AAG, WEEK 3: EXERCISE

Let k be a field. Let $U = \mathbf{A}^2_k \setminus \{0\}$ be the open subscheme of $\mathbf{A}^2_k$, where $0 \in \mathbf{A}^2_k$ is the closed point corresponding to maximal ideal (x, y) of $k[x, y]$. (Warning: U is not an affine scheme).

(1) Show $U = D(x) \cup D(y)$, where $D(x)$, $D(y)$ are the standard opens in $\mathbf{A}^2_k$;
(2) Let R be a k-algebra. Let $(s, t) \in R^2$, and $f\colon \operatorname{Spec} R \to \mathbf{A}^2_k$ the morphism of k-schemes corresponding to $k[x, y] \to R$ with $x \mapsto s$ and $y \mapsto t$. Show that φ factors over $U \hookrightarrow \mathbf{A}^2_k$ if and only if $\operatorname{Spec} R = D(s) \cup D(t)$;
(3) Show $U(R) = \{(s, t) \in R^2 \mid Rs + Rt = R\}$;
(4) Show $U(K) = \{(s, t) \in K^2 \mid (s, t) \neq (0, 0)\}$ for every field extension $k \subset K$.

In (3) and (4) the notation $U(R)$ stands for $\operatorname{Hom}(\operatorname{Spec} R, U)$ in the category of schemes *over* k.