Lecture 3. Cap set problem and slice rank

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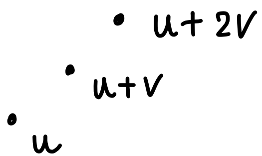
1. Cap set problem

- $x, y, z \in (\mathbb{Z}/3\mathbb{Z})^n$ form a line if $(x,y,z) = (u, u+v, u+2v)$ for some $u, v \in (\mathbb{Z}/3\mathbb{Z})^n$, $v \neq 0$.



- How large can a line-free subset $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ be?

- $2.2202^{n-o(n)} \leq |A| \leq 2.755^{n+o(n)}$

[Edel, Tyrell, DeepMind]

construction

tensor obstruction via subrank

[Ellenberg–Gijswijt]

Lines in $\mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/3\mathbb{Z})^n$

$$(x,y,z) = (u,\ u+v,\ u+2v),\ v \neq 0 \iff \{x,y,z\} = \{1,2,3\}$$

$$I,J,K \in (\mathbb{Z}/3\mathbb{Z})^n \text{ are a line} \iff \forall i \in [n],\ I_i = J_i = K_i \text{ or } \{I_i, J_i, K_i\} = \{1,2,3\} \quad (*)$$

Cap set tensor

$$C := e_1 \otimes e_2 \otimes e_3 + \text{permutations}$$
$$+\ e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 + e_3 \otimes e_3 \otimes e_3 \in \mathbb{F}^3 \otimes \mathbb{F}^3 \otimes \mathbb{F}^3$$

$$C^{\otimes n} = \sum \{ e_I \otimes e_J \otimes e_K : I,J,K \in [3]^n,\ (*) \}$$

Lemma If $A \subseteq (\mathbb{Z}/3\mathbb{Z})^n$ is line-free, then $C^{\otimes n}|_{A \times A \times A}$ is diagonal.

So, $|A| \leq Q(C^{\otimes n})$.

How can we upper bound this?

Matching

$\Psi \subseteq [d] \times [d] \times [d]$ is called a **matching** if $\forall x \neq y \in \Psi$, $\forall i \in [3]$, $x_i \neq y_i$.

Example $\Psi = \mathrm{supp}(\langle d \rangle) = \{(i,i,i) : i \in [d]\}$ is a matching.

Induced matchings

Let $\Phi \subseteq [d] \times [d] \times [d]$. We call $\Psi \subseteq \Phi$ **induced** if $\exists A_1, A_2, A_3 \subseteq [d]$, $\Psi = \Phi \cap (A_1 \times A_2 \times A_3)$.

Example $\{(0,0,0), (1,1,1)\} \subseteq \{(0,0,0), (1,1,1), (2,2,2), (1,2,3) + \text{perm}\}$ is induced

Lemma Let $\Psi \subseteq \Phi$ be an induced matching. Then $|\Psi| \leq Q(T)$ for any tensor T with support Φ.

Proof • $\Psi = \Phi \cap (A_1 \times A_2 \times A_3)$ is a matching

• $\Psi = \mathrm{supp}(T|_{A_1 \times A_2 \times A_3})$ so $\langle |\Psi| \rangle \leq T$. $\square$

2. Slice rank, asymptotic analysis

$C = 111 + 222 + 333 + 123 + \text{perm.} \in \mathbb{F}^3 \otimes \mathbb{F}^3 \otimes \mathbb{F}^3$

How can we upper bound $Q(C^{\otimes n})$? (exercise)

Example For $\mathbb{F} = \mathbb{C}$, $C \cong \langle 3 \rangle$. Then $Q(C^{\otimes n}) = 3^n$, bad bound!

We take $\mathbb{F} = \mathbb{F}_3$!

Lemma $C \cong 110 + 101 + 011 + 002 + 020 + 200$ (over $\mathbb{F}_3$)

Hint: $C : [3] \times [3] \times [3] \to \mathbb{F}_3 \cdot (x, y, z) \mapsto \delta_0(x+y+z)$

$\delta_0(x+y+z) = 1 - (x+y+z)^2 = 1 - x^2 - y^2 - z^2 + xy + yz + xz$

This will be the right tensor to analyse, but we need a new tool.

Recall: tensor rank one: $u \otimes v \otimes w$

Slice rank

- Slice rank one: $\sum_i u \otimes v_i \otimes w_i$, $\sum_i u_i \otimes v \otimes w_i$ or $\sum_i u_i \otimes v_i \otimes w$
- $SR(T) := \min \{ r : T = \sum_{j=1}^{r} T_j,\ T_j \text{ has slice rank one} \}$

Alt:

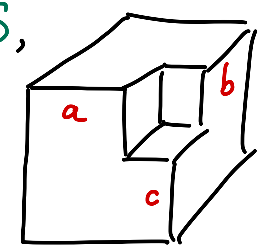
- $SR(T) = \min \{ a+b+c : T \cong S,\ S \in \mathbb{F}^a \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3} + \mathbb{F}^{n_1} \otimes \mathbb{F}^{b} \otimes \mathbb{F}^{n_3} + \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{c} \}$
- $SR(T) = \min \{ R_{(1)}(S_1) + R_{(2)}(S_2) + R_{(3)}(S_3) : T = S_1 + S_2 + S_3 \}$

Lemma

(1) $SR(\langle r \rangle) = r$ [Tao]

(2) $S \leq T \Rightarrow SR(S) \leq SR(T)$.

Visually: $T \cong S$,

$S =$ 

Corollary $Q(T) \leq SR(T)$

Now we need to upper bound $SR(C^{\otimes n})$.

Asymptotic analysis:

$$C \cong 110 + 101 + 011 + 002 + 020 + 200$$

$$C^{\otimes n} \cong \sum \left\{ IJK : \forall i \in [n],\ I_i + J_i + K_i = 2 \right\}$$

$\sum_i I_i \le \frac{2}{3}n$
or $\sum_i J_i \le \frac{2}{3}n$
or $\sum_i K_i \le \frac{2}{3}n$

$$= \sum_{I : \sum_i I_i \le \frac{2}{3}n} \underline{\qquad} + \sum_{J : \sum_i J_i \le \frac{2}{3}n} \underline{\qquad} + \sum_{K : \sum_i K_i \le \frac{2}{3}n} \underline{\qquad}$$

$\#$

$$R_{(1)} \le \sum \left\{ \binom{n}{a,b,c} : a+b+c = n,\ b + 2c \le 2n/3 \right\} \le O(2.755^n)$$

So $SR(C^{\otimes n}) \le O(2.755^n)$.

We will later see structured ways to determine $\lim_{n \to \infty} SR(T^{\otimes n})^{1/n}$.

3. Geometric rank, analytic rank, G-stable rank

We have $Q \leq SR \leq R_{(\ell)}$ What other upper bounds are there on Q?

Geometric rank $T: \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \to \mathbb{C}^{n_3}$ [Moshkovitz-Kopparty-Z]

$$GR(T) = \operatorname{codim} \{ (x,y) : T(x,y) = 0 \}$$

Analytic rank $T: \mathbb{F}_p^{n_1} \times \mathbb{F}_p^{n_2} \to \mathbb{F}_p^{n_3}$ [Gowers-Wolf, Lovett]

$$AR(T) = -\log_p \frac{|\{(x,y) : T(x,y) = 0\}|}{p^{n_1+n_2}} \overset{!}{=} -\log_p \mathop{\mathbb{E}}_{x,y,z} \omega_p^{T(x,y,z)}$$

G-stable rank [Derksen]

$$R^G(T) = \sup_{S \cong T} \min_i \frac{\|S\|_2^2}{\|S_{(i)}\|_\sigma^2}$$

← spectral norm = largest singular value

← ith flattening matrix

Theorem SR, GR, AR, R^G are related up to a constant (over appropriate fields). [Moshkovitz, Cohen, Zhu]

Still, they each have their own applications, e.g.

Theorem $\underline{Q}(\langle n,n,n\rangle) = \lceil \frac{3}{4}n^2 \rceil = GR(\langle n,n,n\rangle)$ $(< SR(\langle n,n,n\rangle) = n^2)$

Yet, they can be quite different from subrank:

Theorem For a random $n \times n \times n$ tensor T,

- $Q(T) \sim \sqrt{n}$
- $SR(T) = n$.

Theorem. $Q \gtrsim \sqrt{SR}$ [Qiyuan Chen - Ke Ye 2025]

4. Asymptotic subrank

For several applications (including cap sets and matrix mult.) the following parameter will be interesting.

Recall: $Q(T) := \max\{r : \langle r \rangle \leq T\}$

Asymptotic subrank $\underset{\sim}{Q}(T) := \lim_{n\to\infty} Q(T^{\otimes n})^{1/n} \overset{!}{=} \sup_n Q(T^{\otimes n})^{1/n}$.

Lemma $\underset{\sim}{Q}(T) \leq \liminf_{n\to\infty} SR(T^{\otimes n})^{1/n}$ (We don't know if the limit always exists...)

Theorem $\underset{\sim}{Q}(C) = 2.755...$

Thus analyzing slice rank of $C^{\otimes n}$ will not give any better bound on cap sets...

Where does the $\underset{\sim}{Q}$ lower bound come from?

5. Tight tensors

There is a general lower bound on $\underset{\sim}{Q}$ for so-called tight tensors, and this lower bound is optimal (Thursday).

Tightness.

- $\Phi \subseteq [n] \times [n] \times [n]$ is tight if there are injective maps $f_i : [n] \to \mathbb{Z}$ such that $\forall u \in \Phi$, $\sum_i f_i(u_i) = 0$.
- $T \in \mathbb{F}^{n \times n \times n}$ is tight if $\exists S \cong T$, $\mathrm{supp}(S)$ is tight.

Example These are tight:

- Any matrix Pf: Any matrix is isomorphic to $\sum_{i=1}^{r} e_i \otimes e_i$. $f_1(i) = i$, $f_2(i) = -i$
- $\langle r \rangle$ Pf: $\langle r \rangle = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$. $f_1(i) = i$, $f_2(i) = i$, $f_3(i) = -2i$
- $\langle a, b, c \rangle$ (exercise)

Theorem Let $T \in \mathbb{F}^{n \times n \times n}$ have tight support $\Phi \subseteq [n] \times [n] \times [n]$.

Then $$\log_2 \underset{\sim}{Q}(T) \geq \max_{P \in \mathcal{P}(\Phi)} \min_{i \in [3]} H(P_i)$$

Where $\mathcal{P}(\Phi)$ are the prob. distr. on Φ, P_i the ith marginal, and H the Shannon entropy.

Given Φ this is usually easy to compute.

Example. • $W = 001 + 010 + 100$ is tight. $\underset{\sim}{Q}(W) \overset{(=)}{\geq} 2^{h(1/3)} \approx 1.88$

• $C \cong 011 + 101 + 110 + 002 + 020 + 200$ is tight. $\underset{\sim}{Q}(C) \overset{(=)}{\geq} 2.755\ldots$

• $\langle a,b,c \rangle$ is tight. $\underset{\sim}{Q}(\langle a,b,c \rangle) \geq \min\{ab, bc, ac\}$.

In fact, for tight Φ, above can be obtained by induced matchings in $\Phi^{\otimes n}$.

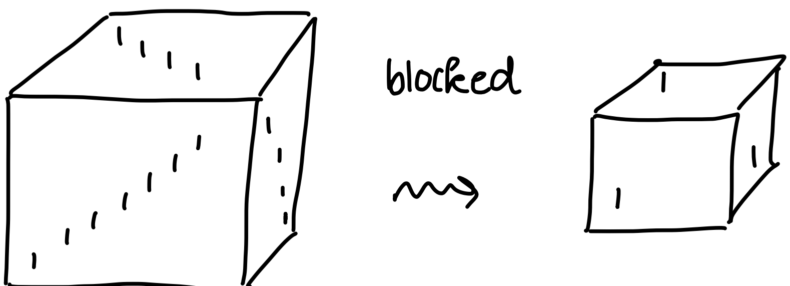
Proof relies on arithmetic progression-free sets. Also: the theorem is optimal!

6. Laser method

CW tensors $$CW_q := \sum_{i=1}^{q} 0ii + i0i + ii0 \in \mathbb{F}^{q+1} \otimes \mathbb{F}^{q+1} \otimes \mathbb{F}^{q+1}.$$

Lemma $\underline{R}(CW_q) \le q+2$ Cheap! How to get matrix mult out of this?

CW_q is not tight (for $q \geq 2$), but it is tight if we squint . . and consider the block tensor, grouping $\{0\}, \{1, \ldots, q\}$ in each coordinate:



CW_q $\qquad$ W

$$W^{\otimes n} \geq \langle 1.88^{n-o(n)} \rangle = r(n)$$

$$CW_q^{\otimes n} \geq \bigoplus_{i=1}^{r(n)} \langle a_i, b_i, c_i \rangle$$

Asymptotic sum inequality : $\omega \leq 2.41$

. . . $\omega \leq 2.37$