AAG: ASSIGNMENT 1 (DUE NOV 6)

Exercise 1. Let k be a field. Let Z be an integral scheme. Let $i\colon Z \to \mathbf{P}^2_k$ be a closed immersion. Show that one of the following statements holds

(1) $Z \cong \operatorname{Spec} l$ for l a finite field extension of k
(2) $Z \cong \operatorname{Proj} k[X,Y,Z]/(F)$ for an irreducible homogeneous $F \in k[X,Y,Z]$
(3) $Z \cong \mathbf{P}^2_k$.

Exercise 2. Let k be a field. Let $f\colon X \to \mathbf{A}^1_k$ be a morphism of finite type. Assume $f^{-1}(y) \neq \emptyset$ for infinitely many $y \in \mathbf{A}^1_k$. Show that $f^{-1}(\eta) \neq \emptyset$, with η the generic point of $\mathbf{A}^1_k$.

(Hint: first reduce to X affine and integral.)

Exercise 3. Let X be a separated scheme. Let $R_1 \to R_2$ be an injective ring homomorphism. Show that the map $X(R_1) \to X(R_2)$ is injective. Give an example of a non-separated X and an injective ring homomorphism $R_1 \to R_2$ such that $X(R_1) \to X(R_2)$ is not injective.

Exercise 4. Let R be a commutative ring and $f \in R[X]$ a *monic* polynomial of degree ≥ 1.

(1) Show that $\operatorname{Spec} R[X]/(f) \to \operatorname{Spec} R$ is surjective,
(2) Show that $\operatorname{Spec} R[X]/(f) \to \operatorname{Spec} R$ is closed,
(3) Show that $\operatorname{Spec} R[X]/(f) \to \operatorname{Spec} R$ is proper.

(Hint for (2): Let $Z \subset \operatorname{Spec} R[X]/(f)$ be a closed subset corresponding to a prime ideal $\mathfrak{p}$. Let $\mathfrak{q} = R \cap \mathfrak{p}$. Consider the map $\operatorname{Spec}(R/\mathfrak{q})[X]/(f) \to \operatorname{Spec} R/\mathfrak{q}$.)