{R}[A]$

Proof:

To prove: $\leq$

$\leq \dim \mathrm{Sym}^n(V) = \binom{n+d^3-1}{d^3-1}$

- Let $T \in \bar{A}$. Then $T^{\otimes n} = \sum_{i=1}^{\mathrm{poly}(n)} \alpha_i S_i^{\otimes n}$ with $S_i \in A$ (Lemma 3)

- $\underset{\sim}{R}(T^{\otimes n}) \leq \sum_{i=1}^{\mathrm{poly}(n)} \underset{\sim}{R}(S_i^{\otimes n}) \leq \mathrm{poly}(n) \cdot \underset{\sim}{R}[A]^n$ (subadditivity of $\underset{\sim}{R}$)

- $\underset{\sim}{R}(T) \leq \underset{\sim}{R}[A]$ (take nth root and $n \to \infty$) $\square$

4. Consequences

$$\mathcal{R} := \{ \underset{\sim}{R}(T) : T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, \ d \in \mathbb{N} \}$$

What can we say about the structure of $\mathcal{R}$?

Theorem $\mathcal{R}$ is closed under applying any polynomial $p \in \mathbb{N}[x]$.
(So $\mathcal{R}$ has "many" elements.)

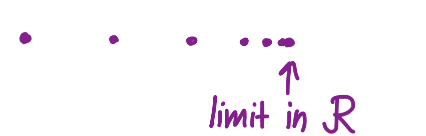
Proof If ϕ maximizes $\max_{\phi \in X} \phi(T)$ then it also maximizes $\max_{\phi \in X} \phi(p(T))$, and $\phi(p(T)) = p(\phi(T))$. $\square$

Theorem $\mathcal{R}$ is well-ordered

(Every non-increasing sequence stabilizes; discrete from above.)

Theorem $\mathcal{R}$ is complete over $\mathbb{C}$

(Euclidean-closed)



Theorem $\mathcal{R} := \{ \underset{\sim}{R}(T) . T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, d \in \mathbb{N} \}$ is well-ordered

Lemma $\forall d$, $\mathcal{R}_d := \{ \underset{\sim}{R}(T) . T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \}$ is well-ordered

Proof Let $A_r = \{ T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \cdot \underset{\sim}{R}(T) \leq r \}$.

Let $r_1 \geq r_2 \geq \dots \in \mathcal{R}_d$.

Then $A_{r_1} \supseteq A_{r_2} \supseteq \cdots$ is a descending chain of Zariski-closed sets.

By noetherianity, this stabilizes: $\exists N, \forall n \geq N, A_{r_n} = A_{r_{n+1}}$.

Let $T_N \in V$ such that $\underset{\sim}{R}(T_N) = r_N$.

Then $\forall n \geq N$, $T_N \in A_{r_n}$. Then $r_n \geq r_N$ so $r_n = r_N$. $\square$

Proof of Theorem Let $r_1 \geq r_2 \geq \dots \in \mathcal{R}$. Then $\forall i, \exists T_i$ concise, $\underset{\sim}{R}(T_i) = r_i$, so that $T_i \in \mathbb{F}^{d_1} \otimes \mathbb{F}^{d_2} \otimes \mathbb{F}^{d_3}$ for some $d_1, d_2, d_3 \leq r_i$.
Then all r_i are contained in the union of $\{ \underset{\sim}{R}(T) : T \in \mathbb{F}^{d_1} \otimes \mathbb{F}^{d_2} \otimes \mathbb{F}^{d_3} \}$ over all $d_i \leq r_1$, which is well-ordered (finite union). $\square$.

Theorem $\mathcal{R} := \{ \underset{\sim}{R}(T) . T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d, d \in \mathbb{N} \}$ is complete

Def. $\underset{\sim}{R}[X] := \sup_{T \in X} \underset{\sim}{R}(T)$.

Lemma. Let X non-empty and Zariski-closed Then $\exists T \in X$, $\underset{\sim}{R}(T) = \underset{\sim}{R}[X]$.

Proof: Baire category theorem

Lemma $\forall d$, $\mathcal{R}_d := \{ \underset{\sim}{R}(T) . T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \}$ is complete

Proof $\mathcal{R}_d$ is well-ordered, so decreasing sequences in $\mathcal{R}_d$ have a limit in $\mathcal{R}_d$

Let $r_1 < r_2 < \cdots \in \mathcal{R}_d$ converge to $r \in \mathbb{R}$.

$V_{\leq r} := \{ T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d : \underset{\sim}{R}(T) \leq r \}$ is Zariski-closed.

$\underset{\sim}{R}[V_{\leq r}] \leq r$ (def) and $\underset{\sim}{R}[V_{\leq r}] \geq r_i$ $\forall i$ so $\underset{\sim}{R}[V_{\leq r}] = r$.

So $\exists$ $T \in V_{\leq r}$, $\underset{\sim}{R}(T) = \underset{\sim}{R}[V_{\leq r}] = r$, so $r \in \mathcal{R}_d$. $\square$

Proof of Theorem. Similar finite union argument as before. $\square$.

Open problems

(1) Is $\mathcal{R} := \{ \underset{\sim}{R}(T) : T \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d,\ d \in \mathbb{N} \}$ discrete from below?

(2) Is $\{ T \in V : \underset{\sim}{R}(T) \leq r \}$ an irreducible variety?

Strassen's asymptotic rank conjecture $\Rightarrow$ (2) $\Rightarrow$ (1)

5. Discreteness

We don't know if $\underset{\sim}{R}$ is discrete.

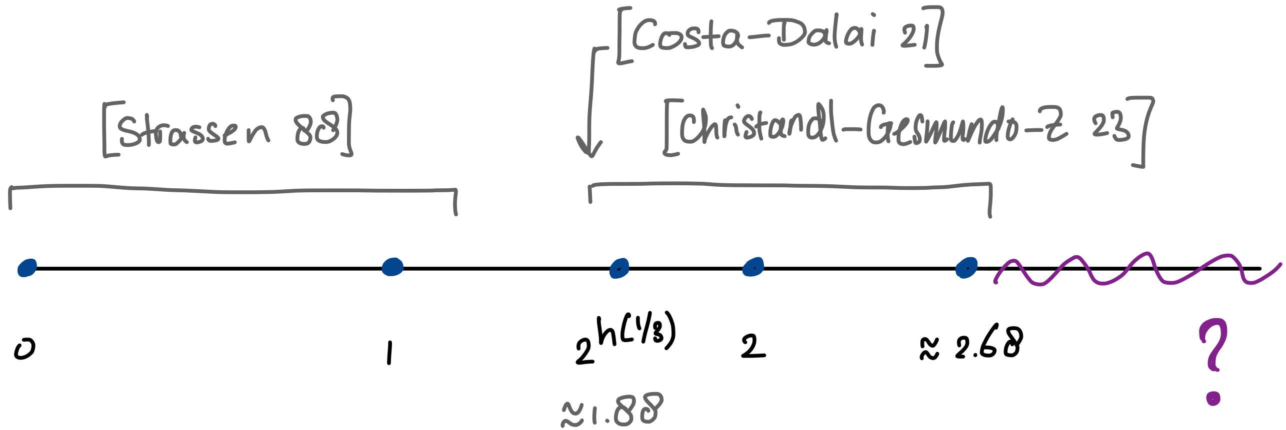
Theorem $\{ \underset{\sim}{SR}(T) : T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n, n \in \mathbb{N} \}$ is discrete.

Ingredients

- $\underset{\sim}{SR}(T) = \min_\theta F_\theta(T)$ quantum functionals
- $\forall n$, $\{ \Delta(T) : T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \}$ is finite

  moment polytope
- $\{ \underset{\sim}{SR}(T) : T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \}$ is finite (so discrete)
- $\forall T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ concise, $\underset{\sim}{Q}(T) \geq \min(n_1, n_2, n_3)^{1/3}$
- $\forall T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{c}$ concise and $n_1 \geq N(c)$, then $\underset{\sim}{Q}(T) = c$.

Known: Values of $\underset{\sim}{Q}$ and $\underset{\sim}{SR}$



- Countably many values over $\mathbb{C}$

  [Blatter - Draisma - Rupniewski 22a]

- Well-ordered over finite fields (no accumulation points from above)

  [Blatter - Draisma - Rupniewski 22b]

6. Asymptotic spectrum distance

arxiv.org/abs/2404.16763

Distance $d(S,T) := \sup\limits_{F \in X} |F(S) - F(T)|$.

Lemma d is a distance on asymptotic equivalence classes.

Lemma. $d(S,T) \leq a/b \iff \langle b\rangle \otimes S \lesssim \langle b\rangle \otimes S \oplus \langle a\rangle$

Lemma. If $S_1, S_2, \ldots \to T$, then $\underset{\sim}{R}(S_i) \to \underset{\sim}{R}(T)$ and $\underset{\sim}{Q}(S_i) \to \underset{\sim}{Q}(T)$.

Proof Let $\varepsilon > 0$. There is N s.t for all $i > N$ and all $F \in X$, $|F(S_i) - F(T)| < \varepsilon$

Let $F_{S_i}, F_T \in X$ s.t. $F_{S_i}(S_i) = \Theta(S_i)$ and $F_T(T) = \Theta(T)$ (duality)

$\Theta(T) = F_T(T) > F_T(S_i) - \varepsilon \geq \Theta(S_i) - \varepsilon$ (same with T and S_i swapped) $\square$

Can we approximate $\langle n,n,n\rangle$ by tensors that are easier to understand?

Recent work: this works well for graphs!