The Asymptotic Spectrum Distance, graph limits, and the Shannon capacity

Jeroen Zuiddam

joint work with David de Boer and Pjotr Buys

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- Despite this, even small instances have remained open, e.g. odd cycles of length ≥ 7

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Survey: Wigderson–Zuiddam (2025)

- This talk: building on asymptotic spectrum duality, we develop new graph limit approach to Shannon capacity via asymptotic spectrum distance

- Graph limit approach: To determine Shannon capacity of a "hard" graph, construct a sequence of easier to analyse graphs converging to it in asymptotic spectrum distance:

$$G_1, G_2, \ldots \longrightarrow C_7 \quad \Rightarrow \quad \Theta(G_1), \Theta(G_2), \ldots \longrightarrow \Theta(C_7)$$

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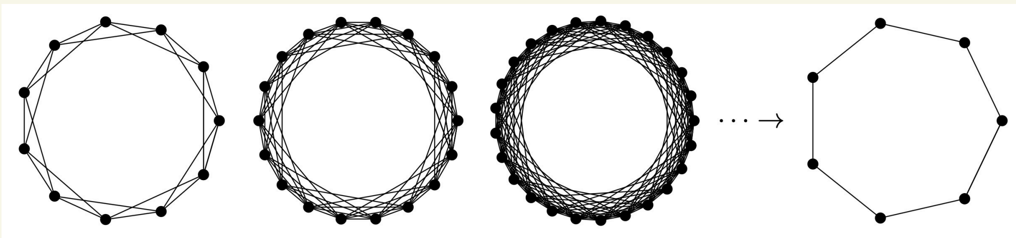
Questions:

- How to construct converging sequences?
- Where to look for graphs that are "easier to analyse"?

We answer both!

Main results

(1) General construction of converging sequences in asymptotic spectrum distance



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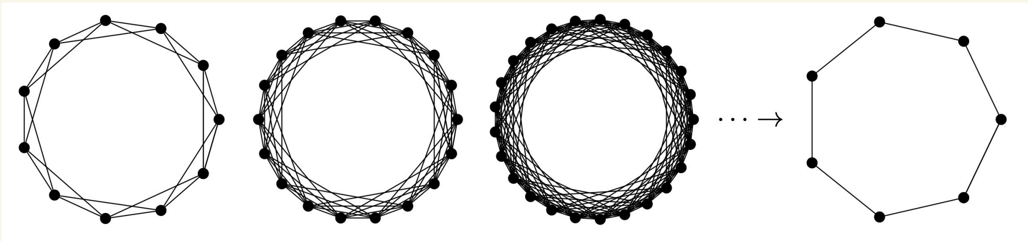

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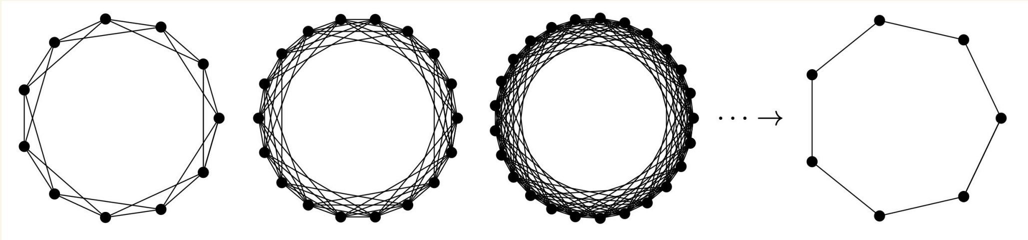

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(2) Cauchy sequences of finite graphs that do not converge to any finite graph

→ Infinite graphs as limit points (Borsuk-like, dynamical systems)

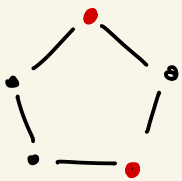
(3) Obtain all best-known lower bounds on Shannon capacity of small odd cycles from "finite version" of graph limit approach

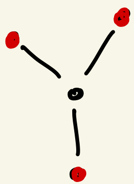
→ New lower bound $\Theta(C_{15}) \geq 7.30139$

1. Shannon capacity and asymptotic spectrum distance
2. Converging sequences
3. Infinite graphs as limit points
4. New lower bound for C_{15}

Basics

Independent set :





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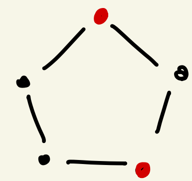 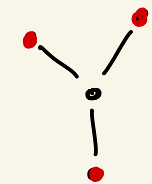

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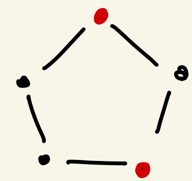

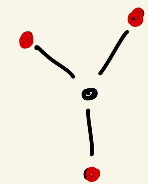

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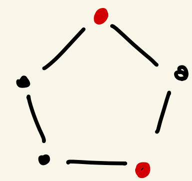

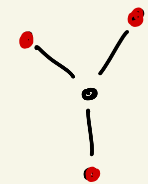

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Cohomomorphism $G \leq H$: if there is a map $V(G) \to V(H)$ preserving independent sets

1. Shannon capacity and asymptotic spectrum distance

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Def. Asymptotic spectrum X := set of all functions F: graphs $\to \mathbb{R}_{\geq 0}$

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(3) $(E_b \boxtimes G)^{\boxtimes n} \leq ((E_b \boxtimes H) \sqcup E_a)^{\boxtimes (n + o(n))}$ "

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<u>Def</u> [Zhu, Hell-Nešetřil] Fraction graph $E_{a/b}$ has vertex set $\mathbb{Z}/a\mathbb{Z}$ and an edge $u \sim v$ iff $|u-v| < b \pmod a$

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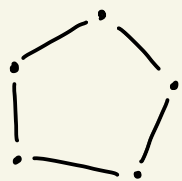
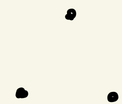
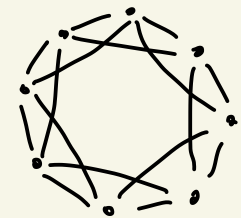
$E_{\frac{5}{2}} =$ 

$E_{\frac{3}{1}} =$ 

$E_{\frac{8}{3}} =$ 

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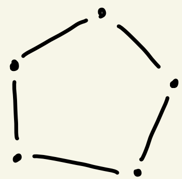
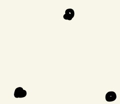
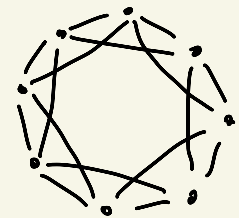
$E_{\frac{5}{2}} =$  $E_{\frac{3}{1}} =$  $E_{\frac{8}{3}} =$ 

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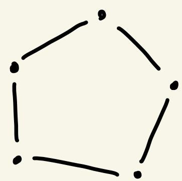
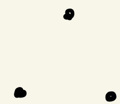
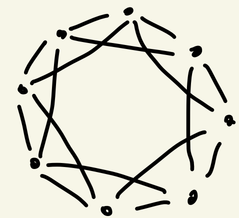
$E_{\frac{5}{2}} =$ 

$E_{\frac{3}{1}} =$ 

$E_{\frac{8}{3}} =$ 

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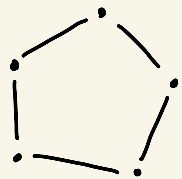
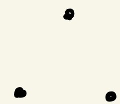
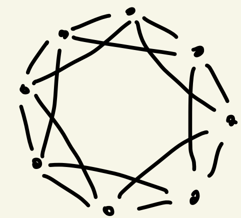
$E_{\frac{5}{2}} =$ 

$E_{\frac{3}{1}} =$ 

$E_{\frac{8}{3}} =$ 

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Theorem B For any irrational $r \geqslant 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy.

Ingredients

Lemma 1. [Vrana] G vertex transitive, $S \subseteq V(G)$, $F \in X$, then

$$F(G[S]) \leq F(G) \leq \frac{|G|}{|S|} \cdot F(G[S]).$$

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Consequence: $F(E_{p'/q'}) \le F(E_{p/q}) \le \frac{p}{p-1} F(E_{p'/q'})$

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$$\mathcal{F}(E_{a/b}) \leq \mathcal{F}(E_{c_n/d_n}) \leq \frac{c_n}{c_n - 1} \mathcal{F}(E_{a/b})$$

Let $n \to \infty$, then $c_n \to \infty$. $\square$

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So:

$$F(E_{a/b}) \leq F(E_{c_n/d_n}) \leq \frac{c_n}{c_n - 1} F(E_{a/b})$$

Let $n \to \infty$, then $c_n \to \infty$. $\square$

Theorem B For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy.

Proof sketch:

Theorem A For any $a/b \geq 2$, if $c_n/d_n \to a/b$ from above, then $E_{c_n/d_n} \longrightarrow E_{a/b}$.

Proof sketch: Let a, b coprime.

There are x, y with $x \cdot b - y \cdot a = 1$.

Then $c_n \cdot b - d_n\, a = 1$ for $c_n = x + a \cdot n$ and $d_n = y + b \cdot n$

So:

$$F(E_{a/b}) \leq F(E_{c_n/d_n}) \leq \frac{c_n}{c_n - 1} F(E_{a/b})$$

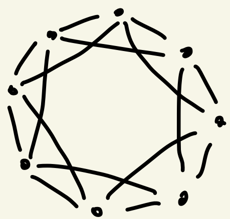
Let $n \to \infty$, then $c_n \to \infty$. $\square$

Theorem B For any irrational $r \geq 2$, if $c_n/d_n \to r$, then E_{c_n/d_n} is Cauchy.

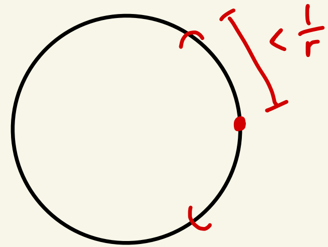
Proof sketch: Continued fraction convergents: $\frac{p_0}{q_0} < \frac{p_2}{q_2} \cdots < r < \cdots \frac{p_3}{q_3} < \frac{p_1}{q_1}$

Satisfy: $q_n \cdot p_{n-1} - p_n \cdot q_{n-1} = (-1)^n$. $\square$

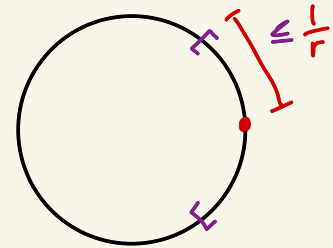
3. Infinite graphs as limit points



$E_{8/3}$

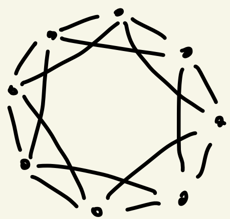


E_r^{o}

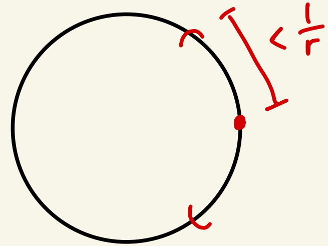


E_r^{c}

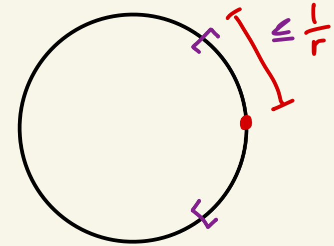
3. Infinite graphs as limit points



$E_{8/3}$



E_r°



E_r^{c}

Theorem

For any irrational $r \geq 2$, $d(E_r^{\circ}, E_r^{c}) = 0$ and if $a_n/b_n \longrightarrow r$, then $E_{a_n/b_n} \longrightarrow E_r^{\circ}$

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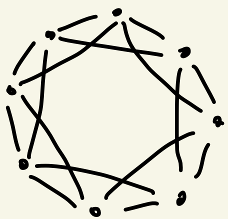
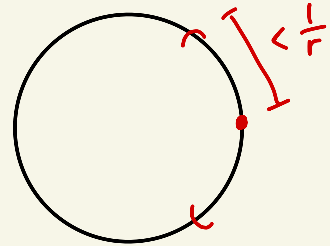
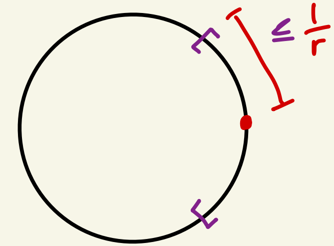


$E_{8/3}$



E_r^{o}



E_r^{c}

Theorem

For any irrational $r \geq 2$, $d(E_r^{o}, E_r^{c}) = 0$ and if $a_n/b_n \to r$, then $E_{a_n/b_n} \to E_r^{o}$

Theorem

$d(E_{p/q}^{c}, E_{p/q}^{o}) = 0$ iff $a_n/b_n \to p/q$ from below $\Rightarrow E_{a_n/b_n} \to E_{p/q}$.

4. New lower bound for C_{15}

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- "Finite version" of graph limit approach:

auxiliary graph H close to target graph

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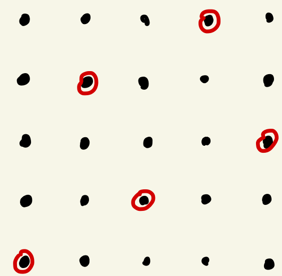
- Orbit independent sets:

$$\Theta(C_5) = \sqrt{5}$$

$$\alpha(C_5^{\boxtimes 2}) = 5$$



$$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$$

(Shannon 1956)

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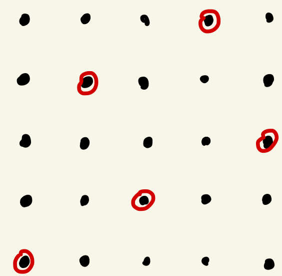

$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$

(Shannon 1956)

G	H	orbit independent set in $H^{\boxtimes k}$	reduction	$\leq \Theta(G)$
$E_{5/2}$	$E_{5/2}$	$\{t \cdot (1,2) : t \in \mathbb{Z}_5\}$	$H = G$	2.23 [Sha56]
$E_{7/2}$	$E_{382/108}$	$\{t \cdot (1,7,7^2,7^3,7^4) : t \in \mathbb{Z}_{382}\}$	$G \leq H$	3.25 [PS19]
$E_{9/2}$	$E_{9/2}$	$\{s \cdot (1,0,2) + t \cdot (0,1,4) : s,t \in \mathbb{Z}_9\}$	$H = G$	4.32 [BMR+71]
$E_{11/2}$	$E_{148/27}$	$\{t \cdot (1,11,11^2) : t \in \mathbb{Z}_{148}\}$	$H \leq G$	5.28 [BMR+71]
$E_{13/2}$	$E_{247/38}$	$\{t \cdot (1,19,117) : t \in \mathbb{Z}_{247}\}$	$H \leq G$	6.27 [BMR+71][18]
$E_{15/2}$	$E_{2873/381}$	$\{t \cdot (1,15,1073,1125) : t \in \mathbb{Z}_{2873}\}$	$G \leq H$	7.30 (Section 6.2)

Some open problems:

(1) Determine $\Theta(G)$ via graph limit approach?

(2) Convergence from below for fraction graphs?

(3) Are infinite graphs complete?

(4) What other problems allow asymptotic spectrum duality/distance?