Tensors and their applications: foundations and recent advances

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Asymptotic spectrum duality

[Strassen, 1987 - 1991]

Applications

- matrix multiplication
- combinatorics
- entanglement

...

Asymptotic spectrum duality

[Strassen, 1987 - 1991]

Applications $\rightsquigarrow$ Tensor ranks

- matrix multiplication
- combinatorics
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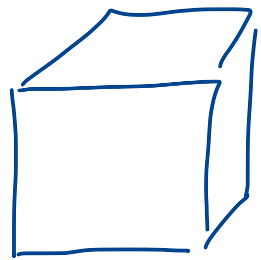
...



Tensor

Asymptotic spectrum duality

[Strassen, 1987 - 1991]

Applications $\rightsquigarrow$ Tensor ranks $\rightsquigarrow$ Asymptotic behaviour

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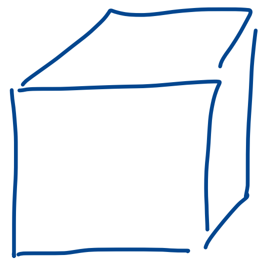

Tensor

$$\lim_{n\to\infty} R(T^{\otimes n})^{1/n}$$

Asymptotic spectrum duality

[Strassen, 1987 - 1991]

Applications $\rightsquigarrow$ Tensor ranks $\rightsquigarrow$ Asymptotic behaviour $\rightsquigarrow$ Asymptotic spectrum

- matrix multiplication
- combinatorics
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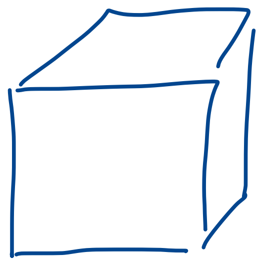
- ...



Tensor

$$\lim_{n\to\infty} R(T^{\otimes n})^{1/n}$$

X

$\longleftarrow$ Asymptotic spectrum duality

The more we know about X, the more tools we have to solve our applications

Bounds, barriers, structure

Survey [Wigderson - Zuiddam 2025]

Week 1 Master class

1. Tensors, ranks, applications
2. Matrix multiplication and asymptotic rank
3. Cap set problem and asymptotic subrank
4. Asymptotic spectrum duality and consequences
5. Quantum information and quantum functionals

Week 2 PhD school

6. Asymptotic spectrum duality: proof
7. Representation theory: Schur–Weyl and moment polytopes
8. Quantum functionals: properties from representation theory
9. Convexity of asymptotic spectra and applications
10. Semicontinuity and discreteness of asymptotic ranks

Lecture 6 Asymptotic spectrum duality: proof.

1. Asymptotic tensor problems and duality
2. Abstract duality
3. Proof
   - 3.1 Strassen preorder
   - 3.2 Closure
   - 3.3 Extension
   - 3.4 Totality
4. Compactness

1. Asymptotic tensor problems and duality

$T \in \mathbb{F}^{n\times n\times n}$, $S \in \mathbb{F}^{m\times m\times m}$

$\langle r\rangle \in \mathbb{F}^{r\times r\times r}$ diagonal tensor

$S \leq T :\Leftrightarrow \exists A_i,\ S = (A_1\otimes A_2\otimes A_3)\cdot T$

$R(T) := \min\{r\in\mathbb{N} : T \leq \langle r\rangle\}$

$Q(T) := \max\{r\in\mathbb{N} : \langle r\rangle \leq T\}$.

Asymptotic:

$S \lesssim T :\Leftrightarrow S^{\otimes k} \leq T^{\otimes(k+o(k))}$

$\underset{\sim}{R}(T) := \lim_{k\to\infty} R(T^{\otimes k})^{1/k}$

$\underset{\sim}{Q}(T) := \lim_{k\to\infty} Q(T^{\otimes k})^{1/k}$

matrix multiplication

$\underset{\sim}{R}(\langle 2,2,2\rangle) = 2^{\omega}$, $\quad \langle a,b,c\rangle := \sum_{i\in[a]}\sum_{j\in[b]}\sum_{k\in[c]} e_{ij}\otimes e_{jk}\otimes e_{ki}$

cap set problem

$\underset{\sim}{Q}(C) = 2.755$, $\quad C := 111 + 222 + 333 + 123 +$ permutations, over $\mathbb{F}_3$

entanglement (asymptotic SLOCC) $\quad S \lesssim T$?

Asymptotic behaviour

$\underset{\sim}{R}$

$\lesssim$

$\underset{\sim}{Q}$

$\longleftrightarrow$

duality

"Asymptotic spectrum"

X

Constructing elements of X

- flattening ranks
- support functionals
- quantum functionals

moment polytopes

Structure of X

- Connectedness
- Convexity
- Semicontinuity

Algebraic geometry

$\mathbb{C}[x_1, \ldots, x_n] \ni 3 + x_1^2 + x_2 x_3$

$I = (p_1 = 0, p_2 = 0, \ldots)$ eqs.

$\mathcal{Z}(I) = \{a : p_1(a) = 0, p_2(a) = 0, \ldots\}$

$\subseteq \mathbb{C}^n$ zero locus

Hilbert:

- $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$
- $q = 0$ on $\mathcal{Z}(I)$

$\Rightarrow \exists k, q^k \in I$

Real geometry

$\mathbb{N}[T_1, T_2, \ldots, T_n] \ni \langle 3 \rangle \oplus T_1^{\otimes 2} \oplus T_2 \otimes T_3$

$I = (p_1 \leq q_1, p_2 \leq q_2, \ldots) = \leq$ ineqs.

$N(I) = \{a : p_1(a) \leq q_1(a), p_2(a) \leq q_2(a), \ldots\}$

$\subseteq \mathbb{R}^n_{\geq 0}$ nonnegative locus $= X$

Strassen:

- $\mathcal{I}(N(I)) = \lesssim$
- $q_1 \leq q_2$ on $N(I)$

$\Rightarrow q_1^k \leq q_2^{k + o(k)}$ in I

Algebraic geometry

$$I \subseteq \sqrt{I} = \bigcap_{\mathfrak{m} \supseteq I} \mathfrak{m}$$

proper maximal ideals

$$\downarrow$$

$$(x_1 - a_1, x_2 - a_2, \ldots)$$

evaluation in $\mathbb{C}^n$

Real geometry

$$\leq \;\subseteq\; \underset{\sim}{\leq} = \bigcap_{\hat{\leq} \supseteq \leq} \hat{\leq}$$

"maximal Strassen preorders"

$$\downarrow$$

$$\phi \in X : T_1 \mapsto a_1,\ T_2 \mapsto a_2, \ldots$$

evaluation in $\mathbb{R}^n_{\geq 0}$

2. Duality

Notation

- $(R, +, \cdot, 0, 1)$ = semiring of tensors with $\oplus, \otimes, \langle 0\rangle, \langle 1\rangle$ ← isomorphism classes of
- $\leq$ restriction
- $\lesssim$ asymptotic restriction: $S \lesssim T :\Leftrightarrow S^k \leq 2^{o(k)} T^k$
$\Leftrightarrow S^k \leq T^{k+o(k)}$

Asymptotic spectrum

$$X = \{\phi : R \to \mathbb{R}_{\geq 0} : \phi(0)=0, \phi(1)=1, \forall S,T\ \phi(ST)=\phi(S)\phi(T),$$
$$\phi(S+T) = \phi(S)+\phi(T),\ S \leq T \Rightarrow \phi(S) \leq \phi(T)\}.$$

Main Theorem $\forall S, T,\quad S \lesssim T \Leftrightarrow \forall \phi \in X, \phi(S) \leq \phi(T)$

$(\Rightarrow)$ $S^k \leq 2^{o(k)} T^k \Rightarrow \phi(S^k) \leq \phi(2^{o(k)} T^k) \Rightarrow \phi(S)^k \leq 2^{o(k)} \phi(T)^k \Rightarrow \phi(S) \leq \phi(T)$ □

Corollary $\forall T$, $\underset{\sim}{Q}(T) = \min_{\phi \in X} \phi(T)$ and $\underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$.

3. Proof.

3.1 Strassen preorder

Preorder

Reflexive and transitive relation $\lesssim$

Strassen preorder

Preorder $\lesssim$ such that:

(1) $\forall S, T, U,\ S \lesssim T \Rightarrow SU \lesssim TU$ and $S + U \lesssim T + U$

(2) $\forall n, m \in \mathbb{N},\ n \leq m$ in $\mathbb{N} \Longleftrightarrow n \lesssim m$ (i.e. $\langle n \rangle \lesssim \langle m \rangle$)

(3) $\forall S \neq 0, \exists n,\ 1 \lesssim S \lesssim n$ "Archimedean"

Example

Restriction on tensors! (also: degeneration, asymptotic restriction)

3.2 Closure

Closure For any Strassen preorder $\leq$ define the **closure** $\underset{\sim}{\leq}$ by
$S \underset{\sim}{\leq} T :\Longleftrightarrow S^n \leq 2^{o(n)} T^n$.

Closed We call $\leq$ **closed** if $\underset{\sim}{\leq} \; = \; \leq$.

Lemma $\underset{\sim}{\leq}$ is closed. (i.e. $\underset{\approx}{\leq} \; = \; \underset{\sim}{\leq}$)

Lemma If $\leq$ is closed, then $\forall S, T, U$

(1) $S + U \leq T + U \Rightarrow S \leq T$ *additive cancellation*

(2) $U \neq 0$ and $SU \leq TU \Rightarrow S \leq T$ *multiplicative cancellation*

(3) $\forall n,\ nS \leq nT + U \Rightarrow S \leq T$ *gap property*

$\exists n,\ nS \not\leq nT + 1 \Leftarrow S \not\leq T$

3.3 Extension

Lemma (One-step extension) Let $\lesssim$ be closed. Suppose $T \not\lesssim S$.
Then $\exists$ Strassen preorder $\lesssim \subseteq \lesssim'$ such that $S \lesssim' T$

Proof Define $\lesssim'$ by $U \lesssim' V :\Leftrightarrow \exists W,\ U + WT \lesssim V + WS$.

Check:

- extension: $\lesssim \subseteq \lesssim'$ (pick $W = 0$)
- $S \lesssim' T$ (pick $W = 1$, $S + T \lesssim T + S \Rightarrow S \lesssim' T$)
- Strassen preorder (exercise)

e.g. to prove $n \lesssim' m \Rightarrow n \leq m$ in $\mathbb{N}$:

$\exists W,\ n + WT \lesssim m + WS$. Suppose $n > m$.
Then $1 + WT \lesssim WS$ (additive cancellation for closed preorder)
So $WT \lesssim WS$. So $T \lesssim S$ (multiplicative cancellation $\not\to$) $\square$

Lemma (Maximal extension) Every Strassen preorder has a maximal Strassen extension (which is then closed and total).

Proof Let $\leqslant$ be a Strassen preorder.

We will use Zorn's lemma.

Let $\mathcal{P}$ be the set of all Strassen preorders extending $\leqslant$ ordered by inclusion.

Let $\mathcal{C} \subseteq \mathcal{P}$ be any chain. Then $\bigcup \mathcal{C}$ is in $\mathcal{P}$ and contains all preorders in $\mathcal{C}$.

Then by Zorn's lemma $\mathcal{P}$ contains a maximal element.

Maximal $\Rightarrow$ closed (otherwise the closure is larger).

Maximal & closed $\Rightarrow$ total (otherwise can do one-step extension) $\square$.

3.4 Totality

Let $\lesssim$ be a Strassen preorder

Fractional rank $\rho(T) := \inf\{\frac{n}{m} : mT \lesssim n\}$

Fractional subrank $\sigma(T) := \sup\{\frac{n}{m} : n \lesssim mT\}$

Lemma.
- $\sigma \leq \rho$
- ρ is sub-add, sub-mult, norm., $\lesssim$-mon.
- σ is super-add, super-mult, norm., $\lesssim$-mon.

Theorem If $\lesssim$ is total, then $\sigma = \rho$ and is thus a $\lesssim$-mon. hom.

Proof Suppose $\sigma(T) < \frac{n}{m} < \rho(T)$

Then $n \not\lesssim mT$

So $mT \lesssim n$ (totality) so $\rho(T) \leq \frac{n}{m}$ $\lightning$ $\square$.

Theorem $S \lesssim T \iff \forall \phi \in X,\ \phi(S) \le \phi(T)$.

Proof $\Rightarrow$ is easy

$\Leftarrow$ Suppose $S \not\lesssim T$. (To prove: $\exists \phi \in X,\ \phi(S) \not\le \phi(T)$.)

By the gap property, $\exists n$, $nS \not\lesssim nT + 1$

Extend $\lesssim$ to a Strassen preorder $\lesssim'$ such that $nT+1 \lesssim' nS$

Extend $\lesssim'$ to a (maximal, closed) total Strassen preorder $\lesssim''$.

Then $nT+1 \lesssim'' nS$

Let $\phi = \kappa = \rho$ be the corresponding $\lesssim''$-mon. hom, $\phi \in X$.

Then $\phi(nT+1) \le \phi(nS)$ so $\phi(T) + \frac{1}{n} \le \phi(S)$ so $\phi(T) < \phi(S)$ □.

4. Compactness

We said at some point that $\underset{\sim}{R}(T) = \max_{\phi \in X} \phi(T)$.
To write an actual "max" there we need to know that X is compact.

Topology

$$X = \{\phi \cdot \text{ mon hom.}\} \subseteq \mathbb{R}_{\geq 0}^{\mathcal{R}}$$

$$\forall T \in \mathcal{R},\ ev_T : X \to \mathbb{R}_{\geq 0} : \phi \mapsto \phi(T)$$

We give X the coarsest topology that makes all ev_T continuous.

Theorem. X is compact (and Hausdorff)

Proof The conditions for mon. hom can be written as intersections of inverse images of closed sets in $\mathbb{R}_{\geq 0}$ under maps ev_T. $\square$

Conclusion

- Asymptotic spectrum duality is more general
  - tensors, $\oplus$, $\otimes$, $\langle 0 \rangle$, $\langle 1 \rangle$, restriction : asymptotic (sub) rank
  - graphs, $\sqcup$, $\boxtimes$, K_0, K_1, co-homomorphism : Shannon capacity

    $\vdots$
- Asymptotic spectrum distance : $d(S,T) = \max_{\phi \in X} |\phi(S) - \phi(T)|$
- next :
  - Rep. theory, quantum functionals $\in X$
  - Structure of X