Algebraic realizations of cyclic isometries of the rational K3 lattice

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Abstract

We prove: For all n and all n-cylic isometries $\phi\colon \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ of the rational K3 lattice there exists an algebraic realization of ϕ, i.e. marked algebraic K3 surfaces (S, η_S) and (M, η_M), where M is a moduli space of sheaves on S, and a Hodge isometry $\psi\colon H^2(S, \mathbb{Q}) \to H^2(M, \mathbb{Q})$ such that $\phi = \eta_M \circ \psi \circ \eta_S^{-1}$.

1 Facts about cyclic isometries

We need the existence of a triple $((S, \eta_S), (M, \eta_M), \psi)$ to have a point in our moduli space from which to start deforming. The moduli space will be seen to be covered by "twistor lines" and deformations of our examples along these lines will be described in a later talk.

We begin by recalling our essential definition.

Definition 1.1. Let L_1 and L_2 be lattices. An isometry $\varphi\colon L_1 \otimes_{\mathbb{Z}} \mathbb{Q} \to L_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ is called n-cyclic if

$$\frac{L_1}{\varphi^{-1}(L_2) \cap L_1} \cong \mathbb{Z}/n\mathbb{Z}$$

By $O(L)$ we denote the (integral) isometry group of L and by $O(L_{\mathbb{Q}})$ the (rational) isometry group of $L_{\mathbb{Q}}$. By Λ we denote the K3 lattice $U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$.

Definition 1.2. For any $\phi \in O(\Lambda_{\mathbb{Q}})$ of n-cyclic type its *double orbit* $[\phi]$ is given by $O(\Lambda)\phi\, O(\Lambda) \subseteq O(\Lambda_{\mathbb{Q}})$.

The following result says that all n-cyclic isometries of the K3 lattice are conjugate to each other using integral isometries. This will be helpful later, since we can just produce a Hodge isometry associated to *some* n-cyclic isometry of the K3 lattice (not necessarily the the one we started with), and then conjugate it from there.

Proposition 1.3 (Proposition 3.3 in [Bus15])**.** *Let ϕ_1 and ϕ_2 be rational isometries of $\Lambda_{\mathbb{Q}}$ of n-cyclic type. Then $[\phi_1] = [\phi_2]$.*

The proof of Proposition 1.3 is lattice-theoretic and very technical. We will skip it.

2 Mukai's construction of Hodge isometries

Let S be a K3 surface. In addition to the usual weight 2 Hodge structure on $H^2(S,\mathbb{Z})$ there is another weight 2 structure on $H^\bullet(S,\mathbb{Z})$, originally defined by Mukai. Observe that since we are on a K3 surface the cohomology ring is equal to the even cohomology ring, i.e. we have $H^\bullet(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$.

Definition 2.1. Define a Hodge structure $\widetilde{H}(S,\mathbb{Z})$ on $H^\bullet(S,\mathbb{Z})$ as follows:

$$\begin{aligned}
\widetilde{H}^{2,0}(S,\mathbb{C}) &= H^{2,0}(S,\mathbb{C}) \\
\widetilde{H}^{1,1}(S,\mathbb{C}) &= H^0(S,\mathbb{C}) \oplus H^{1,1}(S,\mathbb{C}) \oplus H^4(S,\mathbb{C}) \\
\widetilde{H}^{0,2}(S,\mathbb{C}) &= H^{0,2}(S,\mathbb{C})
\end{aligned}$$

We endow $\widetilde{H}(S,\mathbb{Z})$ with the pairing $(\cdot,\cdot)$ of Mukai vectors.

Remark 2.2. Observe that $\widetilde{H}^{1,1}(S,\mathbb{C}) = \bigoplus_{p=0}^{2} H^{p,p}(S,\mathbb{C})$. This means that Hodge isometries of the twiddled Hodge structure naturally arise by intersecting with Chern classes (which are pure of type (p,p)). It also means that Mukai vectors are elements of $\widetilde{H}^{1,1}(S,\mathbb{Z})$.

The pairing can be written explicitly as follows. Let $\alpha = (a_0, a_2, a_4)$ and $\beta = (b_0, b_2, b_4) \in H^\bullet(S,\mathbb{Z})$. Then

$$(\alpha,\beta) = -a_0 b_4 + a_2 b_2 - a_4 b_0$$

Now fix a Mukai vector v such that $M = M(v)$ is itself a K3 and there exists a universal locally free sheaf $\mathcal{E}$ on $S \times M$. We write $v = (r, c_1, c_1^2/2 - c_2 + r)$ and denote the projection maps of $S \times M$ by

$$\begin{array}{ccc}
S \times M & \xrightarrow{\pi_M} & M \\
{\scriptstyle \pi_S}\downarrow & & \\
S & &
\end{array}$$

The starting point of our construction is the following "Fourier–Mukai type" map.

Definition 2.3. We construct a map

$$f_{\mathcal{E}^\vee} \colon H^\bullet(S,\mathbb{Q}) \to H^\bullet(M,\mathbb{Q})$$

by setting

$$f_{\mathcal{E}^\vee}(-) = \pi_{M,*}(v^\vee(\mathcal{E}) \cdot \pi_S^*(-))$$

Remark 2.4. The same can be done for a quasi-universal sheaf $\mathcal{E}$. Then the Mukai vector $v^\vee(\mathcal{E})$ has to be multiplied by $1/s$ where s is the similitude of $\mathcal{E}$.

Let $v^\perp$ be the orthogonal complement of the Mukai vector v in $\widetilde{H}(S,\mathbb{Z})$. Since $(v,v) = 0$ we have $v \in v^\perp$. Then $(v^\perp/\mathbb{Z}v) \otimes \mathbb{Q}$ is free of rank 22. The following theorem was originally proven by Mukai.

Theorem 2.5 ([HL10], Prop. 6.1.13 and Theorem 6.1.14). *The map $f_{\mathcal{E}^\vee}$ is a Hodge isometry $\widetilde{H}(S,\mathbb{Z}) \to \widetilde{H}(M,\mathbb{Z})$. Furthermore, it induces a Hodge isometry $\widetilde{f}_{\mathcal{E}^\vee} : v^\perp/\mathbb{Z}v \to H^2(M,\mathbb{Z})$, independent of the choice of universal sheaf $\mathcal{E}$.*

Remark 2.6. It is not hard to see that there is actually a Hodge isometry $H^2(S,\mathbb{Q}) \to (v^\perp/\mathbb{Z}v) \otimes \mathbb{Q}$ and hence $f_{\mathcal{E}^\vee}$ induces a rational Hodge isometry between $H^2(S,\mathbb{Q})$ and $H^2(M,\mathbb{Q})$. First define

$$\varphi \colon H^2(S,\mathbb{Q}) \to \widetilde{H}(S,\mathbb{Q}), \quad w \mapsto (0, w, c_1 \cdot w/r)$$

Since

$$(\varphi(w), v) = -w \cdot c_1 + \frac{c_1 \cdot w}{r} \cdot r = 0$$

we have $\operatorname{im}(\varphi) \subseteq v^\perp \otimes \mathbb{Q}$. Clearly $v \notin \operatorname{im}(\varphi)$ since $r > 0$, hence the induced map $H^2(S,\mathbb{Q}) \xrightarrow{\varphi} (V/\mathbb{Z}v) \otimes \mathbb{Q}$ is an injective map between vector spaces of the same dimension, hence an isomorphism. We see that $H^{2,0}(S,\mathbb{C})$ is mapped to $\widetilde{H}^{2,0}(S,\mathbb{C})$ and the intersection pairing is preserved since $((0, w_1, c_1 \cdot w_1/r), (0, w_2, c_1 \cdot w_2/r)) = w_1 \cdot w_2$.

3 The kappa class and the induced Hodge isometry

We will later deform $S \times M$ together with the universal sheaf to obtain the main theorem. Since sheaves do in general not stay "untwisted" under these deformations, we need to consider *twisted sheaves*. The definition and properties will be given in a later talk; for now think of a twisted sheaf on a scheme X as a collection of sheaves on an open cover of X, glued together by a Čech 2-cocyle. We reformulate Mukai's result for twisted sheaves so that we can use it later in greater generality.

Let $\mathcal{E}$ be a twisted locally free sheaf of rank r on a compact complex manifold X. One can check that the sheaf $\mathcal{E}^{\otimes r} \otimes \det(\mathcal{E}^\vee)$ is an untwisted locally free sheaf of rank r^r (we cannot check this here since we do not even have a definition). Hence the notion of Chern class makes sense. Let

$$\operatorname{Sqrt}_r(x) = r + \frac{1}{r^r}(x - r^r) + \frac{1-r}{r^{2r+1}}(x - r^r)^2 + \dots$$

be the formal power series of the r-th root function centered at r^r. Since our sheaf has rank r^r its Chern character has constant term r^r, so we can apply Sqrt_r to it. Then we set

$$\kappa(\mathcal{E}) = \operatorname{Sqrt}_r\left(\operatorname{ch}(\mathcal{E}^{\otimes r} \otimes \det(\mathcal{E}^\vee))\right)$$

If we take $\mathcal{E}$ to actually be an untwisted locally free sheaf then we can expand

$$\begin{aligned} \operatorname{ch}(\mathcal{E}^{\otimes r} \otimes \det(\mathcal{E}^\vee)) &= \operatorname{ch}(\mathcal{E})^r \operatorname{ch}\left(\det(\mathcal{E})^\vee\right) \\ &= \operatorname{ch}(\mathcal{E})^r \exp(-c_1(\mathcal{E})) \end{aligned}$$

hence we can take the r-th root and arrive at

$$\kappa(\mathcal{E}) = \operatorname{ch}(\mathcal{E}) \exp(-c_1(\mathcal{E})/r)$$

The class $\mu = \kappa(\mathcal{E}^\vee)\sqrt{\mathrm{Td}(S \times M)}$ induces a map $\varphi_{\mathcal{E}^\vee} \colon H^\bullet(S, \mathbb{Q}) \to H^\bullet(M, \mathbb{Q})$ by push-pull and intersecting, i.e. $\varphi_{\mathcal{E}^\vee}(c) = \pi_{M,*}(\mu \cdot \pi_S^*(c))$. Let $\psi_{\mathcal{E}}$ be the composition

$$H^2(S, \mathbb{Q}) \xrightarrow{\iota} H^\bullet(S, \mathbb{Q}) \xrightarrow{\varphi_{\mathcal{E}^\vee}} H^\bullet(M, \mathbb{Q}) \xrightarrow{\mathrm{pr}_2} H^2(M, \mathbb{Q})$$

This map $\psi_{\mathcal{E}}$ will later be shown to be of cyclic type.

Theorem 3.1. *Let $\mathcal{E}$ be a universal sheaf on $S \times M$. Then the map $\psi_{\mathcal{E}}$ is a Hodge isometry between $H^2(S, \mathbb{Q})$ and $H^2(M, \mathbb{Q})$.*

In order to prove this, we first analyze the map $\varphi_{\mathcal{E}^\vee}$. Note that the universal sheaf $\mathcal{E}$ is untwisted, hence we can expand the kappa class. Let $\alpha = c_1(\mathcal{E}|_{S \times \{m\}})/r$ and $\beta = c_1(\mathcal{E}|_{\{s\} \times M})/r$. Then $c_1(\mathcal{E})/r = \alpha + \beta$ and we have

$$\kappa(\mathcal{E}^\vee)\sqrt{\mathrm{Td}\, S \times M} = \nu(\mathcal{E}^\vee) e^{\pi_S^* \alpha} e^{\pi_M^* \beta}$$

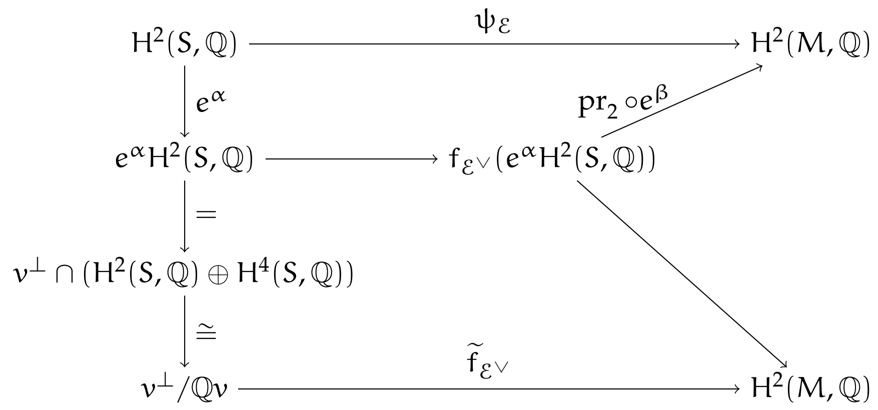
Hence using the projection formula we can express $\varphi_{\mathcal{E}^\vee}$ as

$$\begin{aligned} \varphi_{\mathcal{E}^\vee}(c) &= \pi_{M,*}(\pi_S^*(c) \cdot \nu(\mathcal{E}^\vee) \cdot e^{\pi_S^* \alpha} e^{\pi_M^* \beta}) \\ &= e^\beta \cdot \pi_{M,*}(\pi_S^*(c \cdot e^\alpha) \cdot \nu(\mathcal{E}^\vee)) \end{aligned}$$

i.e. $\varphi_{\mathcal{E}^\vee} = e^\beta \circ f_{\mathcal{E}^\vee} \circ e^\alpha$ where e^α and e^β are the maps given by multiplication with the respective class. **Observation**: Multiplication by $e^\alpha = 1 + \alpha + \alpha^2$ is a Hodge isometry of $\widetilde{H}(S, \mathbb{Q})$. We can check

$$(w_1 \cdot e^\alpha, w_2 \cdot e^\alpha) = -\int w_1^\vee \cdot e^{-\alpha} \cdot w_2 \cdot e^\alpha = -\int w_1^\vee \cdot w_2 = (w_1, w_2)$$

Now consider the following diagram



We make two further observations:

- $e^\alpha H^2(S, \mathbb{Q})$ is contained in $\nu^\perp$, since $(e^\alpha \cdot c, \nu) = -r\alpha \cdot c + c \cdot c_1 = 0$ for $c \in H^2(S, \mathbb{Q})$.
- $e^\alpha H^2(S, \mathbb{Q}) \cap \mathbb{Q}\nu = \{0\}$. This is clear since $r > 0$ is the H^0-piece of ν.

Since $e^\alpha H^2(S,\mathbb{Q})$ is a 22-dimensional subspace of the 23-dimensional vector space $v^\perp$ and $v \notin e^\alpha H^2(S,\mathbb{Q})$ we have $v^\perp = \mathbb{Q}v \oplus e^\alpha H^2(S,\mathbb{Q})$. This explains the isomorphisms on the left side of the diagram. In order to prove that $\psi_\mathcal{E}$ is a Hodge isometry we just have to prove that $\mathrm{pr}_2 \circ e^\beta \circ f_{\mathcal{E}^\vee}$ is a Hodge isometry on $e^\alpha H^2(S,\mathbb{Q})$.

Lemma 3.2. The image of $v^\perp$ under $f_{\mathcal{E}^\vee}$ has no $H^0(M,\mathbb{Q})$-component.

Proof. Anything nontrivial in $H^0(M,\mathbb{Q})$ comes from pushing down classes in $H^0(M) \otimes H^4(S)$, and since in the expression $f_{\mathcal{E}^\vee}(c) = \pi_{M,*}(\pi_S^*(c) \cdot v(\mathcal{E}))$ the $H^0(M) \otimes H^\bullet(S)$-component of $v(\mathcal{E}^\vee)$ is just $v^\vee$, the part in $H^0(M) \otimes H^4(S)$ is $(v,c) = 0$. □

Now looking at $\mathrm{pr}_2 \circ e^\beta \colon H^2(M,\mathbb{Q}) \oplus H^4(M,\mathbb{Q}) \to H^2(M,\mathbb{Q})$ we see that it is in fact the same as the projection map, and the Mukai form is preserved. Hence $\mathrm{pr}_2 \circ e^\beta \circ f_{\mathcal{E}^\vee}$ being a Hodge isometry is equivalent to

- $f_{\mathcal{E}^\vee}$ being a Hodge isometry on $e^\alpha H^2(S,\mathbb{Q})$ and
- the image of $e^\alpha H^2(S,\mathbb{Q})$ under $f_{\mathcal{E}^\vee}$ has empty intersection with $H^4(M,\mathbb{Q})$.

Looking again at the diagram, we invoke the theorems of Mukai showing that $f_{\mathcal{E}^\vee}$ and $\widetilde{f}_{\mathcal{E}^\vee}$ are Hodge isometries. Then we see that, just by dimension reasons, the intersection of the image of $f_{\mathcal{E}^\vee}$ with $H^4(M,\mathbb{Q})$ is empty. Hence $\psi_\mathcal{E}$ is a Hodge isometry as well.

4 The isometry $\psi_\mathcal{E}$ is of cyclic type

We fix a Mukai vector $v = (r, \alpha, s)$ with $(v,v) = 0$ and such that there exists a universal sheaf on $S \times M$. As we have just proved, the class $\widetilde{Z}_{\mathcal{E}^\vee} = \kappa(\mathcal{E}^\vee)\sqrt{\mathrm{Td}(S \times M)}$ determines $f_{\mathcal{E}^\vee}$ and hence the rational Hodge isometry $\psi_\mathcal{E} \colon H^2(S,\mathbb{Q}) \to H^2(M,\mathbb{Q})$. Let L be the cohomology lattice $L = H^2(S,\mathbb{Z})$ and let $I_{\psi_\mathcal{E}} = \psi_\mathcal{E}^{-1}(H^2(M,\mathbb{Z})) \cap L$. We want to show that $L/I_{\psi_\mathcal{E}}$ is a cyclic group.

We begin by computing $\sqrt{\mathrm{Td}(M \times S)}$. If $[S]$ and $[M]$ denote the Poincaré dual of a point of S and M, respectively, then:

$$\begin{aligned}\sqrt{\mathrm{Td}(S \times M)} &= \sqrt{\pi_S^* \mathrm{Td}(S)}\sqrt{\pi_M^* \mathrm{Td}(M)} \\ &= \sqrt{1 + 2\pi_S^*[S]}\sqrt{1 + 2\pi_M^*[M]} \\ &= (1 + \pi_S^*[S])(1 + \pi_M^*[M]) \\ &= 1 + \pi_S^*[S] + \pi_M^*[M] + \pi_S^*[S] \cdot \pi_M^*[M]\end{aligned}$$

The map $\psi_\mathcal{E}$ is determined completely by the H^6-part of the image of $\varphi_{\mathcal{E}^\vee}$ in $H^\bullet(S \times M)$, i.e. the degree 6 part of

$$\pi_S^*(c) \cdot \kappa(\mathcal{E}^\vee) \cdot \sqrt{\mathrm{Td}(S \times M)}$$

where $c \in H^2(S,\mathbb{Q})$. Since $\sqrt{\mathrm{Td}(S \times M)}$ lives in degrees 0, 4 and 8 and $\pi_S^*(c) \in H^2(S \times M)$ the degree 6 part of this expression consists of $\pi_S^*(c) \cdot \kappa_2(\mathcal{E}^\vee)$ and

$$\begin{aligned}\pi_S^*(c) \cdot (\pi_S^*[S] + \pi_M^*[M]) &= \pi_S^*(c \cdot [S]) + \pi_S^*(c) \cdot \pi_M^*[M] \\ &= \pi_S^*(c) \cdot \pi_M^*[M]\end{aligned}$$

since $c \cdot [S] \in H^6(S, \mathbb{Z}) = 0$. But if we push down the rest we get

$$\pi_{M,*}(\pi_S^*(c) \cdot \pi_M^*[M]) = [M] \cdot \pi_{M,*}\pi_S^*(c) = 0$$

by the push-pull formula since $\pi_{M,*}$ sends $H^2(S \times M, \mathbb{Z})$ to 0. Hence $\psi_{\mathcal{E}}$ is completely determined by $\kappa_2(\mathcal{E}^\vee)$, which can be calculated as

$$\begin{aligned}
\kappa_2(\mathcal{E}^\vee) &= \Big[\mathrm{ch}(\mathcal{E}^\vee)\exp(c_1(\mathcal{E})/r)\Big]_2 \\
&= \left[(r - c_1(\mathcal{E}) + \mathrm{ch}_2(\mathcal{E}) + \cdots)\left(1 + \frac{c_1(\mathcal{E})}{r} + \frac{c_1^2(\mathcal{E})}{2r^2} + \cdots\right)\right]_2 \\
&= \mathrm{ch}_2(\mathcal{E}) - \frac{c_1^2(\mathcal{E})}{2r}
\end{aligned}$$

We want to find out when exactly $\pi_{M,*}(\kappa_2(\mathcal{E}^\vee) \cdot \pi_S^*(c))$ is integral for integral classes $c \in H^2(S, \mathbb{Z})$. We already know that $\mathrm{ch}_2(\mathcal{E})$ is integral, since $c_2(\mathcal{E})$ is and $c_1^2(\mathcal{E})$ is even because the intersection form is even. Therefore we only have to find out when $c_1^2(\mathcal{E})/2r$ is integral. Let $\beta = c_1(\mathcal{E}^\vee|_{\{s\}\times M}) \in H^2(M) \otimes H^0(S)$. We have

$$\begin{aligned}
\frac{c_1^2(\mathcal{E})}{2r} &= \frac{(\pi_S^*(\alpha) + \pi_M^*(\beta))^2}{2r} \\
&= \frac{\pi_S^*(\alpha^2)}{2r} + \frac{\pi_S^*(\alpha) \cdot \pi_M^*(\beta)}{r} + \frac{\pi_M^*(\beta^2)}{2r}
\end{aligned}$$

The first term induces the zero map because $\pi_S^*(\alpha^2) \cdot \pi_S^*(c)$ is the pullback of a degree 6 class on S (and there are none). The third term also induces the zero map by

$$\pi_{M,*}(\pi_M^*(\beta^2) \cdot \pi_S^*(c)) = \beta^2 \cdot \pi_{M,*}\pi_S^*(c) = 0$$

since $\pi_{M,*}$ sends $H^2(S \times M, \mathbb{Z})$ to 0, as before.

So the only relevant term is the second one and it induces a map

$$\begin{aligned}
\pi_{M,*}\left(\frac{\pi_S^*(\alpha) \cdot \pi_M^*(\beta)}{r} \cdot \pi_S^*(c)\right) &= \frac{1}{r}\beta \cdot \pi_{M,*}\pi_S^*(\alpha \cdot c) \\
&= (\alpha, c)\frac{\beta}{r}
\end{aligned}$$

We write $\alpha = k \cdot x$ and $\beta = j \cdot y$ where $x \in L$ and $y \in H^2(M, \mathbb{Z})$ are primitive classes. Hence the induced map is given by

$$\gamma \mapsto \frac{jk(x, \gamma)}{r} y$$

Then $\psi_{\mathcal{E}}$ takes integral values precisely on the sublattice

$$\begin{aligned}
I_{\psi_{\mathcal{E}}} &= \left\{\gamma \in L \text{ s.t. } \frac{jk}{r}(\gamma, x) \text{ is integral}\right\} \\
&= \left\{\gamma \in L \text{ s.t. } (\gamma, x) \text{ is divisible by } \frac{r}{\gcd(jk, r)}\right\}
\end{aligned}$$

Now we consider $L/I_{\psi_{\mathcal{E}}}$. By the unimodularity of L we can choose $w \in L$ such that $(w, x) = 1$. Then for every $\gamma \in L$ our previous map sends

$$\gamma - (x, \gamma)w \mapsto \frac{jk(x, \gamma) - jk(x, \gamma)(x, w)}{r} y = 0$$

so $\psi_{\mathcal{E}}$ is certainly integral for elements of this form. Hence L is generated by $I_{\psi_{\mathcal{E}}}$ and w. This implies $L/I_{\psi_{\mathcal{E}}}$ is generated by $[w]$, hence is a cyclic group. This group has order $r/\gcd(jk, r)$ which means $\psi_{\mathcal{E}}$ is cyclic of this order.

We are now in a position to prove the main theorem of this talk:

Theorem 4.1 (Conclusion 3.10 in [Bus15])**.** *For any isometry $\phi\colon \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ of n-cyclic type there exist marked projective K3 surfaces (S, η_S) and (M, η_M), where M is a moduli space of rank n sheaves on S, slope-stable with respect to an ample divisor $H \in \mathrm{Pic}(S)$, a universal locally free family $\mathcal{E}$ over $S \times M$ and a rational Hodge isometry $\psi\colon H^2(S, \mathbb{Q}) \to H^2(M, \mathbb{Q})$ induced by $\kappa(\mathcal{E})\sqrt{\mathrm{Td}(S \times M)}$ such that $\phi = \eta_M \circ \psi \circ \eta_S^{-1}$.*

We are going to use the following result (which is more or less a collection of results from the previous talk):

Lemma 4.2 (Corollary 4.6.7 in [HL10])**.** Let S be a K3 surface and H an ample line bundle on S. If $\gcd(r, c_1 \cdot H, c_1^2/2 - c_2) = 1$ then there exists a universal family on the moduli space M of sheaves with this Mukai vector, and $M^s = M$. Hence M is a smooth and irreducible K3 surface, parametrizes stable locally free sheaves and the universal sheaf $\mathcal{E}$ is itself locally free.

Select any integer $n > 0$. We will construct the objects in our theorem, starting with S, M and a Hodge isometry. Choose an $s > 0$ with $\gcd(n, s) = 1$. and choose a general K3 surface S in the moduli space $\mathcal{F}_{sn+1}$ of K3 surfaces of genus $sn + 1$. Then $\mathrm{Pic}(S) = \mathbb{Z}H$ for some ample class H. We have $H^2 = 2 \cdot (sn + 1) - 2 = 2sn$. Consider the Mukai vector $v = (n, H, s)$. Then

$$(v, v) = -2ns + H^2 = 0$$

and

$$\begin{aligned} \gcd(n, H^2, H^2/2 - s) &= \gcd(n, 2sn, sn - s) \\ &= \gcd(n, sn - s) \\ &= \gcd(n, s) \\ &= 1 \end{aligned}$$

so the hypotheses of Lemma 4.2 are satisfied. This gives us our M. Since M is Hodge isometric to S we have $\mathrm{Pic}\, M = \mathbb{Z}\widehat{H}$ for a primitive ample class $\widehat{H}$. By Proposition 1.1 and Theorem 1.2 of [Muk99] we have $\widehat{H}^2 = H^2$.

Remark 4.3. Mukai proves this using specialization to a particular K3 surface of Picard rank 2.

By the same results we can twist the universal sheaf $\mathcal{E}$ by a line bundle coming from M in such a way that

$$c_1(\mathcal{E}) = \pi_S^*(H) + j\pi_M^*(\widehat{H})$$

for an integer j with $js \equiv 1 \mod n$. This implies $\gcd(j, n) = 1$. Setting $r = n$, our discussion of $\psi_{\mathcal{E}}$ yields a Hodge isometry between S and M which is of n-cyclic type. Set $\psi = \psi_{\mathcal{E}}$.

Now we apply the double orbit theorem. Take any markings η_1 and η_2 of S and M, respectively. This means we have three Hodge isometries

$$\eta_1\colon H^2(S,\mathbb{Z}) \to \Lambda, \quad \eta_2\colon H^2(M,\mathbb{Z}) \to \Lambda, \quad \psi\colon H^2(S,\mathbb{Q}) \to H^2(M,\mathbb{Q})$$

Then $\phi_1 = \eta_2 \circ \psi \circ \eta_1^{-1}\colon \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ is a rational n-cyclic isometry. By the result on double orbits, Proposition 1.3, we have $[\phi_1] = [\phi]$, i.e. there exist $a, b \in O(\Lambda)$ with $\phi = a\phi_1 b$. Then $\eta_S = b^{-1}\eta_1$ and $\eta_M = a\eta_2$ are the markings we were looking for.

References

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