Lecture 9 Convexity of asymptotic spectra

1. Convexity questions
2. Finite-dimensional picture
3. Examples and applications
4. Log-convexity for matrix multiplication
5. Log-star-convexity for rectangular matrix multiplication

1. Convexity questions

Asymptotic spectrum $X = \{\phi \in \mathrm{Hom}(\text{tensors}, \mathbb{R}_{\geq 0}) : \leq\text{-monotone}\}$.

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- Support functionals too in X (oblique).

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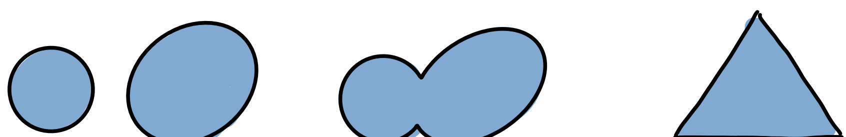
- Quantum functionals interpolate between the flattening ranks
- Support functionals too in X (oblique).

We don't know if we can do this between arbitrary points in X.



Open problem Is X: disconnected connected stronger structure ?

2. Finite-dimensional picture

$X \subseteq \mathbb{R}_{\geq 0}^{\text{tensors}}$

infinite-dimensional

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fix tensor T of interest

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Lemma $X(T_1, \ldots, T_n) \subseteq \mathbb{R}^n_{\geq 0}$ is compact (i.e. closed and bounded).

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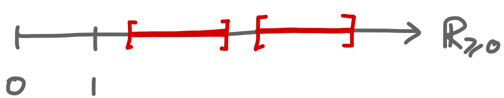

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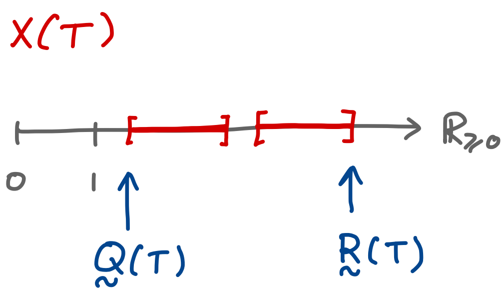

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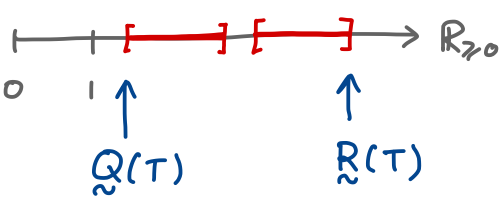
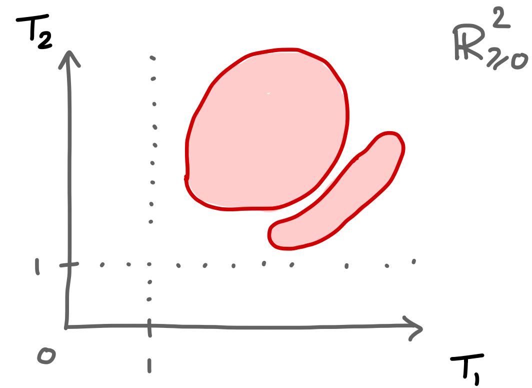


$X(T_1, T_2)$



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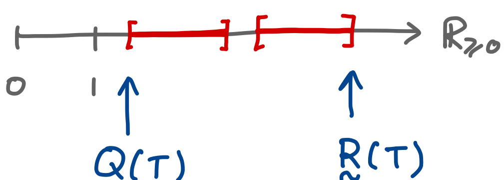

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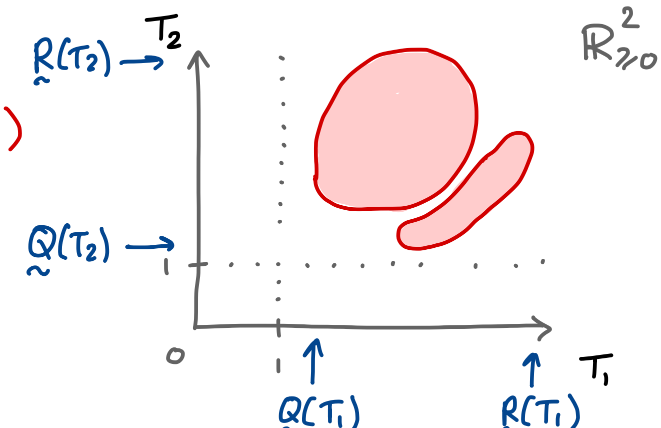

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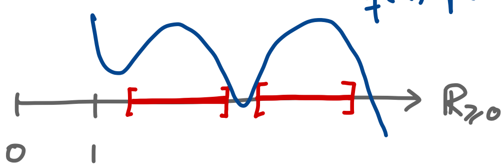

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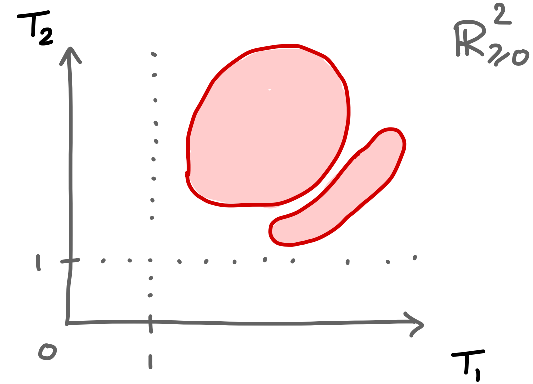

What other information do we get?

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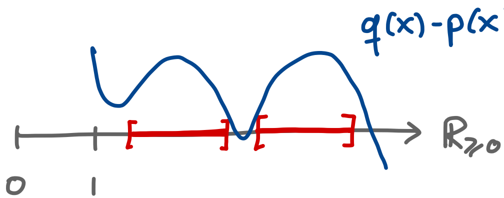

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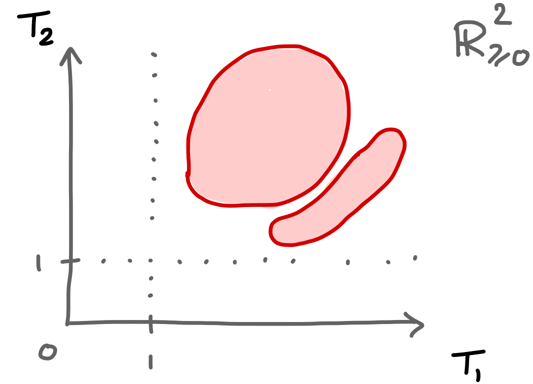
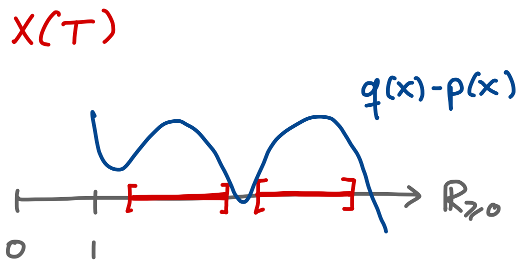
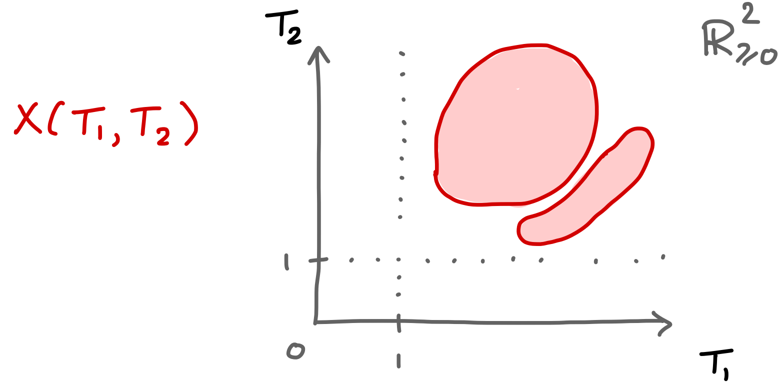


What other information do we get?

Theorem. $\forall p, q \in \mathbb{N}[x]$, $\quad p(T) \lessapprox q(T) \iff p(x) \leq q(x)$ on $X(T)$

$\iff 0 \leq q(x) - p(x)$ on $X(T)$.





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Proof Duality: $p(T) \underset{\sim}{\leq} q(T) \iff \forall \phi \in X,\ \phi(p(T)) \leq \phi(q(T))$

$$\phi(p(T)) = p(\phi(T)), \qquad \phi(q(T)) = q(\phi(T)) \qquad \square$$

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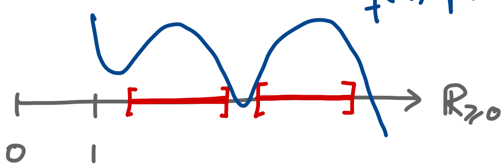

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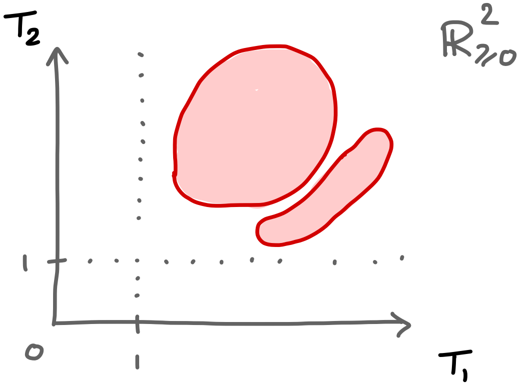

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Theorem $\forall p, q \in \mathbb{N}[x_1, x_2]$, $p(T_1, T_2) \lesssim q(T_1, T_2) \iff 0 \leq q(x_1, x_2) - p(x_1, x_2)$ on $X(T_1, T_2)$.

etc.

3. Examples and applications

Theorem $\quad X(W) = [2^{h(1/3)} \approx 1.88,\ 2]$ $\quad$ proven using support functionals

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Theorem $X(\langle 2,2,2\rangle)$ is connected, so equals $[4, 2^{\omega}]$

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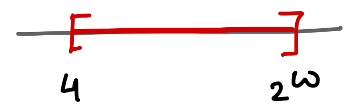

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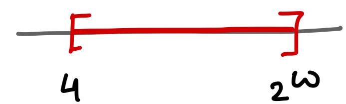

Actually we can do more

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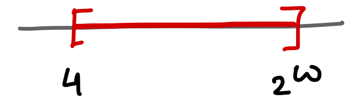

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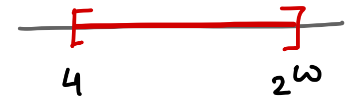

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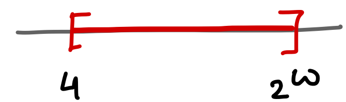

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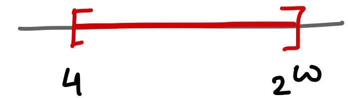

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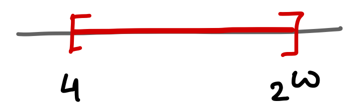

Proof

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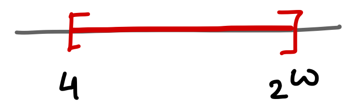

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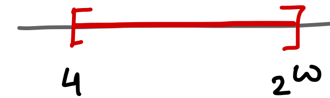
$$\iff \forall x \in X(\langle 2,2,2\rangle),\ \sum_i x^{\log n_i} \le \sum_j x^{\log m_j} \qquad \blacksquare$$

$$\bigoplus_i \langle n_i, n_i, n_i \rangle \overset{(1)}{\lesssim} \bigoplus_j \langle m_j, m_j, m_j \rangle \iff 0 \le \overbrace{\sum_j x^{\log m_j} - \sum_i x^{\log n_i}}^{(2)} \text{ on } X(\langle 2,2,2 \rangle)$$



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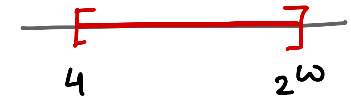
How is this useful? Upper bounds on ω.



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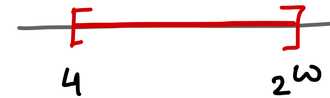

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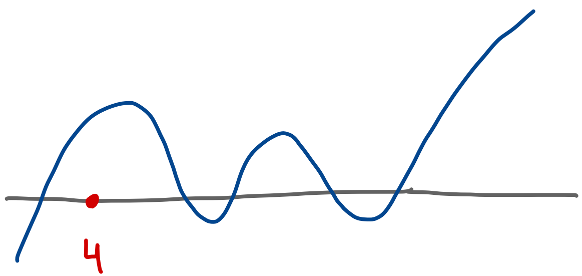
Suppose we prove (1) and the function (2) looks like:



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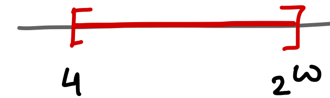

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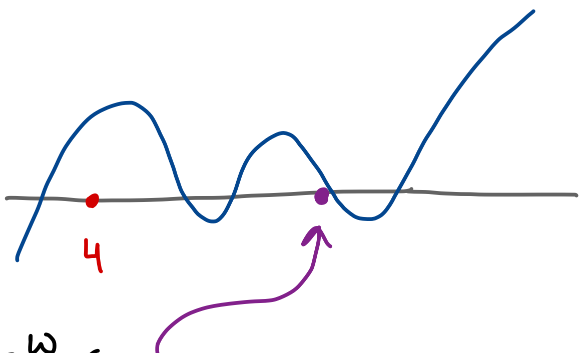

Then $2^\omega \le$

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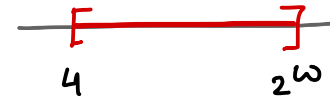

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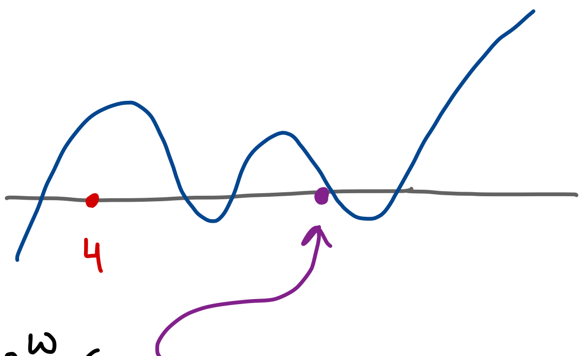

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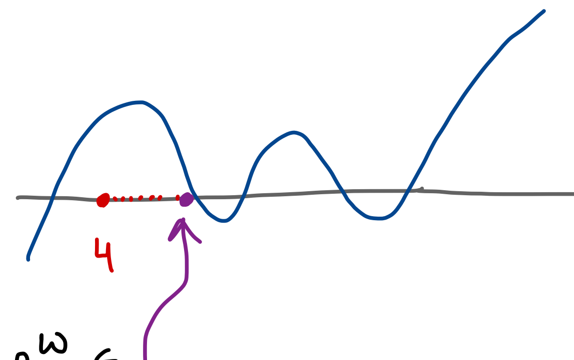
If we <u>do</u>:



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4.1 Proof outline

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Proof conv log X is characterized by linear inequalities on log X

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Proof conv log X is characterized by linear inequalities on log X so $2^{\operatorname{conv} \log X}$ by "monomial inequalities". □.

So to prove log-convexity of say $X = X(\langle 2,2,2 \rangle)$ it suffices to to prove $X = \tilde{X}$, so it suffices to show X is charaterized by monom. ineq.

Monomials

$f = \sum_{i=0}^{d} f_i x^i \in \mathbb{N}[x]$. $\quad \mathrm{Mon}(f) = \{ f_i x^i : i = 0, \ldots, d \}$

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Let $\varepsilon \to 0$. $\square$.

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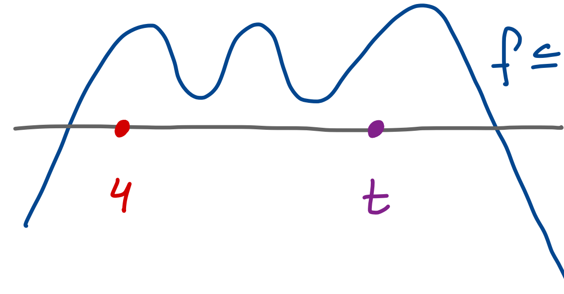

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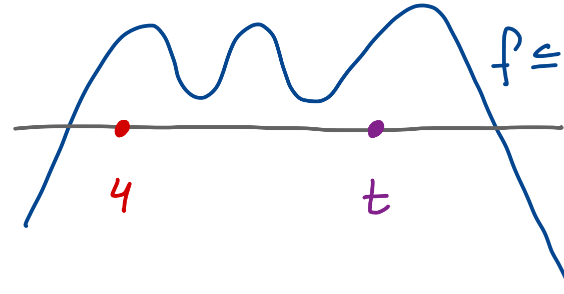

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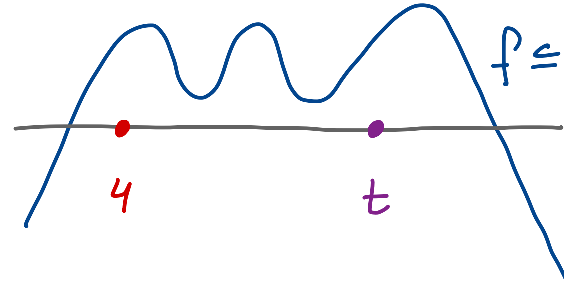

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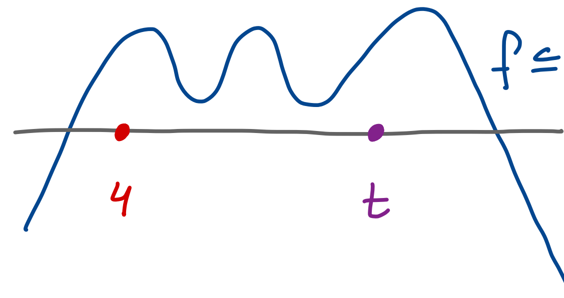

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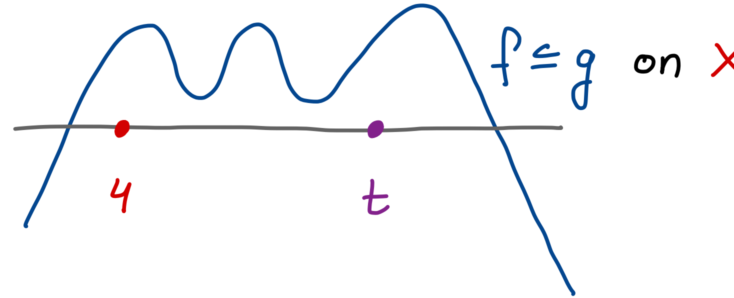

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evaluated at $\langle 2,2,2\rangle$

4.2 Compression

It turns out we need to do this.

theorem. If $\ell \cdot \langle 2,2,2 \rangle^{\otimes k} \leq S \oplus \mathcal{U}$ and $R_1(S) \leq \ell \cdot (2^n)^{k-2}$, then $\lceil \frac{\ell}{2} \rceil \cdot \langle 2,2,2 \rangle^{\otimes k-1} \leq \mathcal{U}$.

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Three ideas:

- stabilizer of $\langle a,b,c \rangle$
- local projection of $\langle a,b,c \rangle$ to $\langle a',b',c' \rangle$
- Matrix subspace lemma

Stabilizer

$$\langle n,n,n \rangle = (A,B,C) \mapsto \mathrm{Tr}(ABC)$$

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Lemma

$\langle n,n,n\rangle \in \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2} \otimes \mathbb{F}^{n^2}$ is invariant under

$$\mathbb{F}^{n^2} \to \mathbb{F}^{n^2}: \quad A \mapsto XAY^{-1}$$

$$\mathbb{F}^{n^2} \to \mathbb{F}^{n^2}: \quad B \mapsto YBZ^{-1}$$

$$\mathbb{F}^{n^2} \to \mathbb{F}^{n^2}: \quad C \mapsto ZCX^{-1}$$

Local projection

$$\langle a, b, c \rangle = \sum_{i \in [a]} \sum_{j \in [b]} \sum_{k \in [c]} e_{ij} \otimes e_{jk} \otimes e_{ki} \in \mathbb{F}^{ab} \otimes \mathbb{F}^{bc} \otimes \mathbb{F}^{ca}$$

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$$P_{b',c'} : \mathbb{F}^{bc} \to \mathbb{F}^{b'c'} : e_{jk} \mapsto e_{jk}\,\delta_{j\le b'}\,\delta_{k\le c'}$$

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Lemma

$$(\mathrm{Id} \otimes P_{b',c'} \otimes P_{c',a'})\langle a,b,c\rangle = \langle a', b', c'\rangle$$

Matrix subspace lemma

$\mathbb{F}$ large enough

Lemma Let $V \subseteq \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$, $d = \dim V$.

Suppose $d \le (b_1+1)(b_2+1)-1$ for some $0 \le b_i \le n_i$

Then there exist subspaces $W_i \subseteq \mathbb{F}^{n_i}$, $\dim(W_i) = n_i - b_i$

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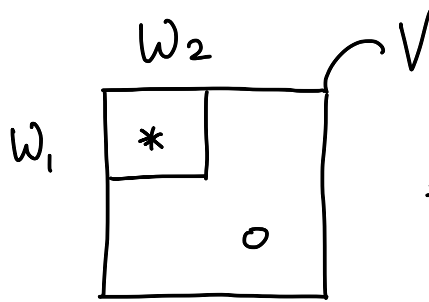

$\Rightarrow \quad * = 0$

Compression

Lemma. If $\langle n_1, n_2, n_3 \rangle \leq S \oplus \mathcal{U}$ and $R_1(S) < \prod_i (n_i - m_i + 1)$ for some $0 \leq m_i \leq n_i$. Then $\langle m_1, m_2, m_3 \rangle \leq \mathcal{U}$

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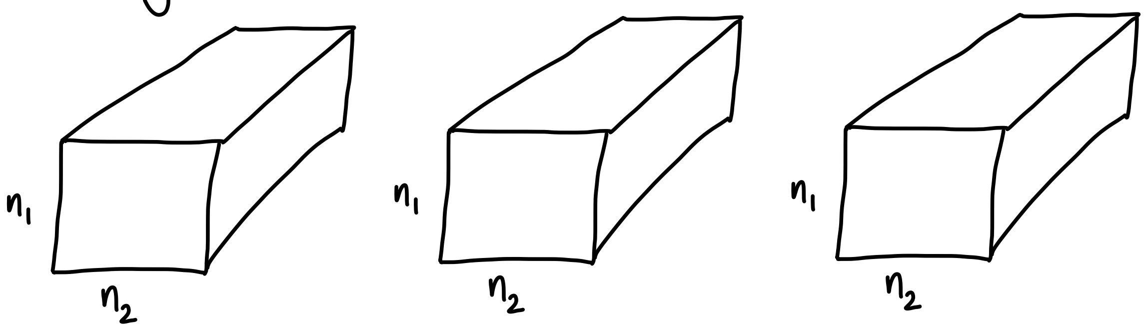

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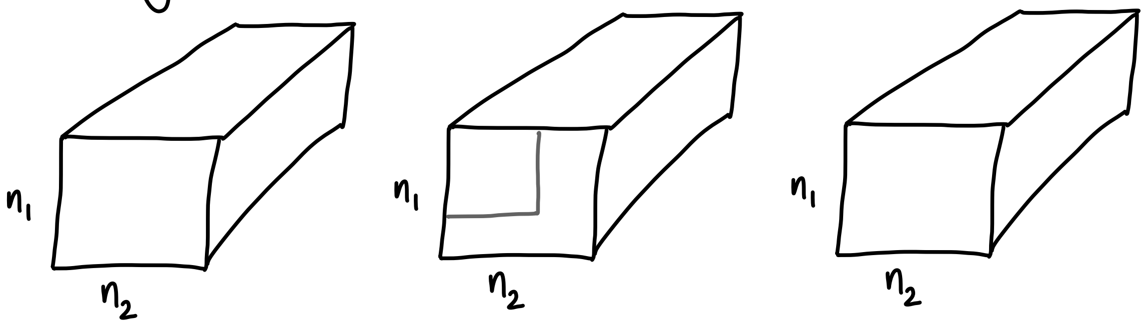

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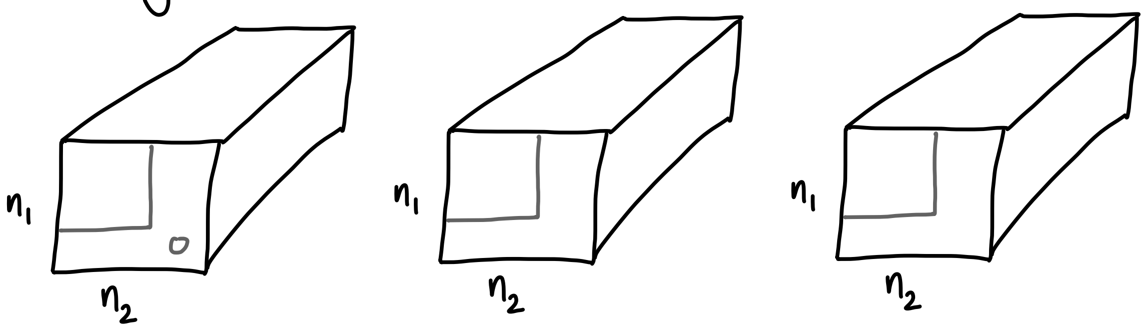

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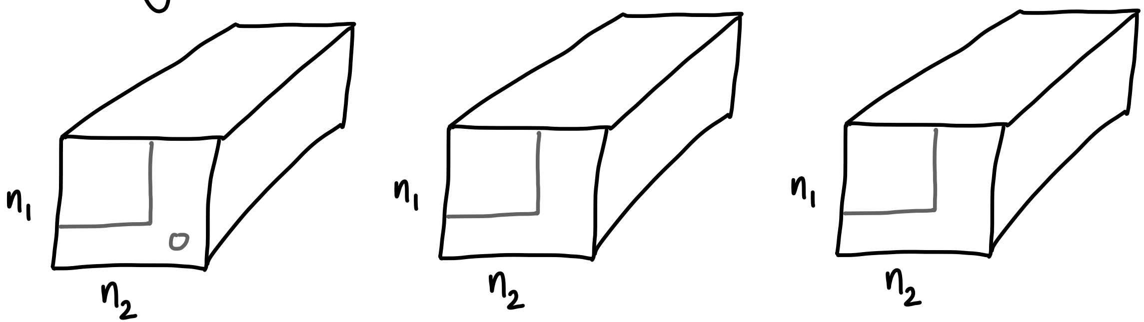

Theorem. If $p \cdot \langle 2,2,2\rangle^{\otimes k} \leq S \oplus \mathcal{U}$ and $R_1(S) \leq p \cdot (2^n)^{k-2}$, then $\lceil \frac{p}{2} \rceil \cdot \langle 2,2,2\rangle^{\otimes k-1} \leq \mathcal{U}$.

5. Log-star-convexity for rectangular matrix multiplication

Theorem $X(\langle 2,2,2\rangle)$ is connected

Strassen proves something stronger

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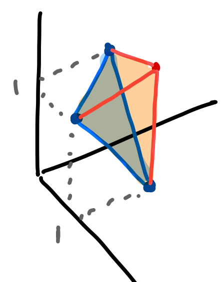

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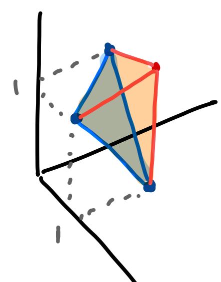

$$\bigoplus_i \langle a_i, b_i, c_i\rangle \lessapprox \bigoplus_i \langle d_i, e_i, f_i\rangle \Rightarrow$$
$$\sum_i (a_i b_i c_i)^{\omega/3} \leq \sum_i (d_i e_i f_i)^{\omega/3}$$

Open problem Is it <u>convex</u>?