Lecture 5 quantum information & the quantum functionals

1. Quantum states
2. Entanglement
3. Quantum operations
4. Marginal density matrices
5. The quantum functionals

1. Quantum states

Quantum particles <u>states</u>:

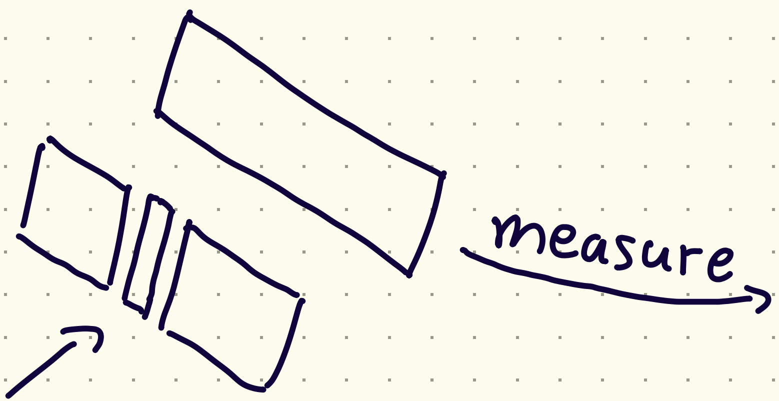
- continuous   e.g. photon location ($\mathbb{R}^3$)
- time-dependent

- discrete   e.g. electron spin

  0   1

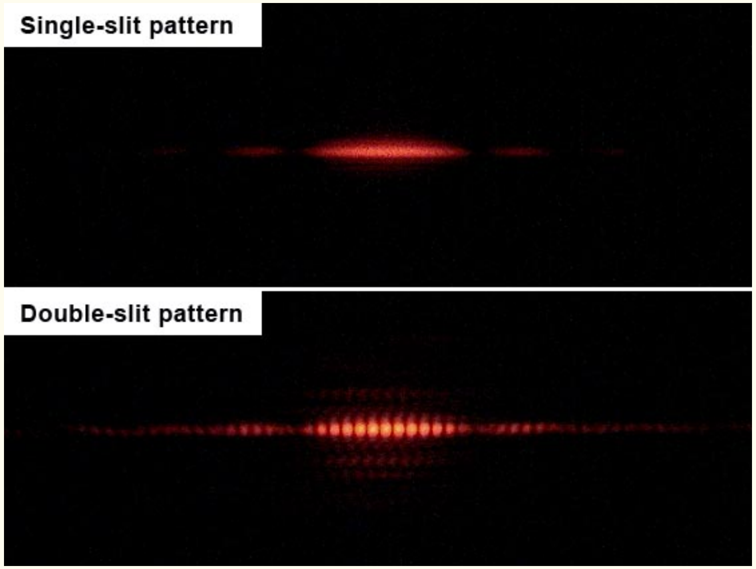
  "qubit" e_1 e_2

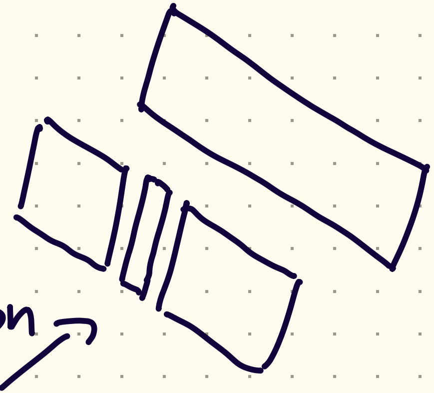
Behaviour: double slit experiment

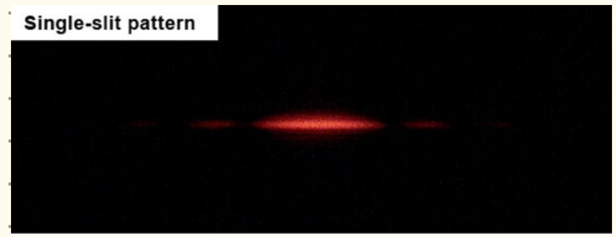


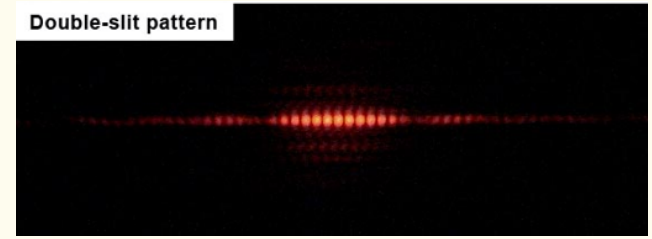
photon

(one-by-one)



photon






Quantum facts

① states can be in superposition

what about a random coin flip? e.g. state $p_1 e_1 + p_2 e_2 \in \mathbb{R}^2$

$p_i \geq 0, \sum p_i = 1$

↳ no, **interference** with itself

"amplitude"

② amplitudes are in $\mathbb{C}$: $v_1 e_1 + v_2 e_2$, $v_i \in \mathbb{C}$

③ measurement finds e_i with prob. $|v_i|^2$

and "collapses" the state

Def. a (discrete) quantum state is a $v \in \mathbb{C}^n$ s.t. $\|v\| = 1$

2. Multipartite states and entanglement

Two particles : states (i,j), each with amplitude

$\longrightarrow \quad \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}, \quad \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$, etc.

↳ "partial system"

Ex. $\frac{1}{\sqrt{2}} e_1 \otimes e_1 \otimes e_1 + \frac{1}{\sqrt{2}} e_2 \otimes e_2 \otimes e_2$

$\xrightarrow{\text{measure 3rd qubit}}$ get e_2 (prob. $\frac{1}{2}$)

$\xrightarrow{\text{state becomes}}$ $e_2 \otimes e_2 \otimes e_2$

we say $\frac{1}{\sqrt{2}}\langle 2\rangle$ is <u>entangled</u>

Def. Measurement :

- get e_i with prob. $\|(e_i e_i^T \otimes I \otimes I) \cdot T\|$
- then the state collapses to $c\,(e_i e_i^T \otimes I \otimes I) \cdot T$

↳ scale to norm 1

Entanglement is a resource.

a measure : $\min\{ r \mid T = \sum_{i=1}^{r} v_i \otimes u_i \otimes w_i \}$ tensor rank !

3. Quantum operations

- Map states in $\mathbb{C}^n$ to states : $n \times n$ unitaries

↳ quantum computers can do any approximately (in theory)

- Systems in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$ can be separated

↳ say in different labs

- Operations: $U_1 \otimes U_2 \otimes U_3$

- More general: restriction !
  - allowing local measurement,
  - local state creation,
  - random success probability

"stochastic local operations and classical communication (SLOCC)"

We get a resource theory of entanglement

- $T \leqslant S$ means S is "more strongly entangled" than T
  $T \trianglelefteq S$ approximately
- $Q(T)$, $\underline{Q}(T)$, $R(T)$, $\underline{R}(T)$ relate T to $\langle r \rangle$, the "golden standard" of entangled states
- $\underset{\sim}{Q}(T)$ is called distillable entanglement, $\underset{\sim}{R}(T)$ entanglement cost

4. Marginal density matrices

What if we obtain $v \in \mathbb{C}^n$ or $w \in \mathbb{C}^n$ with prob. $\frac{1}{2}$?

Def. given states $v_1, \dots, v_s \in \mathbb{C}^n$ and a prob. dist. p, their density matrix is

$$\sum_{i=1}^{s} p_i v_i v_i^* \in \mathbb{C}^{n \times n}$$

i.e. a positive semi-definite matrix of trace 1

Def. $T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$. Flatten to $M \in \mathbb{C}^{n_1 \times n_2 n_3}$.
The first marginal density matrix is $T_1 := MM^* \in \mathbb{C}^{n_1 \times n_1}$

Lemma T_i is PSD and has trace 1 (exercise)

Def. $T \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$. Flatten to $M \in \mathbb{C}^{n_1 \times n_2 n_3}$. The *first marginal density matrix* is $T_1 := MM^* \in \mathbb{C}^{n_1 \times n_1}$

Ex. $T = \frac{1}{\sqrt{2}} \langle 2 \rangle$

$$T_1 = \tfrac{1}{2} \overbrace{(e_1 \otimes (e_1^* \otimes e_1^*) + e_2 \otimes (e_2^* \otimes e_2^*))}^{M} (e_1^* \otimes (e_1 \otimes e_1) + e_2^* \otimes (e_2 \otimes e_2))$$

$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Marginals desribe "local knowledge" of T:

Lemma Suppose we measure factor 2 and 3, obtaining the state $v_{i,j} \otimes e_i \otimes e_j$ with prob. $p_{i,j}$. Then

$$T_1 = \sum_{i=1}^{n_2} \sum_{j=1}^{n_3} p_{i,j} \, v_{i,j} v_{i,j}^*$$

(exercise)

5. The quantum functionals

Def. The moment polytope of T is ↗ or "entanglement polytope"

$$\Delta(T) := \{(\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \mid T' \in \overline{GL \cdot T}, \|T'\| = 1\} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$$

↳ $\geq$-sorted eigenvalues
↳ prob. distribution!

Theorem (Ness-Mumford-Brion)
$\Delta(T)$ is a (convex, compact) polytope

Def. let $(\theta_1, \theta_2, \theta_3)$ be a prob. dist.
The quantum functionals F_θ are

$$F_\theta(T) := 2^{E_\theta(T)} \qquad E_\theta(T) := \max_{(p^1,p^2,p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p_i)$$

Theorem (Christandl-Vrana-Zuiddam, 2018)
F_θ is a spectral point

$$\Delta(T) := \left\{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \;\middle|\; T' \in \overline{GL \cdot T},\ \|T'\| = 1 \right\}$$

$$F_\Theta(T) := 2^{E_\Theta(T)}, \qquad E_\Theta(T) := \max_{(p^1, p^2, p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p^i)$$

Lemma $T \leq S \implies F_\Theta(T) \leq F_\Theta(S)$

Proof: transitivity of $\trianglelefteq$ $\square$

Lemma $F_\Theta(\langle 1 \rangle) = 1$

Proof: $T_i' = [1]$ for all $T' \in \overline{GL \cdot T}$, so

$\Delta(T) = \{(e_1, e_1, e_1)\}$ $\square$

$$\Delta(T) := \left\{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \;\middle|\; T' \in \overline{GL \cdot T},\ \|T'\| = 1 \right\}$$

$$F_\Theta(T) := 2^{E_\Theta(T)}, \qquad E_\Theta(T) := \max_{(p^1,p^2,p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p^i)$$

Lemma $F_\Theta(T \otimes S) \geq F_\Theta(T) F_\Theta(S)$

Proof:
- take $(p^1, p^2, p^3) \in \Delta(T)$, $p^i = \mathrm{spec}(T_i')$,
  $(q^1, q^2, q^3) \in \Delta(S)$, $q^i = \mathrm{spec}(S_i')$, optimal
- Then $T' \otimes S' \in \overline{GL \cdot (T \otimes S)}$, $\|T' \otimes S'\| = 1$

Claim $(T' \otimes S')_i = T_i' \otimes S_i'$ (exercise)

- $\mathrm{spec}((T' \otimes S')_i)$ is $(p_1 q_1, p_1 q_2, \ldots)$ but sorted
- $H(\;\cdot\;) = -\sum_{i,j} p_i q_j \log(p_i q_j) = -\sum_{i,j} p_i q_j \log(p_i) - \sum_{i,j} q_j p_i \log(q_j)$
  $= H(p) + H(q)$

$$\Longrightarrow E_\Theta(T \otimes S) \geq E_\Theta(T) + E_\Theta(S)$$

$\square$

$$\Delta(T) := \{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \mid T' \in \overline{GL \cdot T}, \|T'\| = 1 \}$$

$$F_\Theta(T) := 2^{E_\Theta(T)}, \qquad E_\Theta(T) := \max_{(p^1,p^2,p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p^i)$$

Lemma $F_\Theta(T \oplus S) \geq F_\Theta(T) + F_\Theta(S)$

Proof: similar, see lecture notes □

How about $F_\Theta(T \otimes S) \leq F_\Theta(T) F_\Theta(S)$
$F_\Theta(T \oplus S) \leq F_\Theta(T) + F_\Theta(S)$?

→ next week

using representation theory !!

$$\Delta(T) := \left\{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \mid T' \in \overline{GL \cdot T}, \|T'\| = 1 \right\}$$

$$F_\Theta(T) := 2^{E_\Theta(T)}, \qquad E_\Theta(T) := \max_{(p^1,p^2,p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p^i)$$

Def. T is *free* if $T \cong S$ with

$$S_{i,j,k} \neq 0 \implies \begin{cases} S_{i',j,k} = 0 & \forall i' \neq i \\ S_{i,j',k} = 0 & \forall j' \neq j \\ S_{i,j,k'} = 0 & \forall k' \neq k \end{cases}$$

- $\{\text{tight}\} \subseteq \{\text{oblique}\} \subseteq \{\text{free}\}$
- a generic tensor is non-free

Theorem if T is free, $F_\Theta(T) = \zeta_\Theta(T)$

Question also for non-free tensors?

$$\boxed{\begin{aligned}&\Delta(T) := \left\{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \;\middle|\; T' \in \overline{GL \cdot T},\ \|T'\| = 1 \right\} \\ &F_\theta(T) := 2^{E_\theta(T)}, \quad E_\theta(T) := \max_{(p^1,p^2,p^3)\in\Delta(T)} \sum_{i=1}^{3} \theta_i H(p^i)\end{aligned}}$$

Theorem $\underset{\sim}{SR}(T) = \min_\theta F_\theta(T)$

Question: $\underset{\sim}{SR} \overset{?}{=} \underset{\sim}{Q}$

$$\Delta(T) := \{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \mid T' \in \overline{GL \cdot T}, \|T'\| = 1 \}$$

$$F_\Theta(T) := 2^{E_\Theta(T)}, \qquad E_\Theta(T) := \max_{(p^1,p^2,p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p^i)$$

Prop. if $\Theta = (1,0,0)$ then $F_\Theta(T) = R_1(T)$

Proof: $r := R_1(T)$

flattening of T' $\qquad$ R_1 monotone

$\boxed{\leq}$ $T' \in \overline{GL \cdot T} \Rightarrow R(T') = R(MM^*) \leq R(M) \leq r$

$\Rightarrow \mathrm{spec}(T')$ has $\leq r$ non-zero entries $\Rightarrow H(\cdot) \leq H(\frac{1}{r}, \dots, \frac{1}{r}) \leq \log(r)$

$\boxed{\geq}$ SVD $\Rightarrow T = \sum_{i=1}^{r} v_i \otimes N_i$ with $\{v_i\}$ orthogonal

& $\{N_i\}$ orthonormal

apply restriction $v_i \mapsto \frac{1}{\sqrt{r}} e_i$

$\rightarrow T' = \sum_{i=1}^{r} \frac{1}{\sqrt{r}} e_i \otimes N_i, \quad \|T'\| = 1, \quad T_1' = \sum_{i=1}^{r} \frac{1}{r} e_i e_i^*$

so $H(p^1) = \log r$ $\square$

$$\Delta(T) := \{ (\mathrm{spec}(T_1'), \mathrm{spec}(T_2'), \mathrm{spec}(T_3')) \mid T' \in \overline{GL \cdot T}, \|T'\| = 1 \}$$

$$F_\Theta(T) := 2^{E_\Theta(T)}, \qquad E_\Theta(T) := \max_{(p^1,p^2,p^3) \in \Delta(T)} \sum_{i=1}^{3} \theta_i H(p_i)$$

Prop. if $\Theta = (1,0,0)$ then $F_\Theta(T) = R_1(T)$

Question is $\mathcal{X} = \{ F_\Theta \mid \Theta \}$ for 3-tensors?

- not true for 4-tensors
- if true, $\omega = 2$ (since $F_\Theta(\langle 2,2,2 \rangle) = 2^2$) (exercise)