AAG: ASSIGNMENT 2 (DUE DEC 11)

Final exam. The oral exams are on January 8 and 9. Please indicate on your solution sheet whether you want to participate in the exams.

Exercise 1. You may freely use tag 00NX of the stacks project. Let R be a commutative ring and let M be a finitely presented R-module. Consider the function

$$\delta\colon \operatorname{Spec} R \to \mathbf{Z},\ \mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p}).$$

(1) Show that the set $\{\mathfrak{p} \in \operatorname{Spec} R \colon \delta(\mathfrak{p}) \geq n\}$ is closed in $\operatorname{Spec} R$. (Suggestion: choose a presentation $R^s \to R^t \to M \to 0$ and consider the minors of the matrix defining the map $R^s \to R^t$).
(2) Assume that R is reduced. Show that M is flat if and only if δ is locally constant.
(3) Show that the 'if' part of (2) may fail if R is not reduced.

Exercise 2. Let Y be a locally noetherian and reduced scheme and let $f\colon X \to Y$ be a finite morphism. Consider the function

$$\delta\colon Y \to \mathbf{Z},\ y \mapsto \dim_{\kappa(y)} (f_*\mathcal{O}_X)_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y).$$

(1) Show that f is flat if and only if δ is locally constant.
(2) Let k be a field. In each of the following cases show that f is finite, and determine whether f is flat.
   (a) (Normalization of a cusp). The map $f\colon \operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2, t^3]$ corresponding to the inclusion $k[t^2, t^3] \hookrightarrow k[t]$;
   (b) (Two lines intersecting in a point, mapping to a line). Let $R = k[u]$ and $S = \{(g_1, g_2) \in R \times R \mid g_1(0) = g_2(0)\}$, and let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be the map corresponding to the ring homomorphism $g \mapsto (g, g)$;
   (c) (Two planes intersecting in a point, mapping to a plane). Let $R = k[u, v]$ and $S = \{(g_1, g_2) \in R \times R \mid g_1(0, 0) = g_2(0, 0)\}$, and let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be the map corresponding to the ring homomorphism $g \mapsto (g, g)$.

Exercise 3. Let R be a commutative ring and $I \subset R$ an ideal with $I^2 = 0$. In this exercise you show that the base change map

$$\phi\colon \operatorname{Pic} R \to \operatorname{Pic} R/I$$

is an isomorphism.

(1) Let L be an invertible R-module and $\lambda \in L$. Assume that L/IL is generated by $\bar{\lambda}$. Show that L is generated by λ.
(2) Show that ϕ is injective.
(3) Let $\bar{L}$ be an invertible R/I-module generated by $\bar{s}_0, \ldots, \bar{s}_n$. Use the formal smoothness of $\mathbf{P}^n \to \operatorname{Spec} \mathbf{Z}$ to deduce that there exists an invertible R-module L and $s_0, \ldots, s_n \in L$ generating L, together with an isomorphism $L/IL \to \bar{L}$ sending s_i to $\bar{s}_i$.
(4) Conclude that ϕ is surjective.

Exercise 4.

(1) Let X be a scheme, $U, V \subset X$ open with $X = U \cup V$. Assume $\operatorname{Pic} U$ and $\operatorname{Pic} V$ are trivial. Show that there is an exact sequence
$$1 \to \mathcal{O}_X(X)^\times \to \mathcal{O}_X(U)^\times \times \mathcal{O}_X(V)^\times \to \mathcal{O}_X(U \cap V)^\times \to \operatorname{Pic} X \to 1.$$
(2) (Here you show that in Exercise 3 it is crucial that $\operatorname{Spec} R$ is affine). Let k be a field and $Z = \mathbf{A}^2_k - \{(0,0)\}$. Let $X = Z \times_k \operatorname{Spec} k[\epsilon]/\epsilon^2$. Let $f\colon Z \to X$ be the closed immersion induced by $k[\epsilon]/\epsilon^2 \to k$. Show that $\operatorname{Pic} Z$ is trivial, but $\operatorname{Pic} X$ is not trivial.