Lecture 4 Asymptotic spectrum duality

1. Asymptotic restriction
2. Asymptotic spectrum duality theorem
3. Support functionals
4. Submultiplicativity
5. Oblique tensors
6. Barriers

1. Asymptotic restriction The right order for studying asymptotic problems (like ω).

Asymptotic restriction $S \lesssim T \;:\!\iff\; S^{\otimes n} \le T^{\otimes (n+o(n))}$

Example $\omega = 2 \iff \langle 2,2,2 \rangle \lesssim \langle 4 \rangle$.

Lemma $S^{\otimes n} \le T^{\otimes (n+o(n))} \iff S^{\otimes n} \le \langle 2^{o(n)} \rangle \otimes T^{\otimes n}$

Proof $\Rightarrow$ Let $r = R(T)$. Then $S^{\otimes n} \le T^{\otimes (n+o(n))} \le T^{\otimes o(n)} \otimes T^{\otimes n}$

$\le \langle r^{o(n)} \rangle \otimes T^{\otimes n} = \langle 2^{o(n)} \rangle \otimes T^{\otimes n}$.

$\Leftarrow$ Claim: $\forall T$, either $\underline{\exists i,\ R_i(T) \le 1}$, or $\overbrace{\langle 2 \rangle \le T^{\otimes 2}}^{*}$ (exercise).

Suppose $S^{\otimes n} \le \langle 2^{o(n)} \rangle \otimes T^{\otimes n}$. If $\downarrow$ then $T \sim$ matrix B. Have $R_i(T^{\otimes n}) \le 1$ so $R_i(S^{\otimes n}) \le 2^{o(n)}$ so $R_i(S) \le 1$, and $S \sim$ matrix A. Then $R(A)^n \le 2^{o(n)} R(B)^n$ so $R(A) \le R(B)$ so $A \le B$ i.e. $S \le T$ so $S^{\otimes n} \le T^{\otimes n}$.

Otherwise, (*) so $S^{\otimes n} \le T^{\otimes o(n)} \otimes T^{\otimes n} = T^{\otimes (n+o(n))}$. □

Lemma $S \leq T \Rightarrow S \trianglelefteq T \Rightarrow S \underset{\sim}{\leq} T \Rightarrow \underset{\sim}{R}(S) \leq \underset{\sim}{R}(T)$ (exercise)

$\underset{\sim}{Q}(S) \leq \underset{\sim}{Q}(T)$

$R_{(i)}(S) \leq R_{(i)}(T)$

Use: $S \trianglelefteq T \Rightarrow S^{\otimes n} \leq \langle \mathrm{poly}(n) \rangle \otimes T^{\otimes(n)}$, which we proved before

$\underset{\sim}{\leq}$ is the key to understanding $\underset{\sim}{R}, \underset{\sim}{Q}$ better

$\rightsquigarrow$ Duality

2 Asymptotic spectrum duality theorem

Asymptotic spectrum $X :=$ the set of functions $\phi : \text{tensors} \to \mathbb{R}_{\geq 0}$ such that $\forall S, T$

- $\phi(S \otimes T) = \phi(S)\,\phi(T)$
- $\phi(S \oplus T) = \phi(S) + \phi(T)$
- $S \leq T \Rightarrow \phi(S) \leq \phi(T)$
- $\phi(\langle 1 \rangle) = 1$

In other words, X is the set of restriction monotone semiring homomorphisms.

Example $R_{(i)} \in X$

Lemma $\forall \phi \in X$, $S \lesssim T \Rightarrow \phi(S) \leq \phi(T)$ and $\underset{\sim}{Q}(T) \leq \phi(T) \leq \underset{\sim}{R}(T)$

Theorem (Strassen 1988) $S \lesssim T \iff \forall \phi \in X,\ \phi(S) \leq \phi(T)$.

Theorem (Strassen 1988) $S \underset{\sim}{\leq} T \iff \forall \phi \in X,\ \phi(S) \leq \phi(T).$

Consequences

Theorem $\underset{\sim}{Q}(T) = \min\limits_{\phi \in X} \phi(T)$ and $\underset{\sim}{R}(T) = \max\limits_{\phi \in X} \phi(T)$.

Theorem (Sub/super add/mult) $\underset{\sim}{R}$ is sub mult/add, $\underset{\sim}{Q}$ is super mult/add.

Theorem (Add iff mult.) $\forall S, T,\ \underset{\sim}{R}(S \oplus T) = \underset{\sim}{R}(S) + \underset{\sim}{R}(T)$ (same for $\underset{\sim}{Q}$)

$$\iff \underset{\sim}{R}(S \otimes T) = \underset{\sim}{R}(S)\underset{\sim}{R}(T).$$

Theorem (Asymp. sum ineq.) $\bigoplus\limits_{i=1}^{\ell} \langle a_i, b_i, c_i \rangle \underset{\sim}{\leq} \langle r \rangle \Rightarrow \sum\limits_{i=1}^{\ell} (a_i b_i c_i)^{\omega/3} \leq r$

Theorem $\underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$

Proof $\geq$ clear.

$\leq$
- Suppose $\forall \phi \in X$, $\phi(T) \leq r$.
- $\phi(T^{\otimes n}) \leq \lceil r^n \rceil$ (make RHS integer)
- $T^{\otimes n} \underset{\sim}{\leq} \lceil r^n \rceil$ (duality theorem)
- $T^{\otimes nm} \leq \lceil r^n \rceil^{m + o_n(m)}$ (definition of $\underset{\sim}{\leq}$)
- $\underset{\sim}{R}(T^{\otimes n}) \leq \lceil r^n \rceil^{1 + o_n(m)/m} \to \lceil r^n \rceil$ $(m \to \infty)$
- $\underset{\sim}{R}(T) \leq \lceil r^n \rceil^{1/n} \to r$ $(n \to \infty)$

Theorem $\underset{\sim}{R}(T) = \max\limits_{\phi \in X} \phi(T)$

Theorem (Submult.) $\underset{\sim}{R}(S \otimes T) \leq \underset{\sim}{R}(S)\,\underset{\sim}{R}(T)$.

Proof $\underset{\sim}{R}(S \otimes T) = \phi(S \otimes T) = \phi(S)\,\phi(T) \leq \underset{\sim}{R}(S)\,\underset{\sim}{R}(T)$. $\square$

Theorem. (Add. iff. mult.) $\forall S, T,\ \underset{\sim}{R}(S \oplus T) = \underset{\sim}{R}(S) + \underset{\sim}{R}(T)$

$\iff \underset{\sim}{R}(S \otimes T) = \underset{\sim}{R}(S)\,\underset{\sim}{R}(T)$.

Proof $\Rightarrow$ • $\exists \phi \in X,\ \phi(S \oplus T) = \underset{\sim}{R}(S \oplus T)$.

- $\phi(S) + \phi(T) = \phi(S \oplus T) \overset{!}{=} \underset{\sim}{R}(S \oplus T) = \underset{\sim}{R}(S) + \underset{\sim}{R}(T)$
- $\phi(S) = \underset{\sim}{R}(S)$ and $\phi(T) = \underset{\sim}{R}(T)$.
- $\underset{\sim}{R}(S)\,\underset{\sim}{R}(T) = \phi(S)\,\phi(T) = \phi(S \otimes T) \leq \underset{\sim}{R}(S \otimes T) \leq \underset{\sim}{R}(S)\,\underset{\sim}{R}(T)$

$\Leftarrow$ Similar. $\square$.

Theorem (Asymp. sum ineq.)

- $\bigoplus_{i=1}^{\ell} \langle 2,2,2\rangle^{\otimes m_i} \lesssim \langle r\rangle \Rightarrow \sum_{i=1}^{\ell} (2^{m_i})^{\omega} \leq r$

  Proof $\exists \phi \in X$, $\phi(\langle 2,2,2\rangle) = \underset{\sim}{R}(\langle 2,2,2\rangle) = 2^{\omega}$. $\sum_{i=1}^{\ell} \phi(\langle 2,2,2\rangle)^{m_i} \leq r$. $\square$

- $\bigoplus_{i=1}^{\ell} \langle n_i,n_i,n_i\rangle \lesssim \langle r\rangle \Rightarrow \sum_{i=1}^{\ell} n_i^{\omega} \leq r$

  Proof Claim: $\forall \phi \in X$, $\phi(\langle n,n,n\rangle) = \phi(\langle 2,2,2\rangle)^{\log_2 n}$ (exercise).
  Let ϕ as above: $\sum_{i=1}^{\ell} \phi(\langle 2,2,2\rangle)^{\log_2 n_i} \leq r$. $\square$

- $\bigoplus_{i=1}^{\ell} \langle a_i,b_i,c_i\rangle \lesssim \langle r\rangle \Rightarrow \sum_{i=1}^{\ell} (a_i b_i c_i)^{\omega/3} \leq r$

Proof $2^{\omega} = \phi(\langle 2,2,2\rangle) = \phi(\langle 2,1,1\rangle)\phi(\langle 1,2,1\rangle)\phi(\langle 1,1,2\rangle) = 2^{x_1} \cdot 2^{x_2} \cdot 2^{x_3}$

Claim: $\phi(\langle a_i,b_i,c_i\rangle) = a_i^{x_1} b_i^{x_2} c_i^{x_3}$ (exercise). Then $\sum_{i=1}^{\ell} a_i^{x_1} b_i^{x_2} c_i^{x_3} \leq r$.

The asymptotic spectrum is S_3-invariant: $(1\,2\,3)\phi, (1\,2\,3)^2\phi \in X$ so also

$\sum_{i=1}^{\ell} a_i^{x_3} b_i^{x_1} c_i^{x_2} \leq r$ and $\sum_{i=1}^{\ell} a_i^{x_2} b_i^{x_3} c_i^{x_1} \leq r$. Convexity: $\sum_{i=1}^{\ell} (a_i b_i c_i)^{\frac{x_1+x_2+x_3}{3}} \leq r$

3. Support functionals

What is in the asymptotic spectrum X? There's the flattening ranks. What else? We discuss today the support functionals and tomorrow the quantum functionals.

$$H(q_1, \ldots, q_r) := -\sum_{i=1}^{r} q_i \log q_i .$$

Support functionals for $\theta \in \mathcal{P}([3])$,

$$\zeta^\theta(T) = \min_{S \cong T} \max_{P \in \mathcal{P}(\operatorname{supp} S)} 2^{\sum_i \theta_i H(P_i)}$$

looks familiar?

Theorem $\forall \theta$, ζ^θ is submult, add, mon., norm. [Strassen 1991]

Corollary $\forall \theta$, $\underset{\sim}{Q}(T) \leq \zeta^\theta(T)$

Theorem If T is tight, then $\underset{\sim}{Q}(T) = \min_\theta \zeta^\theta(T)$ $\left[\overset{\text{minimax}}{=} \max_P \min_i 2^{H(P_i)} \right]$

4. Submultiplicativity

$$H_\theta(P) := \sum \theta_i H(P_i)$$

$$\log J^\theta(T) := \min_{S \cong T} \; \max_{P \in \mathcal{P}(\operatorname{supp} S)} H_\theta(P)$$

Lemma $\forall \theta, \forall S, T,$ $\log J^\theta(S \otimes T) \overset{?}{\leq} \log J^\theta(S) + \log J^\theta(T)$

Proof

$$\begin{array}{ccccc}
\log J^\theta(S \otimes T) & \leq & \log J^\theta(S) & + & \log J^\theta(T) \\
\rotatebox{90}{$\geq$} & & \rotatebox{90}{$=$}\,! & & \rotatebox{90}{$=$}\,! \\
\max\limits_{P \in \mathcal{P}(\operatorname{supp} S \otimes T)} H_\theta(P) & & \max\limits_{Q} H_\theta(Q) & & \max\limits_{R} H_\theta(R) \\
\rotatebox{90}{$=$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
H_\theta(P) & \leq & H_\theta(P_{[3]}) & + & H_\theta(P_{3+[3]})
\end{array} \qquad \square$$

5. Oblique tensors

Obliqueness

- $\Phi \subseteq [n]\times[n]\times[n]$ is oblique if it is an antichain in the pointwise ordering.
- $T \in \mathbb{F}^{n\times n\times n}$ is oblique if $\exists S \cong T$, $\mathrm{supp}(S)$ is oblique.

Example

- $\langle r \rangle, \langle a,b,c \rangle$ are oblique.
- In fact, $\{\text{tight}\} \subseteq \{\text{oblique}\}$.

Theorem $\forall \theta$, $J^\theta \in X(\text{oblique})$

We do not know if these are all elements in $X(\text{oblique})$

Example

$$X(W) := \{\phi(W) : \phi \in X\} = [2^{h(1/3)} \approx 1.88, 2] = \{J^\theta(W) : \theta\}$$

6. Barriers

What can we do with these functions? How are matrix multiplication algorithms constructed?

- $\langle n,n,n \rangle \underset{\sim}{\leq} \langle r \rangle$

- $\langle n,n,n \rangle \underset{\sim}{\leq} T \underset{\sim}{\leq} \langle r \rangle$

  $\uparrow$ intermediate tensor

Theorem The best upper bound on ω that can be obtained via T is at least $2\,\dfrac{\log \underset{\sim}{R}(T)}{\log \underset{\sim}{Q}(T)} \geq 2\,\dfrac{\log \max_\theta J^\theta(T)}{\log \min_\theta J^\theta(T)}$

Example CW