Lecture 2 Matrix multiplication and asymptotic rank.

1. Matrix multiplication
2. Tensor product, direct sum
3. Asymptotic rank
4. Asymptotic sum inequality
5. Flattening ranks, conciseness
6. Asymptotic rank conjecture

Previous lecture

restriction $\quad S \leq T \iff S = (A \otimes B \otimes C) \cdot T$

unit tensors $\quad \langle r \rangle = r \times r \times r$ diagonal

rank $\quad R(T) = \min \{ r : T \leq \langle r \rangle \}$

subrank $\quad Q(T) = \max \{ r : \langle r \rangle \leq T \}$

degeneration $\quad S \trianglelefteq T \iff (A(\varepsilon) \otimes B(\varepsilon) \otimes C(\varepsilon)) \cdot T = S + \varepsilon S_1 + \cdots + \varepsilon^e S_e$

border rank $\quad \underline{R}$

border subrank $\quad \underline{Q}$

generic rank $\quad \forall n \ \exists \mathcal{U} \subseteq \mathbb{C}^{n \times n \times n}$, Z-open, non-empty, $\forall T \in \mathcal{U}$,

$$R(T) = \left\lceil \frac{n^3}{3n-2} \right\rceil$$

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bilinear map $\mathbb{F}^{ab} \times \mathbb{F}^{bc} \to \mathbb{F}^{ac}$. $(A, B) \mapsto A \cdot B$

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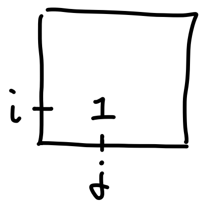
$E_{i,j} =$ 

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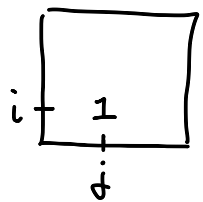
$E_{i,j} =$ 

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Matrix multiplication tensors

$$\langle a,b,c\rangle := \sum_{i\in[a]} \sum_{j\in[b]} \sum_{k\in[c]} \overset{\text{input}}{e_{i,j}} \otimes \overset{\text{input}}{e_{j,k}} \otimes \overset{\text{output (transposed)}}{e_{k,i}} \in \mathbb{F}^{ab} \otimes \mathbb{F}^{bc} \otimes \mathbb{F}^{ca}$$

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Example

$$\langle 1,2,1\rangle = e_{1,1} \otimes e_{1,1} \otimes e_{1,1} + e_{1,2} \otimes e_{2,1} \otimes e_{1,1}$$

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$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\cong e_1\otimes e_1\otimes e_1 + e_2\otimes e_3\otimes e_1 + e_1\otimes e_2\otimes e_2 + e_2\otimes e_4\otimes e_2$$

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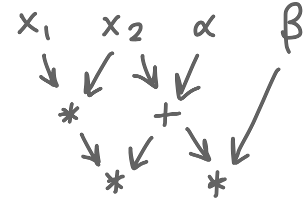
Exponent

arithmetic complexity :

$$\omega := \inf\{\beta : C(\langle n,n,n\rangle) \leq O(n^{\beta})\}.$$



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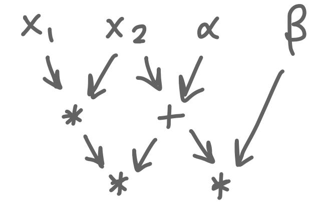

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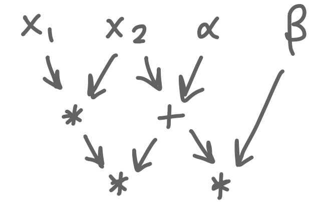

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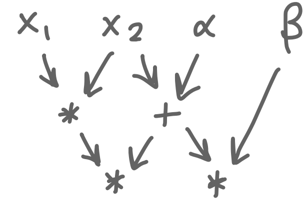

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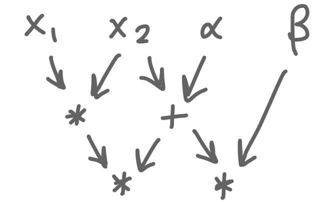

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Block matrices can be multiplied block wise:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

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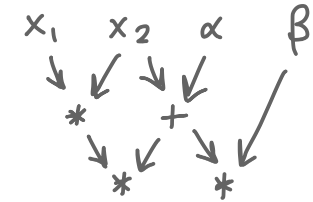

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$$\Rightarrow \quad R(\langle 4,4,4\rangle) \leq 7^2$$

$$R(\langle 8,8,8\rangle) \leq 7^3$$

$$\vdots$$

$$R(\langle 2^n,2^n,2^n\rangle) \leq 7^n = (2^n)^{\log_2 7}$$

$$\omega \leq \log_2 7 \approx 2.81$$

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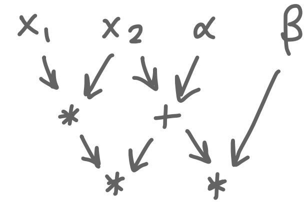

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Lemma

$$R(\langle n,n,n\rangle) \le r \quad \Rightarrow \quad n^{\omega} \le r$$

Example (continued)

$$|0\rangle := e_1 \quad |1\rangle := e_2 \quad |+\rangle := e_1 + e_2 \quad |-\rangle := e_1 - e_2 \qquad \in \mathbb{F}^2$$

$$|xy\rangle := |x\rangle \otimes |y\rangle \qquad \in \mathbb{F}^{2\cdot 2}$$

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$$\xrightarrow{\text{cyclic shift}}$$

$$\langle 2,2,2\rangle \overset{!}{=} - |{-0}\rangle |0+\rangle |11\rangle$$

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$$+\,|{-1}\rangle|1+\rangle|00\rangle + |1+\rangle|00\rangle|{-1}\rangle + |00\rangle|{-1}\rangle|1+\rangle$$

$$+\,\left(|00\rangle + |11\rangle\right)^{\otimes 3} \qquad \text{(invariant)}$$

7 terms in total

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Let $S \in \mathbb{F}^{n_1 \times n_2 \times n_3}$, $T \in \mathbb{F}^{m_1 \times m_2 \times m_3}$

$$S \otimes T := \left(S_{i_1, i_2, i_3} \cdot T_{j_1, j_2, j_3} \right)_{(i_1, j_1), (i_2, j_2), (i_3, j_3)} \in \mathbb{F}^{(n_1 m_1) \times (n_2 m_2) \times (n_3 m_3)}$$

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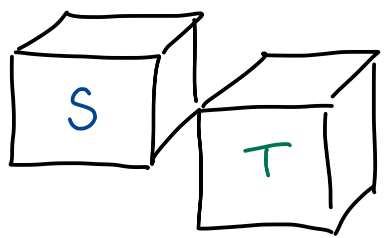

$\in \mathbb{F}^{(n_1 + m_1) \times (n_2 + m_2) \times (n_3 + m_3)}$

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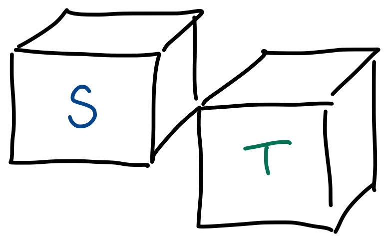

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Example $\langle r \rangle \oplus \langle s \rangle = \langle r + s \rangle$

$\langle a, b, c \rangle \otimes \langle c, d, e \rangle = \langle a \cdot c, b \cdot d, c \cdot e \rangle$ (exercise)

$\langle n \rangle \otimes T = T \oplus \cdots \oplus T$ (n copies)

Lemma $S \leq T$ and $U \leq V$

$\Rightarrow S \otimes U \leq T \otimes V$ and $S \oplus U \leq T \oplus V$ (same for $\trianglelefteq$)

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Corollary
- $R(S \otimes T) \leq R(S)R(T)$ (same for $\underline{R}$)
- $R(S \oplus T) \leq R(S) + R(T)$

(exercises)

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Example

$$\langle 2,2,2 \rangle = \langle 2,1,1 \rangle \otimes \langle 1,2,1 \rangle \otimes \langle 1,1,2 \rangle$$

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Example There are tensors S,T such that

$$R(S \oplus T) < R(S) + R(T)$$

[Shitov 2019, BFPSS 2025]
disproving rank add. conj [Strassen 73]

3. Asymptotic rank

Fekete's lemma

Asymptotic rank $\underset{\sim}{R}(T) = \inf_{n} R(T^{\otimes n})^{1/n} = \lim_{n \to \infty} R(T^{\otimes n})^{1/n}$ [Gartenberg, Strassen].

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Basic properties of $\underset{\sim}{R}$:

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- $\underset{\sim}{R}(T^{\otimes n}) = \underset{\sim}{R}(T)^n$
- $\underset{\sim}{R}(T\otimes S) \le \underset{\sim}{R}(T)\,\underset{\sim}{R}(S)$
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These can be proven directly, but also follow from a general duality, which we will see on Thursday.

How do $\underset{\sim}{R}$, $\underline{R}$ and R relate? So far: $\underline{R} \leq R$, $\underset{\sim}{R} \leq R$

Theorem

$\forall d, \forall T \in \mathbb{F}^{d \times d \times d}, \; R(T^{\otimes n}) \leq \mathrm{poly}(n) \, \underline{R}(T^{\otimes n})$

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$= \sum_{j=0}^{e} \beta_j q(\alpha_j) = q(0) = S$. So $S \leq \langle e+1 \rangle \otimes T$. Argue: $S^{\otimes n} \leq \langle ne+1 \rangle \otimes T^{\otimes n}$ $\square$

Alternative proof:

Lemma

$$\forall T\ \exists m\ \forall n \geq m\ \forall S,\quad S \trianglelefteq T \iff S^{\otimes n} \in \operatorname{span}(GL \cdot T^{\otimes n}) \subseteq \underset{\text{poly(n) dim}}{\underbrace{\mathrm{Sym}^n V}}$$

Proof

$\overline{GL \cdot T}^{Z} = Z(p_1, \ldots, p_\ell)$ for some polynomials p_i.

Let $m = \max_i \deg p_i$.

$$S \trianglelefteq T \iff S \in \overline{GL \cdot T}^{Z} \iff \forall p \in I(GL \cdot T),\ p(S) = 0$$

$$\iff \forall p \in I(GL \cdot T)_{\leq n},\ p(S) = 0$$

$$\iff \forall \text{ linear form } \ell \text{ vanishing on } (GL \cdot T)^{\otimes n},$$

$$\ell(S^{\otimes n}) = 0. \qquad \square$$

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Proof sketch Three ideas:

- Symmetrize: $R(\langle a,b,c\rangle) = R(\langle c,a,b\rangle) = R(\langle b,c,a\rangle)$

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Example (Schönhage) $\underline{R}(\langle k,1,n\rangle \oplus \langle 1,m,1\rangle) \overset{!}{=} kn+1$ for $m=(n-1)(k-1)$.

Taking $k=4$, $n=3$ gives $\omega \le 2.55$

5. Flattening ranks, conciseness $\quad T = (T_{i,j,k}) \in \mathbb{F}^{n_1 \times n_2 \times n_3}$

Flattening matrix

$$T_{(1)} := (T_{i,j,k})_{i \in [n_1],\ (j,k) \in [n_2] \times [n_3]} \in \mathbb{F}^{n_1 \times (n_2 n_3)}$$

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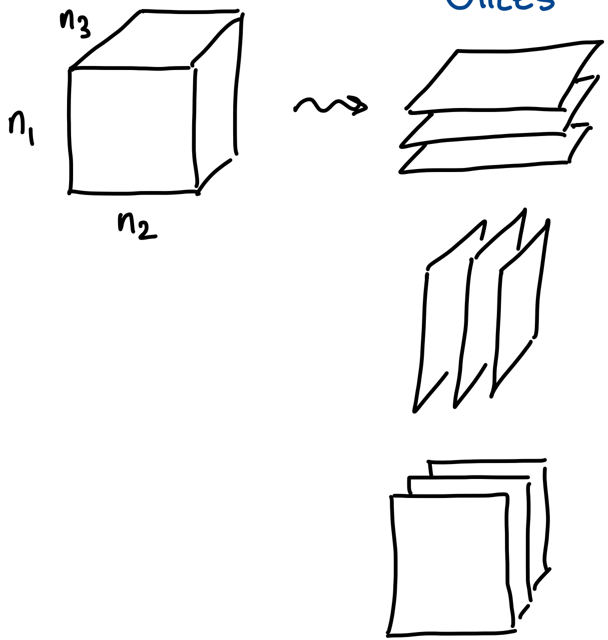
Proof Suppose $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$. Then $T_{(1)} = \sum_{i=1}^{r} u_i \otimes (v_i \otimes w_i)$.
So $R(T_{(1)}) \leq r$.

Alternatively

Slices

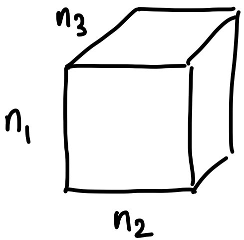


$A_1, \_\_\_, A_{n_1}$

$B_1, \_\_\_, B_{n_2}$

$C_1, \_\_\_, C_{n_3}$

Alternatively



$\rightsquigarrow$

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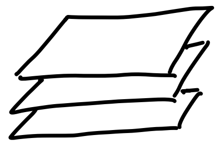
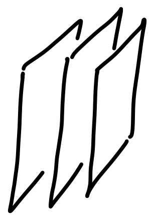
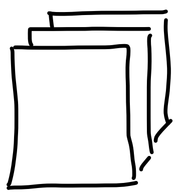


$A_1, \ldots, A_{n_1}$



$B_1, \ldots, B_{n_2}$



$C_1, \ldots, C_{n_3}$

Lemma

$$R_{(1)}(T) \overset{!}{=} \dim \operatorname{span} A_1, \ldots, A_{n_1}$$

$$R_{(2)}(T) \overset{!}{=} \dim \operatorname{span} B_1, \ldots, B_{n_2}$$

$$R_{(3)}(T) \overset{!}{=} \dim \operatorname{span} C_1, \ldots, C_{n_3}$$

Proof hint.

Row rank = column rank $\square$

Recall equivalence: $S \sim T :\Longleftrightarrow S \leq T$ and $T \leq S$

Example

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Lem. Every tensor is equivalent to a concise tensor (in fact, subtensor).

(exercise).

Conciseness is useful for bookkeeping and keeping only the essential part of the tensor.

6. Asymptotic rank conjecture.

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Almost converse: $\underset{\sim}{R}(T) \le n^{2\omega/3}$. Universal families [Kaski–Michałek]

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- $$\underset{\sim}{R}(T)^3 = \underset{\sim}{R}(T^{\otimes 3}) \leq \underset{\sim}{R}(\langle n^2, n^2, n^2 \rangle) = (n^2)^{\omega} \qquad \square$$