Lecture 7: Schur-Weyl duality

1. Representation theory
2. S_n, GL_d and tensors
3. Schur-Weyl duality
4. Moment polytopes: convexity
5. Moment polytopes: indeed a polytope

1. Representation theory

The study of linear spaces through group actions

Def. a representation of a group G is

- a vector space V & • a group hom. $\rho_V : G \to GL(V)$ called the "action" of G

Shorthand: $g \cdot v := \rho(g) v$ $\qquad \rho_V(g \cdot h) = \rho_V(g)\rho_V(h)$

Ex. 1. $G = S_3$ "acts" on $V = \mathbb{C}^3$ by permutation matrices:

$$\rho_{\mathbb{C}^3}((123)) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in GL(V)$$

2. $G = GL_{n_1} \times GL_{n_2} \times GL_{n_3}$ & $V = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$

Def. a subrepresentation is a $W \subseteq V$ that is also a representation
↳ closed under the action of G

Ex. 1. $W = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$, since $\pi \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. only $\{0\}$ and V, since $\operatorname{span}\{GL \cdot T\} = V$ (exercise)
$\quad T \neq 0$

Def. V is irreducible if only $\{0\}$ & V are subrepresentations

Ex. 1. no 2. yes

Irreducible representations are building blocks

Theorem for G (e.g.)

- a finite group, or
- a closed subgroup of GL with $A^* \in G \ \forall A \in G$,

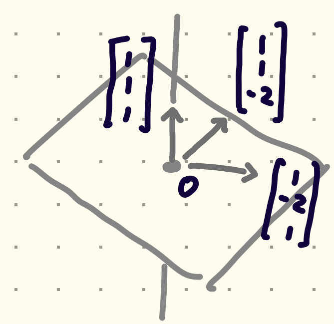
=> "reductive"

V finite dim. $\Rightarrow$ $V = W_1 \oplus \cdots \oplus W_s$

where W_i are irreducible subrepresentations

so $\rho_V(g) \sim \begin{bmatrix} \boxed{\rho_1(g)} & & & 0 \\ & \boxed{\rho_2(g)} & & \\ & & \ddots & \\ 0 & & & \boxed{\rho_s(g)} \end{bmatrix}$

Ex. 1. $\mathbb{C}^3 = \mathrm{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \oplus \mathrm{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}$ $\rho(g) \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{\rho_2(g)}} \\ 0 & & \end{bmatrix}$



For many groups, irreducible representations are classified

Def. V is isomorphic to V' if there exists
$L: V \to V'$ linear bijective s.t. $L\rho(g) = \rho'(g) L \quad \forall g \in G$

i.e. we know some set $\{V_\lambda\}$ of $\cong$-representatives

$$\Rightarrow V = W_1 \oplus \cdots \oplus W_s \cong \bigoplus_\lambda \underbrace{V_\lambda \oplus \cdots \oplus V_\lambda}_{m_\lambda \text{ times}}$$

$$W_i \cong V_{\lambda_i} \qquad \cong \bigoplus_\lambda V_\lambda \otimes \mathbb{C}^{m_\lambda}$$

↳ $\mathbb{C}$ is the trivial repr. $\rho_{\mathbb{C}}(g) = I \quad \forall g$

Def. (abstract) direct sum / tensor product:

$$\rho_{V \oplus V'}(g) := \rho_V(g) \oplus \rho_{V'}(g) \qquad \rho_{V \otimes V'}(g) := \rho_V(g) \otimes \rho_{V'}(g)$$

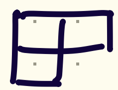
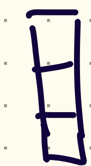
Ex. $\mathbb{C}^3 = \mathrm{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\} \oplus \mathrm{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}\right\} \cong \mathbb{C} \oplus V_{⊞}$

2. S_n, GL_d & tensors

We construct all V_λ for $G = S_n$ using tensors

Def. partition of n: $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d)$, $\lambda_i \in \mathbb{N}$, $\sum_i \lambda_i = 1$

notation: $\lambda \vdash_d n$

Ex. $\lambda = (2,1) \vdash_2 3$  $\lambda = (1,1,1,0) \vdash_4 3$ 

Young diagram

Def. $v_\lambda := (e_1 \wedge \cdots \wedge e_{\lambda_1^*}) \otimes (e_1 \wedge \cdots \wedge e_{\lambda_2^*}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{\lambda_{\lambda_1}^*}) \in (\mathbb{C}^d)^{\otimes n}$

where $\lambda_i^* :=$ "length of the i-th column of λ"

Ex. $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$: $v_{(4,3,1)} = (e_1 \wedge e_2 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \otimes e_1 \quad \in \quad (\mathbb{C}^3)^{\otimes 8}$

Def. $\overset{\in S_n}{\pi} \cdot \overset{\in (\mathbb{C}^d)^{\otimes n}}{v} = \pi \cdot \sum_i v_1^i \otimes v_2^i \otimes \cdots \otimes v_n^i := \sum_i v_{\pi^{-1}(1)}^i \otimes v_{\pi^{-1}(2)}^i \otimes \cdots \otimes v_{\pi^{-1}(n)}^i$

Ex. $(1\,2\,3) \cdot (v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_1 \otimes v_2$

Ex. $v_{(4,3,1)} = (e_1 \wedge e_2 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \otimes e_1 \in (\mathbb{C}^3)^{\otimes 8}$ for the tableau $\begin{ytableau} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 \end{ytableau}$

Theorem any irreducible rep. V of S_n satisfies

$$V \cong V_\lambda := \operatorname{span}\{S_n \cdot v_\lambda\} \quad \text{for some unique } \lambda \vdash_n n$$

$$\subseteq (\mathbb{C}^n)^{\otimes n}$$

Ex. S_2: $v_{\ydiagram{1,1}} = e_1 \wedge e_2 = 11 - 22$ $\qquad v_{\ydiagram{2}} = 11$

S_n: $v_{\ydiagram{1,1,1}} = e_1 \wedge \cdots \wedge e_n$ $\qquad \pi \cdot v_{\ydiagram{1,1,1}} = \operatorname{sgn}(\pi)\, v_{\ydiagram{1,1,1}}$ $\qquad \pi \cdot v_{\ydiagram{3}} = v_{\ydiagram{3}}$

S_3: $v_{\ydiagram{2,1}} = (e_1 \wedge e_2) \otimes e_1 = 121 - 211$

$(12) \cdot v_{\ydiagram{2,1}} = 211 - 121 = -v_{\ydiagram{2,1}}$

$(23) \cdot v_{\ydiagram{2,1}} = 112 - 211 =: \omega$

$(123) \cdot v_{\ydiagram{2,1}} = 112 - 121 = \omega + v_{\ydiagram{2,1}}$

$(13) \cdot v_{\ydiagram{2,1}} = 121 - 112 = -\omega + v_{\ydiagram{2,1}}$

$(321) \cdot v_{\ydiagram{2,1}} = 211 - 112 = -\omega$

so $V_{\ydiagram{2,1}} = \operatorname{span}\{v_{\ydiagram{2,1}}, \omega\}$

Ex. $\mathbb{C}^3 \cong V_{\ydiagram{3}} \oplus V_{\ydiagram{2,1}}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mapsto 111$

$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \mapsto 2\omega - v_{\ydiagram{2,1}}$

$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \mapsto 2v_{\ydiagram{2,1}} - \omega$

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \mapsfrom v_{\ydiagram{2,1}}$

Now we look at $G = GL_d$

Def. $\underset{\substack{\cap \\ GL_d}}{A} \cdot \underset{\substack{\cap \\ (\mathbb{C}^d)^{\otimes n}}}{v} := (A \otimes A \otimes \cdots \otimes A) \cdot v$ "diagonal action"

Def. $v \in V$ is a highest weight vector of weight $\lambda \vdash_d n$ if

$$\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_d \end{bmatrix} \cdot v = t_1^{\lambda_1} \cdots t_d^{\lambda_d} v \quad \& \quad \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \cdot v = v$$

Ex. v_λ, e.g. $v_{(4,3,1)} = (e_1 \wedge e_2 \wedge e_3) \otimes (e_1 \wedge e_2) \otimes (e_1 \wedge e_2) \otimes e_1$

since $\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \cdot e_1 \wedge \cdots \wedge e_s = \sum_{i_1, \dots, i_s \leq s} * \, e_{i_1} \wedge \cdots \wedge e_{i_s} = e_1 \wedge \cdots \wedge e_s$

Prop: every irr. rep. of GL_d has a unique highest weight vector. (up to scaling)

$\rho : GL_d \to GL(V)$ is a polynomial map

Theorem any polynomial irreducible rep. V of GL_d satisfies

$$V \cong \underset{\subseteq (\mathbb{C}^d)^{\otimes n}}{W_\lambda := \operatorname{span}\{GL_d \cdot v_\lambda\}} \quad \text{for some } n > 0 \ \& \ \lambda \vdash_d n$$

Theorem any polynomial irreducible rep. V of GL_d satisfies

$$V \cong W_\lambda := \operatorname{span}\{GL_d \cdot v_\lambda\} \quad \text{for some } n>0 \;\&\; \lambda \vdash_d n$$
$$\subseteq (\mathbb{C}^d)^{\otimes n}$$

Ex. • $\mathbb{C}^2 \otimes \mathbb{C}^2 = \operatorname{span}\{\overset{\text{highest weight vector}}{11}, 22, 12+21\} \oplus \operatorname{span}\{12-21\}$

$= W_{\ydiagram{2}} \oplus W_{\ydiagram{1,1}}$

• $W_{\ydiagram{1}} = \mathbb{C}^d$

• $W_{\ydiagram{1,1,1}} = \mathbb{C}(e_1 \wedge e_2 \wedge \dots \wedge e_d)$ (one-dimensional)

$A \cdot (e_1 \wedge \dots \wedge e_d) = Ae_1 \wedge \dots \wedge Ae_d = \det(A)(e_1 \wedge \dots \wedge e_d)$

• $W_{\ydiagram{3}} = \operatorname{span}\{GL_d \cdot (e_1 \otimes \dots \otimes e_1)\} = \operatorname{span}\{v \otimes \dots \otimes v \mid v \in \mathbb{C}^d\}$

• $W_{\ydiagram{2,1}} = \operatorname{span}\{v_{\ydiagram{2,1}}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot v_{\ydiagram{2,1}}\}$

$v_{\ydiagram{2,1}} = 121 - 211$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot v_{\ydiagram{2,1}} = 212 - 122$

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot v_{\ydiagram{2,1}} = 121 + \cancel{221} + 122 + \cancel{222} - 211 - \cancel{221} - 212 - \cancel{222} = v_{\ydiagram{2,1}} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot v_{\ydiagram{2,1}}$

3. Schur-Weyl duality

How do the actions of S_n & GL_d on $(\mathbb{C}^d)^{\otimes n}$ interact?

Lemma $\pi \cdot (A \cdot v) = A \cdot (\pi \cdot v) \quad \forall \pi \in S_n,\ A \in GL_n,\ v \in (\mathbb{C}^d)^{\otimes n}$

Proof. $\pi \cdot (A \otimes \cdots \otimes A) \cdot (v_1 \otimes \cdots \otimes v_n) = \pi \cdot (Av_1 \otimes \cdots \otimes Av_n)$

$= (A \otimes \cdots \otimes A) \cdot (\pi \cdot (v_1 \otimes \cdots \otimes v_n))$

Corollary $V_\lambda \cong A \cdot V_\lambda$ and $W_\lambda \cong \pi \cdot W_\lambda$

Proof. $\rho(A) : V_\lambda \to A \cdot V_\lambda$ is invertible & commutes with S_n action $\square$

also, $V_\lambda \cap A \cdot V_\lambda = V_\lambda$ or $\{0\}$, since it is a subrepresentation

$$(\mathbb{C}^d)^{\otimes n} \overset{S_n}{\cong} \bigoplus_{\lambda \vdash_d n} V_\lambda \oplus A_2 \cdot V_\lambda \oplus A_3 \cdot V_\lambda \oplus \cdots \oplus A_{m_\lambda} \cdot V_\lambda$$

$$(\mathbb{C}^d)^{\otimes n} \overset{GL_d}{\cong} \bigoplus_{\lambda \vdash_d n} \underset{\rotatebox{90}{$\in$}}{W_\lambda} \oplus \underset{\rotatebox{90}{$\in$}}{\pi_2 \cdot W_\lambda} \oplus \underset{\rotatebox{90}{$\in$}}{\pi_3 \cdot W_\lambda} \oplus \cdots \oplus \underset{\rotatebox{90}{$\in$}}{\pi_{\ell_\lambda} \cdot W_\lambda}$$

$$\operatorname{span}\{ v_\lambda,\ \pi_2 \cdot v_\lambda,\ \pi_3 \cdot v_\lambda,\ \ldots,\ \pi_{\ell_\lambda} \cdot v_\lambda \} = V_\lambda$$

the "λ-isotypic subspace" ISO_λ

$$(\mathbb{C}^d)^{\otimes n} \overset{S_n}{=} \bigoplus_{\lambda \vdash_d n} \boxed{V_\lambda \oplus A_2 \cdot V_\lambda \oplus A_3 \cdot V_\lambda \oplus \cdots \oplus A_{m_\lambda} \cdot V_\lambda}$$

$$(\mathbb{C}^d)^{\otimes n} \overset{Gl_d}{=} \bigoplus_{\lambda \vdash_d n} \boxed{W_\lambda \oplus \pi_2 \cdot W_\lambda \oplus \pi_3 \cdot W_\lambda \oplus \cdots \oplus \pi_{\ell_\lambda} \cdot W_\lambda}$$

$$\mathrm{span}\{ v_\lambda, \ \pi_2 \cdot v_\lambda, \ \pi_3 \cdot v_\lambda, \ \ldots, \ \pi_{\ell_\lambda} \cdot v_\lambda \} = V_\lambda$$

$$\mathrm{span}\{ A_i \cdot v_\lambda, \ A_i \cdot \pi_2 \cdot v_\lambda, \ A_i \cdot \pi_3 \cdot v_\lambda, \ \ldots, \ A_i \cdot \pi_{\ell_\lambda} \cdot v_\lambda \} = A_i \cdot V_\lambda$$

Theorem (Schur-Weyl duality) for all λ

$$\boxed{} = \boxed{} \cong V_\lambda \otimes W_\lambda$$

i.e.

$$(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} \mathrm{ISO}_\lambda \cong \bigoplus_{\lambda \vdash_d n} \underset{\substack{\cong \\ V_\lambda}}{[\lambda]} \otimes \underset{\substack{\cong \\ W_\lambda}}{\{\lambda\}}$$

note: $[\lambda] \otimes \{\lambda'\}$ are irr. rep. of $S_n \times Gl_d$ (excercise)

$\longrightarrow$ but we only get $\lambda' = \lambda$!

$$(\mathbb{C}^d)^{\otimes n} \overset{GL_d}{=} \bigoplus_{\lambda \vdash_d n} \boxed{W_\lambda \oplus \pi_2 \cdot W_\lambda \oplus \pi_3 \cdot W_\lambda \oplus \cdots \oplus \pi_{\ell_\lambda} \cdot W_\lambda}$$

$$\mathrm{span}\{ v_\lambda, \ \pi_2 \cdot v_\lambda, \ \pi_3 \cdot v_\lambda, \ \dots, \ \pi_{\ell_\lambda} \cdot v_\lambda \} = V_\lambda$$

Theorem $(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} \mathrm{ISO}_\lambda \cong \bigoplus_{\lambda \vdash_d n} [\lambda] \otimes \{\lambda\}$

Ex.

- $\langle GL_d \cdot v_{\square\square\cdots\square} \rangle = \{ v \in (\mathbb{C}^d)^{\otimes n} \mid \pi \cdot v = v \}$, the symmetric subspace
- $\langle GL_d \cdot v_{\boxminus} \rangle = \{ v \in (\mathbb{C}^d)^{\otimes n} \mid \pi \cdot v = \mathrm{sign}(\pi)\, v \}$, the antisymmetric subspace
  (if $d \geq n$, if $d = n$ one-dimensional)
- $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong W_{\square\square\square} \otimes V_{\square\square\square} \oplus W_{\boxplus} \otimes V_{\boxplus}$

$$= \langle GL_d \cdot v_{\square\square\square} \rangle \oplus \langle GL_d \cdot v_{\boxplus} \rangle \oplus \langle GL_d \cdot ((23) \cdot v_{\boxplus}) \rangle$$

$V_{\square\square\square} = \langle v_{\square\square\square} \rangle$

$V_{\boxplus} = \langle v_{\boxplus}, (23) \cdot v_{\boxplus} \rangle$

$$= \langle 111, 222, 112 + 121 + 211, 221 + 212 + 122 \rangle$$
$$\oplus \langle 121 - 211, 212 - 122 \rangle$$
$$\oplus \langle 112 - 211, 221 - 122 \rangle$$

Back to our setting

$T \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$

Groups: $GL := GL_{d_1} \times GL_{d_2} \times GL_{d_3}$ & S_n

$$\rho(A,B,C) := A^{\otimes n} \otimes B^{\otimes n} \otimes C^{\otimes n}$$
$$\rho(\pi) := \rho_{(\mathbb{C}^{d_1})^{\otimes n}}(\pi) \otimes \rho_{(\mathbb{C}^{d_2})^{\otimes n}}(\pi) \otimes \rho_{(\mathbb{C}^{d_3})^{\otimes n}}(\pi)$$

$$T^{\otimes n} \in (\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n}$$

$$= \bigoplus_{\substack{\lambda \vdash_{d_1} n,\ \mu \vdash_{d_2} n, \\ \nu \vdash_{d_3} n}} ISO_\lambda \otimes ISO_\mu \otimes ISO_\nu$$

↳ the (λ,μ,ν)-isotypic subspace $ISO_{\lambda,\mu,\nu}$

$$\cong \left(\bigoplus_{\lambda \vdash_{d_1} n} [\lambda] \otimes \{\lambda\}\right) \otimes \left(\bigoplus_{\mu \vdash_{d_2} n} [\mu] \otimes \{\mu\}\right) \otimes \left(\bigoplus_{\nu \vdash_{d_3} n} [\nu] \otimes \{\nu\}\right)$$

$$\cong \bigoplus_{\lambda,\mu,\nu} [\lambda] \otimes [\mu] \otimes [\nu] \otimes \{\lambda\} \otimes \{\mu\} \otimes \{\nu\}$$

$$(\mathbb{C}^{d_1})^{\otimes n} \otimes (\mathbb{C}^{d_2})^{\otimes n} \otimes (\mathbb{C}^{d_3})^{\otimes n} = \bigoplus_{\lambda,\mu,\nu} \mathrm{ISO}_\lambda \otimes \mathrm{ISO}_\mu \otimes \mathrm{ISO}_\nu$$

$$\cong \bigoplus_{\lambda,\mu,\nu} [\lambda]\otimes[\mu]\otimes[\nu] \otimes \{\lambda\}\otimes\{\mu\}\otimes\{\nu\}$$

Comments

- Highest weight vectors: $\mathrm{span}\{\pi_1 \cdot v_\lambda \otimes \pi_2 \cdot v_\mu \otimes \pi_3 \cdot v_\nu \mid \pi_i \in S_n\}$
  $= V_\lambda \otimes V_\mu \otimes V_\nu$

$$\left(\begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}\right) \cdot \overset{\psi}{\omega} = \omega$$

$$\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_{d_1} \end{bmatrix}, \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_{d_2} \end{bmatrix}, \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_{d_3} \end{bmatrix}\right) \cdot \omega = \prod_i t_i^{\lambda_i} \prod_i s_i^{\mu_i} \prod_i r_i^{\nu_i} \, \omega$$

$\hookrightarrow$ weight (λ, μ, ν)

- $[\lambda] \otimes \{\lambda\}$ is an irr. rep. of $S_n \times Gl_d$.

  so is $[\lambda]\otimes[\mu]\otimes[\nu] \otimes \{\lambda\}\otimes\{\mu\}\otimes\{\nu\}$

  of $S_n \times S_n \times S_n \times Gl_{d_1} \times Gl_{d_2} \times Gl_{d_3}$

  $\longrightarrow$ but not of $S_n \times Gl_{d_1} \times Gl_{d_2} \times Gl_{d_3}$

4. Moment polytopes

$$[\lambda]\otimes[\mu]\otimes[\nu]\otimes\{\lambda\}\otimes\{\mu\}\otimes\{\nu\}$$

$$T^{\otimes n} \in \left(\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}\right)^{\otimes n} = \bigoplus_{\lambda,\mu,\nu} \underbrace{\mathrm{ISO}_{\lambda,\mu,\nu}}_{\cong\,[\lambda]\otimes[\mu]\otimes[\nu]\otimes\{\lambda\}\otimes\{\mu\}\otimes\{\nu\}}$$

projections

euclid. closure

Theorem

$$\Delta(T) = \overline{\left\{ \left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n}\right) \;\middle|\; n>0,\ \begin{matrix}\lambda \vdash_{d_1} n \\ \mu \vdash_{d_2} n \\ \nu \vdash_{d_3} n\end{matrix},\ P_{\lambda,\mu,\nu}\, T^{\otimes n} \neq 0 \right\}}$$

Corollary $\Delta(T)$ is convex

We will prove this using highest weight vectors

Lemma $\rho((A,B,C))\,P_{\lambda,\mu,\nu} = P_{\lambda,\mu,\nu}\,\rho((A,B,C))$

- $$(\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3})^{\otimes n} = \bigoplus_{\lambda,\mu,\nu} \underline{\mathrm{ISO}_\lambda} \otimes \mathrm{ISO}_\mu \otimes \mathrm{ISO}_\nu$$

$$\mathrm{ISO}_\lambda = W_\lambda \oplus \pi_2\cdot W_\lambda \oplus \pi_3\cdot W_\lambda \oplus \cdots \oplus \pi_{\ell_\lambda}\cdot W_\lambda$$

$$\rho(A) \sim \begin{bmatrix} \boxed{\rho_{\lambda_1}(A)} & & & & \\ & \ddots & & & \\ & & \boxed{\rho_{\lambda_1}(A)} & & \\ & & & \ddots & \\ & & & & \boxed{\rho_{\lambda_s}(A)} \\ & & & & & \ddots \\ & & & & & & \boxed{\rho_{\lambda_s}(A)} \end{bmatrix}$$

proj. onto

$$P_{\lambda_1} \sim \begin{bmatrix} \boxed{I} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

$\square$

Lemma $P_{\lambda,\mu,\nu} T^{\otimes n} \neq 0 \iff \langle \omega, S^{\otimes n} \rangle \neq 0$ for some $S \cong T$ and h.w.v. ω

Proof. $\boxed{\Leftarrow}$ let $S = (A \otimes B \otimes C) \cdot T$

- $\langle \omega, S^{\otimes n} \rangle = \langle \omega, (A^{\otimes n} \otimes B^{\otimes n} \otimes C^{\otimes n}) \cdot T^{\otimes n} \rangle = \langle \underbrace{(A^{\otimes n} \otimes B^{\otimes n} \otimes C^{\otimes n})^{*} \omega}_{=: \omega'}, T^{\otimes n} \rangle$
- $\omega \in ISO_{\lambda,\mu,\nu} \Rightarrow \omega' \in ISO_{\lambda,\mu,\nu}$

$\boxed{\Rightarrow}$ $ISO_{\lambda} = W_{\lambda} \oplus \pi_2 \cdot W_{\lambda} \oplus \pi_3 \cdot W_{\lambda} \oplus \cdots \oplus \pi_{\ell_\lambda} \cdot W_{\lambda}$

$$\begin{array}{lccccc} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{h.w.v.:} & v_\lambda, & \pi_2 \cdot v_\lambda, & \pi_3 \cdot v_\lambda, & \cdots, & \pi_{\ell_\lambda} \cdot v_\lambda \end{array}$$

- $P : ISO_{\lambda,\mu,\nu} \to \pi_i \cdot W_{\lambda} \otimes \pi'_j \cdot W_{\mu} \otimes \pi''_k \cdot W_{\nu} =: W$

  with $P P_{\lambda,\mu,\nu} \cdot T^{\otimes n} \neq 0$ exists

- W is irr. rep. of $GL_{d_1} \times GL_{d_2} \times GL_{d_3}$

$\Rightarrow \operatorname{span}\{GL \cdot (PP_{\lambda,\mu,\nu} \cdot T^{\otimes n})\} \overset{\text{previous lemma}}{=} \operatorname{span}\{PP_{\lambda,\mu,\nu} \cdot (GL \cdot T^{\otimes n})\}$

$\ni \pi_i \cdot v_\lambda \otimes \pi'_j \cdot v_\mu \otimes \pi''_k \cdot v_\nu$ $\square$

Lemma if v, v' are highest weight vectors, so is $v \otimes v'$
of weight λ, λ' of weight $\lambda+\lambda'$

Proof. (for GL_d)

- $\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \cdot (v \otimes v') = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \cdot v \otimes \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \cdot v' = v \otimes v'$,
- $\begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_d \end{bmatrix} \cdot (v \otimes v') = \prod_i t_i^{\lambda_i + \lambda_i'} (v \otimes v')$

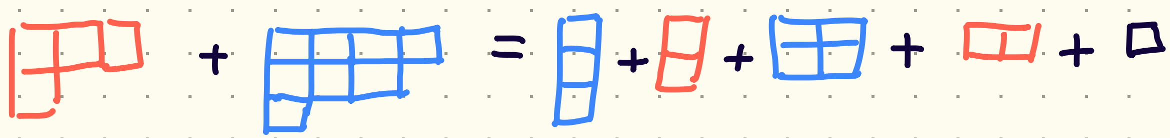
Concrete proof.

$(\pi \cdot v_\lambda) \otimes (\pi' \cdot v_{\lambda'}) =$

$\pi \cdot ((e_1 \wedge \cdots \wedge e_{\lambda_1^*}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{\lambda^*_{\lambda_1}})) \otimes \pi' \cdot ((e_1 \wedge \cdots \wedge e_{\lambda_1'^*}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{\lambda'^*_{\lambda_1'}}))$

$= \sigma \cdot ((e_1 \wedge \cdots \wedge e_{(\lambda+\lambda')^*_1}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{(\lambda+\lambda')^*_{\lambda_1+\lambda_1'}}))$

$= \sigma \cdot v_{\lambda+\lambda'}$



Lemma $P_{\lambda,\mu,\nu} T^{\otimes n} \neq 0 \iff \langle v, S^{\otimes n} \rangle \neq 0$ for some $S \cong T$ and h.w.v. v

Lemma if v, v' are highest weight vectors, so is $v \otimes v'$
of weight λ, λ' of weight $\lambda + \lambda'$

Corollary $\Delta(T) = \overline{\{(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n}) \mid \lambda,\mu,\nu \vdash n,\ P_{\lambda,\mu,\nu} T^{\otimes n} \neq 0\}}$ is convex

Proof. take $\left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n}\right), \left(\frac{\lambda'}{m}, \frac{\mu'}{m}, \frac{\nu'}{m}\right) \in \Delta(T)$

- to show: $\left(\underbrace{\tfrac{1}{2}\left(\tfrac{\lambda}{n} + \tfrac{\lambda'}{m}\right)}, \ldots, \ldots\right) \in \Delta(T)$
  $= \left(\frac{m\lambda + n\lambda'}{2nm}, \ldots, \ldots\right)$
- $\langle v, S^{\otimes n} \rangle \neq 0$ & $\langle v', S^{\otimes m} \rangle \neq 0$ for h.w.v. v and v'
  $S \cong T$ weight λ, μ, ν λ', μ', ν'
- $\langle v^{\otimes m} \otimes v'^{\otimes n}, S^{\otimes nm} \rangle = \langle v, S^{\otimes n} \rangle^m \langle v', S^{\otimes m} \rangle^n \neq 0$
- h.w.v.$^{\checkmark}$! weight is $(m\lambda + n\lambda', \ldots, \ldots)$

$\square$

5. Moment polytopes: a polytope

Theorem

$$\Delta(T) = \left\{ \left(\frac{\lambda}{n}, \frac{\mu}{n}, \frac{\nu}{n}\right) \;\middle|\; n > 0,\ \begin{array}{l} \lambda \vdash_{d_1} n \\ \mu \vdash_{d_2} n \\ \nu \vdash_{d_3} n \end{array},\ P_{\lambda,\mu,\nu} T^{\otimes n} \neq 0 \right\}$$

Corollary $\Delta(T)$ is a polytope

Sketch.
- let ω be a h.w.v.
- $T \mapsto \langle \omega, T^{\otimes n} \rangle$ is a polynomial of degree n
- $\left(\begin{bmatrix} t_1 & & \\ & \ddots & \\ * & & \end{bmatrix}, \begin{bmatrix} s_1 & & \\ & \ddots & \\ * & & \end{bmatrix}, \begin{bmatrix} r_1 & & \\ & \ddots & \\ * & & \end{bmatrix} \right) \cdot T \mapsto \left\langle \left(\begin{bmatrix} t_1 & * \\ & \ddots \end{bmatrix}, \begin{bmatrix} s_1 & * \\ & \ddots \end{bmatrix}, \begin{bmatrix} r_1 & * \\ & \ddots \end{bmatrix} \right) \cdot \omega, T^{\otimes n} \right\rangle$

$$= \prod_{i,j,k} t_i^{\lambda_i} s_j^{\mu_j} r_k^{\nu_k} \langle \omega, T^{\otimes n} \rangle$$

- polynomials with this property also give h.w.v.
  ↳ "covariants"
- the ideal of covariants is finitely generated