Lecture 8 Quantum functionals

1. Asymptotic spectrum recap

2. Moment polytopes and Schur-Weyl recap

3. Quantum functionals recap

4. Kronecker coefficients, semigroup property, entropy inequalities

5. Submultiplicativity

6. Examples

1. Asymptotic spectrum recap

Describing behaviour of large powers of tensors: $T^{\otimes n}$

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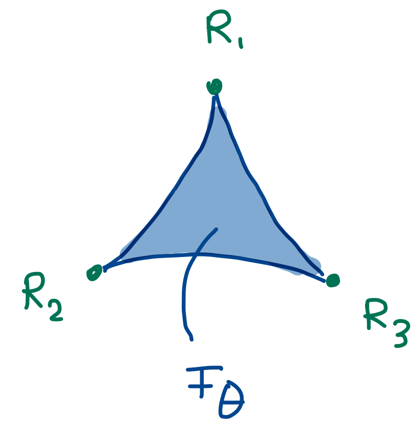

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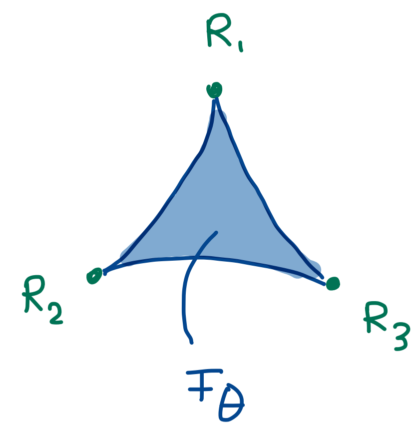

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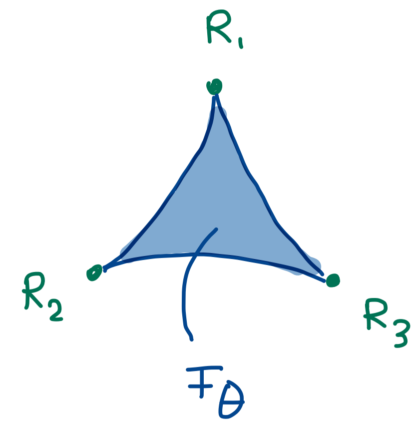

We don't know if this is all of X

$X = \{F_\theta : \theta\} \Rightarrow \omega = 2.$

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$\langle 2 \rangle$, W, $\langle 2,1,1 \rangle$, $\langle 1,2,1 \rangle$, $\langle 1,1,2 \rangle$, $\langle 1 \rangle$, $\langle 0 \rangle$.

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$\rightsquigarrow$ Coarser classification that is meaningful and finite.
Moment polytopes $\Delta(T)$

Moment polytope

$S, T \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3}$ $\qquad$ $S_i \in \mathbb{C}^{d_i \times (d_j d_k)}$ flattening

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ordered list of eigenvalues

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Schur-Weyl decomposition

$$S_n \times (GL_d \times GL_d \times GL_d)$$

$$(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda_i \vdash_d n} [\lambda_1] \otimes [\lambda_2] \otimes [\lambda_3] \otimes \{\lambda_1\} \otimes \{\lambda_2\} \otimes \{\lambda_3\}$$

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We have used the spec definition of $\Delta(T)$ to prove mon., norm., supermult (and superadd can be done similarly).

Remaining: submult (and subadd).

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Kronecker coefficients.

$$S_n \circlearrowright [\lambda] \otimes [\mu] = \bigoplus_{\nu \vdash n} [\nu]^{\oplus k(\lambda, \mu, \nu)}$$ decomposition into S_n-irreps.

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$k((2,1), (2,1), \nu) = 1$ for $\nu = (1,1)$, (2) and $(2,1)$

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S_n-equivariant maps correspond to S_n-invariant tensors

$[\nu] \cong [\nu]^*$ because the characters of S_n-reps are rational.

$$k(\lambda,\mu,\nu) = \dim\left([\lambda]\otimes[\mu]\otimes[\nu]\right)^{S_n}$$

Theorem (Semigroup property)

$$k(\lambda,\mu,\nu)\neq 0 \quad \& \quad k(\lambda',\mu',\nu')\neq 0 \implies k(\lambda+\lambda',\mu+\mu',\nu+\nu')\neq 0$$

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$k(\lambda,\mu,\nu) \neq 0$ & $k(\lambda',\mu',\nu') \neq 0 \Rightarrow k(\lambda+\lambda', \mu+\mu', \nu+\nu') \neq 0$

Proof

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so $\left([\lambda+\lambda']\otimes[\mu+\mu']\otimes[\nu+\nu']\right)^{S_{n+m}}\neq 0$ (Schur-Weyl) $\square$

Entropy inequalities

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$\implies \forall N,\ [N\nu] \subseteq [N\lambda] \otimes [N\mu]$ (def.)

$\implies \forall N,\ \dim[N\nu] \leq \dim[N\lambda] \cdot \dim[N\mu]$

Use $(*)$ and take Nn th root and $N \to \infty$. $\square$

5. Submultiplicativity

$$\Delta(T) = \text{closure}\left\{ \underline{\lambda}/n \,\cdot\, n \in \mathbb{N},\ \lambda_i \vdash n,\ P_{\underline{\lambda}}(T^{\otimes n}) \neq 0 \right\}$$

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Then $P^{d^2}_{\underline{\lambda}} \sum_{\underline{\mu}, \underline{\nu}} (P^d_{\underline{\mu}} \otimes P^d_{\underline{\nu}})(S \otimes T)^{\otimes n} \neq 0$.

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$\square$

Subadditivity

Analogous argument

Kronecker coefficients

Littlewood-Richardson coefficients

$\lambda, \mu \vdash n$

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu \vdash n} [\nu]^{\oplus k(\lambda,\mu,\nu)}$$

- $k(\lambda,\mu,\nu) \neq 0$ & $k(\lambda',\mu',\nu') \neq 0$

  $\Rightarrow k(\lambda+\lambda', \mu+\mu', \nu+\nu') \neq 0$
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Littlewood-Richardson coefficients

$\lambda \vdash a+b$

$$[\lambda] \big\downarrow^{S_{a+b}}_{S_a \times S_b} = \bigoplus_{\substack{\mu \vdash a \\ \nu \vdash b}} \left([\mu] \otimes [\nu]\right)^{\oplus c^{\lambda}_{\mu\nu}}$$

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Analogous argument

Kronecker coefficients	Littlewood-Richardson coefficients
$\lambda, \mu \vdash n$	$\lambda \vdash a+b$
$[\lambda] \otimes [\mu] = \bigoplus_{\nu \vdash n} [\nu]^{\oplus k(\lambda,\mu,\nu)}$	$[\lambda] \downarrow^{S_{a+b}}_{S_a \times S_b} = \bigoplus_{\substack{\mu \vdash a \\ \nu \vdash b}} \left([\mu] \otimes [\nu]\right)^{\oplus c^{\lambda}_{\mu\nu}}$
• $k(\lambda,\mu,\nu) \neq 0$ & $k(\lambda',\mu',\nu') \neq 0$ $\Rightarrow k(\lambda+\lambda', \mu+\mu', \nu+\nu') \neq 0$	• $c^{\lambda}_{\mu\nu} \neq 0$ & $c^{\lambda'}_{\mu'\nu'} \neq 0 \Rightarrow c^{\lambda+\lambda'}_{\mu+\mu'\ \nu+\nu'} \neq 0$
• $k(\lambda,\mu,\nu) \neq 0 \Rightarrow H(\lambda) \leq H(\mu) + H(\nu)$	• $H(\lambda) \leq \frac{\lvert\mu\rvert}{\lvert\mu\rvert+\lvert\nu\rvert} H(\mu) + \frac{\lvert\nu\rvert}{\lvert\mu\rvert+\lvert\nu\rvert} H(\nu) + h\left(\frac{\lvert\mu\rvert}{\lvert\mu\rvert+\lvert\nu\rvert}\right).$

Examples: $3\times 3\times 3$: We computed all moment polytopes

arxiv.org/abs/2510.08336

Vertices	Generic	D	D + W	$D + e_{111}$	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}	T_{24}	T_{25}
$(1, 0, 0 \mid 1, 0, 0 \mid 1, 0, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	
$(1, 0, 0 \mid 1/2, 1/2, 0 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●		
$(1/2, 1/2, 0 \mid 1, 0, 0 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				
$(1/2, 1/2, 0 \mid 1/2, 1/2, 0 \mid 1, 0, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				
$(1/2, 1/2, 0 \mid 1/2, 1/2, 0 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●					
$(1, 0, 0 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●	●	●	●						●			
$(1/3, 1/3, 1/3 \mid 1, 0, 0 \mid 1/3, 1/3, 1/3)$	●		●	●	●	●	●	●	●	●	●	●		●	●			●											
$(1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3 \mid 1, 0, 0)$	●		●	●	●	●	●	●	●	●	●	●		●	●														
$(3/4, 1/4, 0 \mid 1/2, 1/2, 0 \mid 1/2, 1/4, 1/4)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●						
$(3/4, 1/4, 0 \mid 1/2, 1/4, 1/4 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●								
$(1/2, 1/4, 1/4 \mid 3/4, 1/4, 0 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●		●	●			●											
$(1/2, 1/4, 1/4 \mid 1/2, 1/2, 0 \mid 3/4, 1/4, 0)$	●	●	●	●	●	●	●	●	●	●	●	●		●	●			●											
$(1/2, 1/2, 0 \mid 3/4, 1/4, 0 \mid 1/2, 1/4, 1/4)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●						
$(1/2, 1/2, 0 \mid 1/2, 1/4, 1/4 \mid 3/4, 1/4, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●								
$(2/3, 1/3, 0 \mid 2/3, 1/3, 0 \mid 1/3, 1/3, 1/3)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●						
$(2/3, 1/3, 0 \mid 1/3, 1/3, 1/3 \mid 2/3, 1/3, 0)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●								
$(1/3, 1/3, 1/3 \mid 2/3, 1/3, 0 \mid 2/3, 1/3, 0)$	●	●	●	●	●	●	●	●	●	●	●	●		●	●			●											
$(1/2, 1/2, 0 \mid 1/2, 1/2, 0 \mid 1/3, 1/3, 1/3)$	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●					●							
$(1/2, 1/2, 0 \mid 1/3, 1/3, 1/3 \mid 1/2, 1/2, 0)$	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●												
$(1/3, 1/3, 1/3 \mid 1/2, 1/2, 0 \mid 1/2, 1/2, 0)$	●		●	●	●	●	●	●	●	●	●	●		●															
$(2/3, 1/6, 1/6 \mid 2/3, 1/6, 1/6 \mid 1/2, 1/2, 0)$	●		●	●	●	●	●	●	●	●	●	●		●	●														
$(2/3, 1/6, 1/6 \mid 1/2, 1/2, 0 \mid 2/3, 1/6, 1/6)$	●		●	●	●	●	●	●	●	●	●	●		●	●			●											
$(1/2, 1/2, 0 \mid 2/3, 1/6, 1/6 \mid 2/3, 1/6, 1/6)$	●		●	●	●	●	●	●	●	●	●	●	●	●		●	●	●	●	●									
$(2/3, 1/6, 1/6 \mid 1/2, 1/2, 0 \mid 1/3, 1/3, 1/3)$	●	●	●	●	●	●	●	●	●	●	●	●		●															
$(2/3, 1/6, 1/6 \mid 1/3, 1/3, 1/3 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●		●				●											
$(1/2, 1/2, 0 \mid 2/3, 1/6, 1/6 \mid 1/3, 1/3, 1/3)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				●								
$(1/2, 1/2, 0 \mid 1/3, 1/3, 1/3 \mid 2/3, 1/6, 1/6)$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●				●								
$(1/3, 1/3, 1/3 \mid 2/3, 1/6, 1/6 \mid 1/2, 1/2, 0)$	●	●	●	●	●	●	●	●	●	●	●	●		●				●											
$(1/3, 1/3, 1/3 \mid 1/2, 1/2, 0 \mid 2/3, 1/6, 1/6)$	●	●	●	●	●	●	●	●	●	●	●	●		●															
$(1/2, 1/4, 1/4 \mid 1/2, 1/2, 0 \mid 1/2, 1/4, 1/4)$	○	○	○	○	○	○	○	○	○	○	○	○		○	●														
$(1/2, 1/4, 1/4 \mid 1/2, 1/4, 1/4 \mid 1/2, 1/2, 0)$	○	○	○	○	○	○	○	○	○	○	○	○		○	●			○											
$(1/2, 1/2, 0 \mid 1/2, 1/4, 1/4 \mid 1/2, 1/4, 1/4)$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○		●		○								
$(2/3, 1/3, 0 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	○		●		○	○	○	●	●	●		●	○			●			●										
$(1/3, 1/3, 1/3 \mid 2/3, 1/3, 0 \mid 1/3, 1/3, 1/3)$	○		●		○	○	○	●	●	●	●	●			●														
$(1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3 \mid 2/3, 1/3, 0)$	○		●		○	○	●	○	○	●	●	●			●														
$(1/2, 1/2, 0 \mid 1/2, 1/3, 1/6 \mid 1/3, 1/3, 1/3)$	○		●		○	○	○	●	●	●		●	○			●													
$(1/2, 1/2, 0 \mid 1/3, 1/3, 1/3 \mid 1/2, 1/3, 1/6)$	○		●		○	○	○	●	●	●		●	○			●													
$(1/2, 1/3, 1/6 \mid 1/2, 1/2, 0 \mid 1/3, 1/3, 1/3)$	○		●		○	○	○	●	●	●	●	●																	
$(1/2, 1/3, 1/6 \mid 1/3, 1/3, 1/3 \mid 1/2, 1/2, 0)$	○		●		○	○	●	○	○	●	●	●																	
$(1/3, 1/3, 1/3 \mid 1/2, 1/2, 0 \mid 1/2, 1/3, 1/6)$	○		●		○	○	○	●	●	●	●	●																	
$(1/3, 1/3, 1/3 \mid 1/2, 1/3, 1/6 \mid 1/2, 1/2, 0)$	○		●		○	○	●	○	○	●	●	●																	
$(1/2, 1/2, 0 \mid 2/5, 2/5, 1/5 \mid 2/5, 3/10, 3/10)$	○		●		○	○	○	●	●	●		●	○			●													
$(1/2, 1/2, 0 \mid 2/5, 3/10, 3/10 \mid 2/5, 2/5, 1/5)$	○		●		○	○	○	●	●	●		●	○			●													
$(2/5, 2/5, 1/5 \mid 1/2, 1/2, 0 \mid 2/5, 3/10, 3/10)$	○		●		○	○	○	●	●	●	●	●																	
$(2/5, 2/5, 1/5 \mid 2/5, 3/10, 3/10 \mid 1/2, 1/2, 0)$	○		●		○	○	●	○	○	●	●	●																	
$(2/5, 3/10, 3/10 \mid 1/2, 1/2, 0 \mid 2/5, 2/5, 1/5)$	○		●		○	○	○	●	●	●	●	●																	
$(2/5, 3/10, 3/10 \mid 2/5, 2/5, 1/5 \mid 1/2, 1/2, 0)$	○		●		○	○	●	○	○	●	●	●																	
$(3/8, 3/8, 1/4 \mid 1/2, 1/2, 0 \mid 3/8, 3/8, 1/4)$	○		●		○	○	○	●	●	●	●	●																	
$(3/8, 3/8, 1/4 \mid 3/8, 3/8, 1/4 \mid 1/2, 1/2, 0)$	○		●		○	○	●	○	○	●	●	●																	
$(1/2, 1/2, 0 \mid 3/8, 3/8, 1/4 \mid 3/8, 3/8, 1/4)$	○		●		○	○	○	●	●	●		●	○			●													
$(1/2, 1/2, 0 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	●				●	●	●						●																
$(1/3, 1/3, 1/3 \mid 1/2, 1/2, 0 \mid 1/3, 1/3, 1/3)$	●				●	●	●																						
$(1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3 \mid 1/2, 1/2, 0)$	●				●	●		●	●																				
$(3/5, 1/5, 1/5 \mid 3/5, 1/5, 1/5 \mid 2/5, 2/5, 1/5)$	○		○	○	○	○	○	○	○	○	○	○		○	●			○											
$(3/5, 1/5, 1/5 \mid 2/5, 2/5, 1/5 \mid 3/5, 1/5, 1/5)$	○		○	○	○	○	○	○	○	○	○	○		○	●														
$(1/2, 1/3, 1/6 \mid 2/3, 1/6, 1/6 \mid 1/3, 1/3, 1/3)$	○		○	○	○	○	○	○	○	○	●	○			●														
$(1/2, 1/3, 1/6 \mid 1/3, 1/3, 1/3 \mid 2/3, 1/6, 1/6)$	○		○	○	○	○	○	○	○	○	●	○			●														
$(2/5, 2/5, 1/5 \mid 3/5, 1/5, 1/5 \mid 2/5, 2/5, 1/5)$	○	○	○	○	○	○	○	○	○	○	●	○			●														
$(2/5, 2/5, 1/5 \mid 2/5, 2/5, 1/5 \mid 3/5, 1/5, 1/5)$	○	○	○	○	○	○	○	○	○	○	●	○			●														
$(5/9, 2/9, 2/9 \mid 5/9, 2/9, 2/9 \mid 1/3, 1/3, 1/3)$	○		○	○	○	○		○	●																				
$(4/9, 4/9, 1/9 \mid 1/3, 1/3, 1/3 \mid 4/9, 4/9, 1/9)$	○		○	○	○	○	●	○	●	●																			
$(1/3, 1/3, 1/3 \mid 4/9, 4/9, 1/9 \mid 4/9, 4/9, 1/9)$	○		○	○	○	○	●	○	●	●	●																		
$(5/9, 2/9, 2/9 \mid 4/9, 4/9, 1/9 \mid 1/3, 1/3, 1/3)$	○	○	○	○	○	○	○	●		●																			
$(4/9, 4/9, 1/9 \mid 5/9, 2/9, 2/9 \mid 1/3, 1/3, 1/3)$	○	○	○	○	○	○	○	●		●																			
$(3/5, 1/5, 1/5 \mid 2/5, 2/5, 1/5 \mid 1/3, 1/3, 1/3)$	○		○	○	○	○		●																					
$(2/5, 2/5, 1/5 \mid 3/5, 1/5, 1/5 \mid 1/3, 1/3, 1/3)$	○		○	○	○	○		●																					
$(7/15, 4/15, 4/15 \mid 1/3, 1/3, 1/3 \mid 3/5, 1/5, 1/5)$	○		○	○	○	○	●	○		●																			
$(1/3, 1/3, 1/3 \mid 7/15, 4/15, 4/15 \mid 3/5, 1/5, 1/5)$	○		○	○	○	○	●	○		●																			
$(2/3, 1/3, 1/3 \mid 7/15, 7/15, 1/15 \mid 2/5, 2/5, 1/5)$	○		●		○	○	○																						
$(2/3, 1/3, 1/3 \mid 2/5, 2/5, 1/5 \mid 7/15, 7/15, 1/15)$	○		●		○	○	○	●		●																			
$(7/15, 7/15, 1/15 \mid 2/3, 1/3, 1/3 \mid 2/5, 2/5, 1/5)$	○		●		○	○	○																						
$(7/15, 7/15, 1/15 \mid 2/5, 2/5, 1/5 \mid 2/3, 1/3, 1/3)$	○		●		○	○		○	○																				
$(2/5, 2/5, 1/5 \mid 2/3, 1/3, 1/3 \mid 7/15, 7/15, 1/15)$	○		●		○	○	○	●		●																			
$(2/5, 2/5, 1/5 \mid 7/15, 7/15, 1/15 \mid 2/3, 1/3, 1/3)$	○		●		○	○		○	○																				
$(1/2, 1/4, 1/4 \mid 1/2, 1/4, 1/4 \mid 1/2, 1/4, 1/4)$	○	○	○	○	○	○	○	○	○	○	●	●			●														
$(3/7, 3/7, 1/7 \mid 3/7, 3/7, 1/7 \mid 3/7, 2/7, 2/7)$	○		○	○	○	○	○	○	●	○		●																	
$(3/7, 3/7, 1/7 \mid 3/7, 2/7, 2/7 \mid 3/7, 3/7, 1/7)$	○		○	○	○	○	○	○	○	○		●																	
$(3/7, 2/7, 2/7 \mid 3/7, 3/7, 1/7 \mid 3/7, 3/7, 1/7)$	○		○	○	○	○	○	○	○	○	○	●																	
$(5/12, 5/12, 1/6 \mid 1/3, 1/3, 1/3 \mid 1/2, 1/4, 1/4)$	○	○	○	○	○	○	●	●		●																			
$(1/3, 1/3, 1/3 \mid 5/12, 5/12, 1/6 \mid 1/2, 1/4, 1/4)$	○	○	○	○	○	○	●	●		●																			
$(1/2, 1/4, 1/4 \mid 1/2, 1/4, 1/4 \mid 1/3, 1/3, 1/3)$	○		○	○	○	○		●																					
$(2/5, 2/5, 1/5 \mid 2/5, 2/5, 1/5 \mid 2/5, 2/5, 1/5)$	○	○	○	○	○	○	○	○	●	○		●																	
$(5/9, 2/9, 2/9 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	○		○	○	○	●																							
$(1/3, 1/3, 1/3 \mid 5/9, 2/9, 2/9 \mid 1/3, 1/3, 1/3)$	○		○	○	●	●																							
$(1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3 \mid 5/9, 2/9, 2/9)$	○		○	○	●			●																					
$(4/9, 4/9, 1/9 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	○				○	●																							
$(1/3, 1/3, 1/3 \mid 4/9, 4/9, 1/9 \mid 1/3, 1/3, 1/3)$	○				●	●																							
$(1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3 \mid 4/9, 4/9, 1/9)$	○				●			●																					
$(1/2, 1/4, 1/4 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	○		○	○	●																								
$(2/5, 2/5, 1/5 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	○				●																								
$(1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3 \mid 1/3, 1/3, 1/3)$	●	●	●	●																									

$3 \times 3 \times 3$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$
$$= \mathbb{C}[x]/(x^3)$$

$$T_{11} = 113 + 131 + 212 + 321$$
$$= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$T_{14} = 113 + 121 + 132 + 211 + 312$$
$$= \mathbb{C}[x,y]/(x^2, y^2, xy)$$

nurmiev

$3 \times 3 \times 3$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$
$$= \mathbb{C}[x]/(x^3)$$

$$T_{11} = 113 + 131 + 212 + 321$$
$$= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$T_{14} = 113 + 121 + 132 + 211 + 312$$
$$= \mathbb{C}[x,y]/(x^2, y^2, xy)$$

Degeneration Hasse diagram:

$$\begin{array}{ccc} & \not\trianglerighteq & \\ T_{10} & \not\trianglelefteq & T_{11} \\ \trianglerighteq & & \trianglerighteq \\ & T_{14} & \end{array}$$

nurmiev

$3 \times 3 \times 3$

$$T_{10} = 113 + 122 + 131 + 212 + 221 + 311$$
$$= \mathbb{C}[x]/(x^3)$$

$$T_{11} = 113 + 131 + 212 + 321$$
$$= \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$T_{14} = 113 + 121 + 132 + 211 + 312$$
$$= \mathbb{C}[x,y]/(x^2, y^2, xy)$$

Degeneration Hasse diagram:

$$\begin{array}{ccc} & \ntrianglerighteq & \\ T_{10} & \ntrianglelefteq & T_{11} \\ \trianglerighteq & & \trianglerighteq \\ & T_{14} & \end{array}$$

Asymptotic restriction Hasse diagram:

$$\begin{array}{ccc} & \not\gtrsim & \\ T_{10} & \not\lesssim & T_{11} \\ \gtrsim & & \gtrsim \\ & T_{14} & \end{array}$$

quantum functionals

4 x 4 x 4

Vertices			Polytopes							
			Generic	U_4	$(2v_1+v_2+v_3)\oplus e_{111}$	$(D+e_{111})\oplus e_{111}$	M_2	$W\otimes U_2$	$W\otimes W$	$W\oplus W$
(1, 0, 0, 0	1, 0, 0, 0	1, 0, 0, 0)	●	●	●	●	●	●	●	●
(1/2, 1/2, 0, 0	1/2, 1/2, 0, 0	1, 0, 0, 0)	●	●	●	●	●	●	●	●
(1/2, 1/2, 0, 0	1/2, 1/2, 0, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/2, 1/4, 1/4, 0	1/2, 1/2, 0, 0	3/4, 1/4, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/3, 0	2/3, 1/3, 0, 0	2/3, 1/3, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0	1, 0, 0, 0)	●	●	●	●	●	●	●	●
(2/3, 1/6, 1/6, 0	2/3, 1/6, 1/6, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/3, 0	2/3, 1/6, 1/6, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/4, 1/4, 1/4, 1/4	1/2, 1/2, 0, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/6, 1/6	1/3, 1/3, 1/3, 0	5/6, 1/6, 0, 0)	●	●	●	●	●	●	●	●
(1/2, 1/6, 1/6, 1/6	2/3, 1/6, 1/6, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(2/5, 1/5, 1/5, 1/5	2/5, 2/5, 1/5, 0	4/5, 1/5, 0, 0)	●	●	●	●	●	●	●	●
(5/8, 1/8, 1/8, 1/8	1/2, 1/4, 1/4, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/4, 1/4, 1/4, 1/4	1/2, 1/4, 1/4, 0	3/4, 1/4, 0, 0)	●	●	●	●	●	●	●	●
(7/12, 1/4, 1/12, 1/12	1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/2, 1/6, 1/6, 1/6	1/3, 1/3, 1/3, 0	2/3, 1/3, 0, 0)	●	●	●	●	●	●	●	●
(1/2, 1/6, 1/6, 1/6	1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 2/9, 2/9, 2/9	1/3, 1/3, 1/3, 0	2/3, 1/3, 0, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	3/8, 3/8, 1/4, 0	5/8, 3/8, 0, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	1/2, 1/4, 1/4, 0	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(1/3, 2/9, 2/9, 2/9	7/18, 7/18, 2/9, 0	1/2, 1/2, 0, 0)	○	○	○	○		○	●	○
(1/3, 5/18, 5/18, 1/9	1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0)	○	○	○	○		○	●	○
(1/3, 1/3, 1/6, 1/6	1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0)	○	○	○	○	●	○	●	○
(2/7, 2/7, 2/7, 1/7	5/14, 5/14, 2/7, 0	1/2, 1/2, 0, 0)	○	○	○	○		○	●	○
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/3, 0	1/2, 1/2, 0, 0)	●	●	●	●		●		●
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	1, 0, 0, 0)	●	●	●	●	●	●	●	●
(2/5, 2/5, 1/10, 1/10	2/5, 3/10, 3/10, 0	4/5, 1/10, 1/10, 0)	●	●	●	●	●	●	●	●
(4/7, 1/7, 1/7, 1/7	3/7, 3/7, 1/7, 0	5/7, 1/7, 1/7, 0)	●	●	●	●	●	●	●	●
(4/9, 1/3, 1/9, 1/9	1/3, 1/3, 1/3, 0	7/9, 1/9, 1/9, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/6, 1/6	2/3, 1/6, 1/6, 0	2/3, 1/6, 1/6, 0)	●	●	●	●	●	●	●	●
(3/10, 3/10, 1/5, 1/5	2/5, 3/10, 3/10, 0	4/5, 1/10, 1/10, 0)	●	●	●	●	●	●	●	●
(1/3, 1/3, 1/6, 1/6	1/3, 1/3, 1/3, 0	7/9, 1/9, 1/9, 0)	●	●	●	●		●	●	●
(1/3, 2/9, 2/9, 2/9	1/3, 1/3, 1/3, 0	7/9, 1/9, 1/9, 0)	●	●	●	●	●	●	●	●
(1/4, 1/4, 1/4, 1/4	3/8, 3/8, 1/4, 0	3/4, 1/8, 1/8, 0)	●	●	●	●	●	●	●	●
(7/24, 7/24, 5/24, 5/24	1/3, 1/3, 1/3, 0	3/4, 1/8, 1/8, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	2/5, 3/10, 3/10, 0	7/10, 3/20, 3/20, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	1/2, 1/4, 1/4, 0	2/3, 1/6, 1/6, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/3, 0	2/3, 1/4, 1/12, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/3, 0	2/3, 1/6, 1/6, 0)	●	●	●	●	●	●	●	●
(8/27, 8/27, 8/27, 1/9	1/3, 1/3, 1/3, 0	13/27, 13/27, 1/27, 0)	○	○	○	○		○	●	○
(5/12, 5/12, 1/12, 1/12	1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0)	○	○	○	○	○	○	●	○
(3/7, 4/21, 4/21, 4/21	1/3, 1/3, 1/3, 0	3/7, 3/7, 1/7, 0)	○	○	○	○	○	○	●	○
(1/2, 1/6, 1/6, 1/6	1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0)	○	○	○	○	○	●	●	●
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/3, 0	5/12, 5/12, 1/6, 0)	○	○	○	○	●	●		●
(1/3, 1/3, 1/6, 1/6	1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0)	○	○	○	○	○	●		●
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/3, 0	1/3, 1/3, 1/3, 0)	●	●	●	●	●			
(5/8, 1/8, 1/8, 1/8	5/8, 1/8, 1/8, 1/8	1/2, 1/2, 0, 0)	●	●	●	●	●	●	●	●
(3/8, 1/4, 1/4, 1/8	5/8, 1/8, 1/8, 1/8	1/2, 1/2, 0, 0)	●	●	●	●		●	●	●
(3/10, 3/10, 3/10, 1/10	3/5, 1/5, 1/10, 1/10	1/2, 1/2, 0, 0)	●	●	●	●		●	●	●
(2/5, 1/5, 1/5, 1/5	3/5, 1/5, 1/10, 1/10	1/2, 1/2, 0, 0)	●	●	●	●		●	●	●
(1/2, 1/6, 1/6, 1/6	1/2, 1/6, 1/6, 1/6	1/2, 1/2, 0, 0)	○	○	○	○	○	○	●	○
(2/7, 2/7, 2/7, 1/7	4/7, 1/7, 1/7, 1/7	4/7, 3/7, 0, 0)	●	●	●	●		●	●	●

Vertices			Generic	U_4	$(2v_1+v_2+v_3)\oplus e_{111}$	$(D+e_{111})\oplus e_{111}$	M_2	$W\otimes U_2$	$W\otimes W$	$W\oplus W$
(3/10, 3/10, 3/10, 1/10	11/20, 3/20, 3/20, 3/20	1/2, 1/2, 0, 0)	●	●	●	●		●		●
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	3/4, 1/4, 0, 0)	○	○	○	●	○	○		○
(1/4, 1/4, 1/4, 1/4	1/2, 1/4, 1/8, 1/8	5/8, 3/8, 0, 0)	●	●	●	●		●	●	●
(1/3, 1/3, 1/6, 1/6	1/2, 1/6, 1/6, 1/6	1/2, 1/2, 0, 0)	○	○	○	○	○	○	●	○
(5/16, 5/16, 5/16, 1/16	5/16, 5/16, 5/16, 1/16	1/2, 1/2, 0, 0)	○	○	○	○	○	○	●	○
(1/4, 1/4, 1/4, 1/4	1/2, 1/6, 1/6, 1/6	1/2, 1/2, 0, 0)	●	●	●	●		●		●
(2/5, 1/5, 1/5, 1/5	2/5, 1/5, 1/5, 1/5	3/5, 2/5, 0, 0)	○	○	○	○	○	○	●	○
(1/4, 1/4, 1/4, 1/4	3/8, 3/8, 1/8, 1/8	1/2, 1/2, 0, 0)	○	○	○	●	○	○		○
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/4, 1/12	1/2, 1/2, 0, 0)	○	○	○	●		○		○
(1/4, 1/4, 1/4, 1/4	5/12, 1/4, 1/6, 1/6	1/2, 1/2, 0, 0)	○	○	○	●		○		○
(3/10, 7/30, 7/30, 7/30	3/10, 3/10, 3/10, 1/10	1/2, 1/2, 0, 0)	○	○	○	●		○		○
(1/4, 1/4, 1/4, 1/4	3/8, 1/4, 1/4, 1/8	1/2, 1/2, 0, 0)	○	○	○	●	○	○		○
(5/14, 3/14, 3/14, 3/14	5/14, 3/14, 3/14, 3/14	1/2, 1/2, 0, 0)	○	○	○	●	○	○		○
(2/7, 2/7, 3/14, 3/14	2/7, 2/7, 2/7, 1/7	1/2, 1/2, 0, 0)	○	○	○	●	○	○		○
(3/10, 3/10, 1/5, 1/5	3/10, 3/10, 1/5, 1/5	1/2, 1/2, 0, 0)	○	○	○	●	○	○		○
(5/18, 5/18, 5/18, 1/6	5/18, 5/18, 5/18, 1/6	1/2, 1/2, 0, 0)	○	○	○	●	○	○		○
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	1/2, 1/2, 0, 0)	●	●	●		●	●		●
(5/12, 5/12, 1/12, 1/12	3/4, 1/12, 1/12, 1/12	1/2, 1/4, 1/4, 0)	●	●	●	●		●	●	●
(2/5, 2/5, 1/10, 1/10	7/10, 1/10, 1/10, 1/10	3/5, 1/5, 1/5, 0)	●	●	●	●		●	●	●
(5/12, 5/12, 1/12, 1/12	3/4, 1/12, 1/12, 1/12	1/3, 1/3, 1/3, 0)	●	●	●	●	●	●	●	●
(5/14, 5/14, 1/7, 1/7	11/14, 1/14, 1/14, 1/14	3/7, 2/7, 2/7, 0)	●	●	●	●		●	●	●
(1/3, 1/3, 1/6, 1/6	3/4, 1/12, 1/12, 1/12	1/2, 1/4, 1/4, 0)	●	●	●	●		●	●	●
(5/8, 1/8, 1/8, 1/8	5/8, 1/8, 1/8, 1/8	3/8, 3/8, 1/4, 0)	●	●	●	●		●	●	●
(5/12, 1/4, 1/6, 1/6	3/4, 1/12, 1/12, 1/12	1/3, 1/3, 1/3, 0)	●	●	●	●		●	●	●
(5/11, 2/11, 2/11, 2/11	8/11, 1/11, 1/11, 1/11	4/11, 4/11, 3/11, 0)	●	●	●	●		●	●	●
(5/9, 2/9, 1/9, 1/9	2/3, 1/9, 1/9, 1/9	1/3, 1/3, 1/3, 0)	●	●	●	●		●	●	●
(2/5, 1/5, 1/5, 1/5	11/15, 2/15, 1/15, 1/15	1/3, 1/3, 1/3, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	3/4, 1/12, 1/12, 1/12	1/3, 1/3, 1/3, 0)	●	●	●	●	●	●	●	●
(1/3, 2/9, 2/9, 2/9	2/3, 1/9, 1/9, 1/9	4/9, 4/9, 1/9, 0)	●	●	●	●		●	●	●
(1/2, 1/6, 1/6, 1/6	2/3, 1/9, 1/9, 1/9	1/3, 1/3, 1/3, 0)	●	●	●	●		●		●
(1/4, 1/4, 1/4, 1/4	2/3, 1/6, 1/12, 1/12	5/12, 5/12, 1/6, 0)	●	●	●	●		●	●	●
(1/4, 1/4, 1/4, 1/4	5/8, 1/8, 1/8, 1/8	1/2, 3/8, 1/8, 0)	●	●	●	●		●	●	●
(11/36, 11/36, 11/36, 1/12	17/36, 17/36, 1/36, 1/36	1/3, 1/3, 1/3, 0)	○	○	○	○	○	○	●	○
(2/7, 2/7, 2/7, 1/7	4/7, 1/7, 1/7, 1/7	3/7, 3/7, 1/7, 0)	○	○	○	○	○	○	●	○
(1/4, 1/4, 1/4, 1/4	5/12, 5/12, 1/12, 1/12	1/3, 1/3, 1/3, 0)	○	○	○	○	●	●		●
(1/4, 1/4, 1/4, 1/4	1/2, 1/6, 1/6, 1/6	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	3/8, 3/8, 1/8, 1/8	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	5/12, 5/12, 1/6, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	1/2, 1/4, 1/4, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/4, 1/12	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	5/12, 1/4, 1/6, 1/6	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(4/15, 4/15, 4/15, 1/5	2/5, 1/5, 1/5, 1/5	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	3/8, 1/4, 1/4, 1/8	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(2/7, 2/7, 2/7, 1/7	2/7, 2/7, 2/7, 1/7	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(2/7, 2/7, 3/14, 3/14	5/14, 3/14, 3/14, 3/14	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(3/10, 3/10, 1/5, 1/5	3/10, 3/10, 1/5, 1/5	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(1/3, 2/9, 2/9, 2/9	1/3, 2/9, 2/9, 2/9	1/3, 1/3, 1/3, 0)	○	○	○	●	○			
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	1/3, 1/3, 1/3, 0)	●	●	●		●			
(1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4	1/4, 1/4, 1/4, 1/4)	●	●	●	●	●			

$4 \times 4 \times 4$

$\langle 4 \rangle$ diagonal tensor

$\langle 2,2,2 \rangle$ matrix multiplication tensor

$4 \times 4 \times 4$

$\langle 4 \rangle$ diagonal tensor

$\langle 2,2,2 \rangle$ matrix multiplication tensor

Degeneration Hasse diagram:

$$\langle 4 \rangle \overset{\not\trianglerighteq}{\not\trianglelefteq} \langle 2,2,2 \rangle$$

$4 \times 4 \times 4$

$\langle 4 \rangle$ diagonal tensor

$\langle 2,2,2 \rangle$ matrix multiplication tensor

Degeneration Hasse diagram:

$$\langle 4 \rangle \;\overset{\ntrianglerighteq}{\ntrianglelefteq}\; \langle 2,2,2 \rangle$$

Asymptotic restriction Hasse diagram:

$? \Rightarrow \omega = 2$

$$\langle 4 \rangle \;\overset{\gtrsim}{\underset{\lesssim}{}}\; \langle 2,2,2 \rangle$$

tightness

$4 \times 4 \times 4$

$\langle 4 \rangle$ diagonal tensor

$\langle 2,2,2 \rangle$ matrix multiplication tensor

$$\Delta(\langle 2,2,2 \rangle) \subsetneq \Delta(\langle 4 \rangle)$$

Degeneration Hasse diagram:

$$\langle 4 \rangle \quad \begin{matrix} \not\trianglerighteq \\ \not\trianglelefteq \end{matrix} \quad \langle 2,2,2 \rangle$$

Asymptotic restriction Hasse diagram:

? $\Rightarrow \omega = 2$

$$\langle 4 \rangle \quad \begin{matrix} \gtrapprox \\ \lessapprox \end{matrix} \quad \langle 2,2,2 \rangle$$

tightness

$4 \times 4 \times 4$

$\langle 4 \rangle$ diagonal tensor

$\langle 2,2,2 \rangle$ matrix multiplication tensor

Degeneration Hasse diagram:

$$\langle 4 \rangle \quad \cancel{\trianglerighteq} \;\; \cancel{\trianglelefteq} \quad \langle 2,2,2 \rangle$$

$$\Delta(\langle 2,2,2 \rangle) \subsetneq \Delta(\langle 4 \rangle)$$

Asymptotic restriction Hasse diagram:

$? \Rightarrow \omega = 2$

$$\langle 4 \rangle \quad \overset{\gtrsim}{\underset{\lesssim}{}} \quad \langle 2,2,2 \rangle$$

tightness

Theorem $\forall n, \exists x$

$$\Delta(\langle n,n,n \rangle) \qquad \Delta(\langle n^2 \rangle)$$

$$\cancel{\omega} \qquad \omega$$

$$x \qquad x$$

arxiv.org/abs/2503.22633