Sheaves and functions modulo p

lectures on the Woods Hole trace formula*

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Abstract. The Woods Hole trace formula is a kind of Lefschetz fixed point theorem for coherent cohomology on algebraic varieties. We explain how it leads to a characteristic-p-valued version of the sheaves-functions dictionary of Deligne, relating $\mathbf{F}_q$-valued functions on the rational points of varieties over $\mathbf{F}_q$ to coherent modules equipped with a Frobenius structure. We will discuss various applications, including some recent and new results on characteristic p zeta values.

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Introduction

Artin-Scheier theory. Let K be a field of characteristic p. Then for every $a \in K$ the polynomial $X^p - X - a$ is separable, and the additive group $\mathbf{F}_p$ acts transitively on the set of zeroes in a field extension. By Artin-Scheier theory, every Galois extension L/K with $\mathrm{Gal}(L/K) = \mathbf{F}_p$ arises as the splitting field of such a polynomial. In other words, there is an exact sequence

$$0 \longrightarrow \mathbf{F}_p \longrightarrow K \xrightarrow{1-\sigma} K \longrightarrow \mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \mathbf{F}_p) \longrightarrow 0,$$

where σ is the p-th power Frobenius map. In the language of étale cohomology, this can be rewritten without the choice of a separable closure as an exact sequence

$$(1) \qquad 0 \longrightarrow \mathbf{F}_p \longrightarrow K \xrightarrow{1-\sigma} K \longrightarrow \mathrm{H}^1((\mathrm{Spec}\, K)_{\mathrm{et}}, \mathbf{F}_p) \longrightarrow 0.$$

More generally, if S is a scheme over $\mathbf{F}_p$, then we have an exact sequence

$$0 \longrightarrow (\mathbf{F}_p)_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{1-\sigma} \mathbf{G}_{a,S} \longrightarrow 0$$

of sheaves on S_{et}. Here $(\mathbf{F}_p)_S$ denotes the constant sheaf with stalk $\mathbf{F}_p$. Since the étale and Zariski cohomology of $\mathbf{G}_{a,S} = \mathcal{O}_S$ coincide, we obtain a long exact sequence

$$(2) \qquad \cdots \longrightarrow \mathrm{H}^i(S_{\mathrm{et}}, \mathbf{F}_p) \longrightarrow \mathrm{H}^i(S, \mathcal{O}_S) \xrightarrow{1-\sigma} \mathrm{H}^i(S, \mathcal{O}_S) \longrightarrow \cdots$$

relating the mod p étale cohomology of S with the coherent cohomology of S, generalizing (1).

Katz and locally constant coefficients. Let S be a noetherian scheme over $\mathbf{F}_p$. In his paper on p-adic properties of modular forms Nick Katz **[35]** showed that there is an equivalence of categories between

(i) pairs $(\mathcal{F}, \tau)$ consisting of a locally free $\mathcal{O}_S$-module $\mathcal{F}$ and an isomorphism $\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}$ of $\mathcal{O}_S$-modules;

(ii) $\mathbf{F}_p$-modules V on S_{et} that are locally constant of finite rank.

By adjunction the map τ defines an $\mathcal{O}_S$-linear map $\tau_a\colon \mathcal{F} \to \sigma_\star\mathcal{F}$. Since σ is the identity on the underlying topological space of S, we have a natural identification $\sigma_\star\mathcal{F} = \mathcal{F}$. Under this identification τ_a becomes an additive map $\tau_s\colon \mathcal{F} \to \mathcal{F}$ satisfying $\tau_s(fs) = f^p\tau_s(s)$ for all local sections f of $\mathcal{O}_S$ and s of $\mathcal{F}$. (The subscripts a and s to τ stand for *adjoint* and *semi-linear*, respectively.) Any of the three maps τ, τ_a, τ_s determines the other two.

The étale $\mathbf{F}_p$-module V corresponding to $(\mathcal{F}, \tau)$ is defined by a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{F} \xrightarrow{1-\tau_s} \mathcal{F} \longrightarrow 0 \tag{3}$$

of sheaves on S_{et}, and again there is a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(S_{\mathrm{et}}, V) \longrightarrow \mathrm{H}^i(S, \mathcal{F}) \xrightarrow{1-\tau_s} \mathrm{H}^i(S, \mathcal{F}) \longrightarrow \cdots$$

relating the étale cohomology of S with coefficients in V to the Zariski cohomology of the quasi-coherent $\mathcal{O}_S$-module $\mathcal{F}$. This generalizes the long exact sequence (2) to 'twisted' coefficients.

Böckle-Pink and constructible coefficients. A natural problem is now to extend Katz's theorem from locally constant to *constructible* $\mathbf{F}_p$-modules on S_{et}. A strikingly elegant answer was provided by Gebhard Böckle and Richard Pink, in their monograph [**11**]. Let S be a noetherian scheme over $\mathbf{F}_p$. Consider the category $\mathbf{Coh}_\tau\, S$ of pairs $(\mathcal{F}, \tau)$ consisting of a coherent $\mathcal{O}_S$-module $\mathcal{F}$ (not necessarily locally free) and an $\mathcal{O}_S$-linear map $\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}$ (not necessarily an isomorphism). Such a pair $(\mathcal{F}, \tau)$ defines a constructible $\mathbf{F}_p$-module V on S_{et} by the short exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{F} \xrightarrow{1-\tau_s} \mathcal{F} \longrightarrow 0.$$

The resulting functor from $\mathbf{Coh}_\tau\, S$ to the category of constructible $\mathbf{F}_p$-modules is not an equivalence. Indeed, if τ_s is nilpotent, then $1 - \tau_s$ will be an isomorphism and hence $(\mathcal{F}, \tau)$ will be mapped to the zero sheaf. Böckle and Pink prove that the full subcategory consisting of pairs $(\mathcal{F}, \tau)$ with τ_s nilpotent is a thick (or 'Serre') subcategory. They define the category $\mathbf{Crys}\, X$ of Crystals on X as the quotient category, and show that $\mathbf{Crys}\, X$ is equivalent with the category of constructible $\mathbf{F}_p$-modules on S_{et}. They moreover construct functors $f^\star$, $f_!$ and $\otimes$ between categories of crystals, compatible with the corresponding functors between categories of constructible $\mathbf{F}_p$-modules.

A different 'quasi-coherent' description of the category of constructible $\mathbf{F}_p$-modules is due to Emerton and Kisin [**19, 20**].

Sheaves and functions. Let X be a scheme of finite type over $\mathbf{F}_q$, and $\mathcal{F}$ a constructible ℓ-adic sheaf on X. For every $x \in X(\mathbf{F}_q)$ the sheaf $x^\star\mathcal{F}$ on $(\operatorname{Spec}\mathbf{F}_q)_{\mathrm{et}}$ is a $\mathbf{Q}_\ell$-vector space equipped with a Frobenius endomorphism Frob. Taking traces, we obtain a function

$$\mathrm{tr}_{\mathcal{F}}\colon X(\mathbf{F}_q) \to \mathbf{Q}_\ell,\, x \mapsto \mathrm{tr}_{\mathbf{Q}_\ell}\left(\mathrm{Frob} \mid x^\star\mathcal{F}\right),$$

called the *trace function* of $\mathcal{F}$. The "dictionnaire faisceaux-fonctions" of Grothendieck and Deligne expresses the effect of various functors applied to constructible ℓ-adic sheaves on their trace functions, see [**39**, §1]. We just give two important examples. If $\mathcal{F}$ and $\mathcal{G}$ are constructible ℓ-adic sheaves on X then

$$\mathrm{tr}_{\mathcal{F}\otimes\mathcal{G}}\, x = (\mathrm{tr}_{\mathcal{F}}\, x)\cdot(\mathrm{tr}_{\mathcal{G}}\, x)$$

for all $x \in X(\mathbf{F}_q)$. If $f\colon X \to Y$ is a proper map between schemes of finite type over $\mathbf{F}_q$ then we have

$$\sum_n (-1)^n\, \mathrm{tr}_{\mathrm{R}^n f_\star\mathcal{F}}\, y = \sum_{\substack{x\in X(\mathbf{F}_q)\\ f(x)=y}} \mathrm{tr}_{\mathcal{F}}\, x$$

by the Lefschetz trace formula, and more generally, for a separated $f\colon X \to Y$, provided one replaces $\mathrm{R}^n f_\star$ by $\mathrm{R}^n f_!$. The dictionary constitutes a powerful tool for proving combinatorial identities between characteristic-zero valued functions on $\mathbf{F}_q$-points of varieties over $\mathbf{F}_q$ using ℓ-adic cohomology.

Deligne has similarly shown that the Lefschetz trace formula also holds for étale cohomology with mod p coefficients [**16**, pp. 125–128]. The proof is based on the exact sequence (3), and on the Woods Hole trace formula in coherent cohomology [**27**, III.6.12]. In fact, the theory of of Böckle and Pink now gives a full "sheaves-functions dictonary", translating between cohomological constructions with coherent sheaves equipped with a Frobenius endomorphism and combinatorics of $\mathbf{F}_q$-valued functions on $\mathbf{F}_q$-points of varieties over $\mathbf{F}_q$. This formalism no longer refers to the étale site, and all statements and proofs can be given in terms of coherent sheaves.

The present lecture notes. The present notes constitute a slightly expanded and more polished version of a series of eight lectures given at the Morningside Center for Mathematics in Beijing in October 2013.

Starting from scratch, we explain the theory of crystals of Böckle and Pink, and how it leads to a sheaves-functions dictionary, translating back and forward between the combinatorics of $\mathbf{F}_q$-valued functions on rational points on varieties over $\mathbf{F}_q$ and the cohomology of coherent sheaves equipped with a Frobenius endomorphism. We illustrate the power of this formalism with a series of applications, ranging from classical results on oscillating sums, and zeta functions modulo p to recent results on special values of characteristic p-valued L-functions.

In *Chapters 1 and 2* we expose part of the theory of crystals of Böckle and Pink. By restricting to $\mathbf{F}_q$-coefficients, by various finiteness assumptions, and by using the theorem of formal functions to give a short new proof of proper base change, we are able to keep the necessary prerequisites to a minimum, and to condense the fundamentals into a relatively concise account.

Chapter 3 contains the central result of these notes: the trace formula for crystals, and the resulting sheaves-functions dictionary. The statement reduces quickly to the special case of projective variety over a finite field. Rather than deducing it from the Woods Hole trace formula of SGA 5, we follow Fulton's very elegant and elementary proof [**22**] to settle this case (filling in a gap in the original argument along the way). By passing to Grothendieck groups of crystals, we both avoid the use of derived categories, and streamline the exposition. *Chapter 4* gives some elementary applications of the trace formula, and *Chapter 5* generalizes the trace formula to crystals with coefficients in various $\mathbf{F}_q$-algebras. Rather than developing the theory with coefficients right from the start we have opted to postpone the introduction of coefficients until Chapter 5, and deducing the general results from their special cases treated in the first three chapters.

We hope our gradual approach in chapters 1–5 will be valued by those who wish to learn to use the sheaves-functions dictionary, but may be intimidated by the large edifice of the full theory of Böckle and Pink.

Chapter 6 computes the cohomology of the "external" symmetric powers of a coherent sheaf on a curve. These are coherent sheaves on the symmetric powers of the curve. In principle this is a special case of a much more general result of Deligne, which expresses the coherent cohomology of symmetric powers on higher-dimensional varieties using simplicial techniques going back to Dold and Puppe. By restricting to dimension 1, we manage to avoid simplicial machinery and obtain a completely explicit statement with a relatively elementary proof. Our proof uses Čech cohomology and Koszul resolutions. The main result in this chapter does not involve Frobenius and holds in arbitrary characteristic. Since it may be of independent interest, care has taken that it can be read independently of the preceding chapters.

In *Chapter 7* we apply the results of Chapter 6 to prove an L-function version of the trace formula of Chapter 5. Since we work with p-torsion coefficients, the charactersitic polynomial of an endomorphism is not determined by the traces of its powers, and we cannot rely on the usual tricks to simply reduce the L-function version to the trace formula for powers of the Frobenius. Rather, we closely mimic Deligne's approach in SGA 4 and SGA 4.5 and use symmetric powers to reduce to the trace formula. A completely different proof is given in Böckle-Pink, based on Serre duality and Anderson's "elementary approach".

We end in *Chapter 8* with an application of the obtained results. We use the main theorem of Chapter 7 to compute special values of L-functions, in particular values of Goss zeta functions at negative integers. The principal result is a generalization of a recent theorem of V. Lafforgue. Under a certain semi-simplicity hypothesis, it expresses special values in terms of extension groups of crystals. It is a characteristic p valued analogue of conjectures and results by K. Kato and Milne and Ramachandran. We end with a simple example showing that the semi-simplicity hypothesis is not always verified. This is contrary to the classical setting of ℓ-adic representations coming from smooth projective varieties over finite fields, where semi-simplicity is conjectured to hold in general.

The appendix gives a self-contained proof of the Woods Hole trace formula for a transversal endomorphism of a proper smooth scheme over a field, using Grothendieck-Serre duality. This is logically independent of the rest of these notes, as these give an independent proof, due to Fulton, for the Frobenius endomorphism. However, since the only published proof of this more general trace formula [**27**, III.6] is rather convoluted, we have decided to include a simpler proof in these notes.

Prerequisites and organization. Although many of the results are closely related to the formalism of étale constructible sheaves, there is no logical dependency, and we do not assume that the reader is familiar with the étale theory. The only prerequisite is familiarity with coherent cohomology at the level of Chapter 3 of Hartshorne [**31**], except for the Leray spectral sequence. We do not make use of derived categories (except for in the appendix, where we need them to state Grothendieck-Serre duality), although throughout the text we retain some of their power and flexibility by an extensive use of Grothendieck groups "$\mathrm{K}_0(-)$".

The *Stacks Project* of Johan de Jong and his collaborators [**45**] is rapidly becoming one the most clear, complete and precise references for the foundations of modern algebraic geometry, and we refer to it extensively.

At the end of each chapter are short sections called 'Notes' and 'Exercises'. The former contains historic remarks, comments on some more advanced topics, and references to the literature. In particular, rather than attributing every single lemma and proposition locally, we indicate the origin of the results here. The exercises are of widely varying level of difficulty. Those that require more background are marked with a ($\star$).

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CHAPTER 1

τ-sheaves, crystals, and their trace functions

We fix a finite field $\mathbf{F}_q$ with q elements. Let X be a scheme over $\mathbf{F}_q$. Denote by $\sigma\colon X \to X$ the Frobenius endomorphism which is the identity on the underlying topological space and is given on functions by

$$\mathcal{O}_X(U) \to \mathcal{O}_X(U),\ r \mapsto r^q.$$

It is a morphism of $\mathbf{F}_q$-schemes.

1. Coherent τ-sheaves

Let X be a scheme over $\mathbf{F}_q$.

Definition 1.1. A *τ-sheaf on X* is a pair $(\mathcal{F}, \tau)$ consisting of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a morphism of $\mathcal{O}_X$-modules

$$\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}.$$

A *morphism of τ-sheaves* $\varphi\colon (\mathcal{F}_1, \tau_1) \to (\mathcal{F}_2, \tau_2)$ is a morphism $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ of $\mathcal{O}_X$-modules such that the square

$$\begin{array}{ccc} \sigma^\star\mathcal{F}_1 & \xrightarrow{\tau_1} & \mathcal{F}_1 \\ \downarrow{\scriptstyle \sigma^\star\varphi} & & \downarrow{\scriptstyle \varphi} \\ \sigma^\star\mathcal{F}_2 & \xrightarrow{\tau_2} & \mathcal{F}_2 \end{array}$$

commutes. The category of τ-sheaves on X is denoted $\mathbf{QCoh}_\tau\, X$.

We will often write $\mathcal{F}$ for the τ-sheaf $(\mathcal{F}, \tau)$, and $\tau_\mathcal{F}$ for the map τ.

Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups. We say that α is q-linear if $\alpha(rs) = r^q\alpha(s)$ for all local sections r of $\mathcal{O}_X$ and s of $\mathcal{F}$.

Proposition 1.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Then the following sets are in natural bijection:*

(1) $\{\tau\colon \sigma^\star\mathcal{F} \to \mathcal{G} \mid \tau$ *is $\mathcal{O}_X$-linear*$\}$,

(2) $\{\tau_a\colon \mathcal{F} \to \sigma_\star\mathcal{G} \mid \tau_a$ *is* $\mathcal{O}_X$*-linear*$\}$,
(3) $\{\tau_s\colon \mathcal{F} \to \mathcal{G} \mid \tau_s$ *is* q*-linear*$\}$.

The subscript a stands for *adjoint*, the s for *semi-linear*.

Proof. By adjunction we have $\mathrm{Hom}_{\mathcal{O}_X}(\sigma^\star\mathcal{F},\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\sigma_\star\mathcal{G})$.

Since σ is the identity on the topological space X we have a canonical isomorphism

$$\alpha\colon \sigma_\star\mathcal{G} \to \mathcal{G}$$

as sheaves of abelian groups. As a map of $\mathcal{O}_X$-modules it is q-linear. The map

$$\tau_s \mapsto \tau_a := \alpha\tau_s$$

gives the bijection between the second and third sets of maps in the proposition. □

Definition 1.3. Assume X is noetherian.* A τ-sheaf $(\mathcal{F},\tau)$ on X is called *coherent* if the underlying $\mathcal{O}_X$-module $\mathcal{F}$ is coherent. A morphism of coherent τ-sheaves is a morphism of τ-sheaves. The category of coherent τ-sheaves on X is denoted $\mathbf{Coh}_\tau\, X$. If R is an $\mathbf{F}_q$-algebra then we will often write $\mathbf{Coh}_\tau\, R$ in stead of $\mathbf{Coh}_\tau\,\mathrm{Spec}\, R$.

Example 1.4. Let $X = \mathrm{Spec}\, R$ for some $\mathbf{F}_q$-algebra R. Let $\mathcal{F}$ be the quasi-coherent $\mathcal{O}_X$-module corresponding to an R-module M. Then $\sigma^\star\mathcal{F}$ corresponds to the R-module

$$R \otimes_{\sigma,R} M,$$

with R-module structure coming from the left factor, and where σ denotes the map $R \to R, r \mapsto r^q$. To give $\mathcal{F}$ the structure of a τ-sheaf is therefore the same as giving an R-linear map

$$\tau\colon R \otimes_{\sigma,R} M \to M.$$

The induced q-linear map τ_s becomes on global sections the map

$$\tau_s\colon M \to M,\ m \mapsto \tau(1 \otimes m)$$

satisfying $\tau_s(rm) = r^q\tau_s(m)$. Conversely, any such map determines a map $\mathcal{F} \to \sigma_\star\mathcal{F}$ of quasi-coherent $\mathcal{O}_X$-modules, and therefore the structure of a τ-sheaf on $\mathcal{F}$.

If R is noetherian then the τ-sheaf $\mathcal{F}$ is coherent if and only if M is a finitely generated R-module.

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*For simplicity, we will restrict ourselves to noetherian schemes whenever dealing with *coherent* $\mathcal{O}_X$-modules. With the necessary care many of the results in this text could be extended to cover τ-sheaves over more general schemes.

Let $f\colon X \to Y$ be a morphism of schemes over $\mathbf{F}_q$. Then $\sigma_Y \circ f = f \circ \sigma_X$. In particular, for a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{F}$ we have a canonical isomorphism

$$\sigma_X^\star f^\star \mathcal{F} \longrightarrow f^\star \sigma_Y^\star \mathcal{F}.$$

Definition 1.5. Let $\mathcal{F} = (\mathcal{F}, \tau)$ be a τ-sheaf on Y. The *pullback* or *base change* $f^\star \mathcal{F}$ of $\mathcal{F}$ along f is the τ-sheaf $(f^\star \mathcal{F}, \tau_{f^\star \mathcal{F}})$, where $\tau_{f^\star \mathcal{F}}$ is the composition

$$\begin{array}{ccc} \sigma_X^\star f^\star \mathcal{F} & \xrightarrow{\ \sim\ } & f^\star \sigma_Y^\star \mathcal{F} \\ & \searrow^{\tau_{f^\star\mathcal{F}}} & \downarrow \\ & & f^\star \mathcal{F}. \end{array}$$

Pullback defines a functor $f^\star\colon \mathbf{QCoh}_\tau\, Y \to \mathbf{QCoh}_\tau\, X$.

Example 1.6. Assume $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} S$ and $f\colon X \to Y$ induced by an $\mathbf{F}_q$-algebra homomorphism $R \to S$. Assume that $\mathcal{F}$ corresponds to an S-module M equipped with a q-linear $\tau_s\colon M \to M$. The map

$$\tau_s'\colon S \otimes_R M \to S \otimes_R M,\ s \otimes m \mapsto s^q \otimes \tau_s(m)$$

is well-defined and q-linear. The pair $(S \otimes_R M,\ \tau_s)$ corresponds to the pull-back $f^\star \mathcal{F}$.

Proposition 1.7. *Let $f\colon X \to Y$ be a morphism of noetherian schemes over $\mathbf{F}_q$. Let $\mathcal{F}$ be a coherent τ-sheaf on Y. Then $f^\star \mathcal{F}$ is a coherent τ-sheaf.*

Proof. Since X is noetherian, $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module [**28**, I.1.6.1] [**45**, Tag 01XY], and therefore the pull-back of any coherent $\mathcal{O}_Y$-module is a coherent $\mathcal{O}_X$-module [**28**, 0.5.3.11] [**45**, Tag 01BM]. □

Proposition 1.8. *Let X be a scheme over $\mathbf{F}_q$. The category $\mathbf{QCoh}_\tau\, X$ is abelian. If X is noetherian then also $\mathbf{Coh}_\tau\, X$ is abelian.*

Proof. Clearly the categories are additive. We need to show that they satisfy

(AB1) Every morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ has a kernel and cokernel,

(AB2) For every morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ the natural map

$$\operatorname{coker}(\ker \varphi \to \mathcal{F}) \to \ker(\mathcal{G} \to \operatorname{coker} \varphi)$$

is an isomorphism.

Since σ is the identity on the underlying topological space of X the functor $\sigma_\star$ on quasi-coherent $\mathcal{O}_X$-modules is exact. In particular, a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of quasi-coherent τ-sheaves induces a commutative diagram of $\mathcal{O}_X$-modules with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \ker\varphi & \longrightarrow & \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G} & \longrightarrow & \operatorname{coker}\varphi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau_a} & & \downarrow{\scriptstyle \tau_a} & & \downarrow{\scriptstyle \tau_a} & & \downarrow{\scriptstyle \tau_a} & & \\
0 & \longrightarrow & \sigma_\star \ker\varphi & \longrightarrow & \sigma_\star\mathcal{F} & \xrightarrow{\sigma_\star\varphi} & \sigma_\star\mathcal{G} & \longrightarrow & \sigma_\star \operatorname{coker}\varphi & \longrightarrow & 0
\end{array}$$

One directly verifies that $(\ker\varphi, \tau_a)$ and $(\operatorname{coker}\varphi, \tau_a)$ determine a kernel respectively cokernel of the morphism φ in $\mathbf{QCoh}_\tau\, X$. Property AB2 is inherited by the same property for the category of quasi-coherent $\mathcal{O}_X$-modules, hence $\mathbf{QCoh}_\tau\, X$ is is abelian.

The coherent $\mathcal{O}_X$-modules form an abelian subcategory of the category of quasi-coherent $\mathcal{O}_X$-modules and the same arguments as above show that $\mathbf{Coh}_\tau\, X$ satisfies AB1 and AB2. $\square$

In the proof we have used the adjoint maps τ_a to produce kernels and cokernels. The main advantage is that $\sigma_\star$ is an exact functor. In general the functor $\sigma^\star$ is not exact[†] and it takes a bit more work to construct kernels and cokernels directly in terms of the maps $\sigma^\star\mathcal{F} \to \mathcal{F}$. Let us, as an example, describe in detail the kernel of a map $\varphi\colon (\mathcal{F}, \tau_\mathcal{F}) \to (\mathcal{G}, \tau_\mathcal{G})$ of τ-sheaves on X. Let $\mathcal{H}$ be the $\mathcal{O}_X$-module which is the kernel of $\varphi\colon \mathcal{F} \to \mathcal{G}$. Consider the commutative diagram of $\mathcal{O}_X$-modules

$$\begin{array}{ccccccc}
 & & \sigma^\star\mathcal{H} & \longrightarrow & \sigma^\star\mathcal{F} & \xrightarrow{\sigma^\star\varphi} & \sigma^\star\mathcal{G} \\
 & & {\scriptstyle \tau_\mathcal{H}}\,\vdots & \searrow & \downarrow{\scriptstyle \tau_\mathcal{F}} & & \downarrow{\scriptstyle \tau_\mathcal{G}} \\
0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F} & \xrightarrow{\varphi} & \mathcal{G},
\end{array}$$

where the bottom row is exact. The map $\sigma^\star\mathcal{H} \to \sigma^\star\mathcal{G}$ is the pullback along σ of the map $\mathcal{H} \to \mathcal{G}$, and hence it is the zero map. It follows that the map $\sigma^\star\mathcal{H} \to \mathcal{F}$ factors over a unique map $\tau_\mathcal{H}\colon \sigma^\star\mathcal{H} \to \mathcal{H}$. The pair $(\mathcal{H}, \tau_\mathcal{H})$ is the kernel of φ in $\mathbf{QCoh}_\tau\, X$. If $\mathcal{F}$ and $\mathcal{G}$ are coherent then so is $\mathcal{H}$ and then $(\mathcal{H}, \tau_\mathcal{H})$ is also the kernel of φ in $\mathbf{Coh}_\tau\, X$.

[†]For example, if $X = \operatorname{Spec} R$ for a noetherian local ring R then the functor $\sigma^\star$ on quasi-coherent $\mathcal{O}_X$-modules is exact if and only if R is regular, see **[37]**. See also Proposition 8.9.

2. The trace function of a coherent τ-sheaf

Let X be a scheme of finite type[‡] over $\mathbf{F}_q$ and $(\mathcal{F},\tau)$ a coherent τ-sheaf on X. For a point $x \in X(\mathbf{F}_q)$ the fiber $x^\star\mathcal{F}$ of $\mathcal{F}$ is a finite-dimensional $\mathbf{F}_q$-vector space and τ_s induces a *linear* endomorphism of $x^\star\mathcal{F}$.

Definition 1.9. The *trace function* of a coherent τ-sheaf $\mathcal{F}$ is the function

$$\operatorname{tr}_{\mathcal{F}}\colon X(\mathbf{F}_q) \to \mathbf{F}_q,\ x \mapsto \operatorname{tr}_{\mathcal{F}} x := \operatorname{tr}_{\mathbf{F}_q}(\,\tau_s \mid x^\star\mathcal{F}\,).$$

Example 1.10. Let X be a scheme of finite type over $\mathbf{F}_q$. The q-linear map $\tau_a\colon \mathcal{O}_X \to \mathcal{O}_X$ that maps a local section r to r^q defines a coherent τ-sheaf $\mathbf{1}_X = (\mathcal{O}_X,\tau)$. We have $\operatorname{tr}_{\mathbf{1}} x = 1$ for all $x \in X(\mathbf{F}_q)$.

Example 1.11. Let X be a scheme of finite type over $\mathbf{F}_q$. Let $f \in \mathcal{O}_X(X)$. There is a natural isomorphism

$$\sigma^\star\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X$$

given on an affine $\operatorname{Spec} R \subset X$ by

$$R \otimes_{\sigma,R} R \to R,\ r \otimes s \mapsto rs^q.$$

Consider the coherent τ-sheaf $\mathcal{F} = (\mathcal{F},\tau)$ with τ being the composition

$$\sigma^\star\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \xrightarrow{f} \mathcal{O}_X.$$

Alternatively, $\mathcal{F}$ is given by the q-linear map $\tau_s\colon \mathcal{F} \to \mathcal{F}$ given on sections by $s \mapsto fs^q$.We have for all $x \in X(\mathbf{F}_q)$ that

$$\operatorname{tr}_{\mathcal{F}} x = f(x)$$

holds in $\mathbf{F}_q$.

Example 1.12. Consider the Serre twisting sheaves $\mathcal{O}(n)$ on projective space $\mathbf{P}^d = \operatorname{Proj} \mathbf{F}_q[x_0,\ldots,x_d]$ over $\mathbf{F}_q$. We have a natural isomorphism of $\mathcal{O}_{\mathbf{P}^d}$-modules

$$\sigma^\star\mathcal{O}(n) \xrightarrow{\sim} \mathcal{O}(qn).$$

Its q-linear counterpart is the map $\mathcal{O}(n) \to \mathcal{O}(qn)$ which on local sections is given by $r \mapsto r^q$. Now let $n \geq 0$ and let $f \in \mathbf{F}_q[x_0,\ldots,x_d]$

[‡]When considering the trace functions of coherent τ-sheaves we will always restrict to schemes of finite type over $\mathbf{F}_q$.

be homogeneous of degree $(q-1)n$. Then f is a global section of $\mathcal{O}((q-1)n)$, so it defines a map

$$\mathcal{O}(-qn) \to \mathcal{O}(-n)$$

and hence a map

$$\tau\colon \sigma^\star \mathcal{O}(-n) \to \mathcal{O}(-n).$$

The trace function of the τ-sheaf $\mathcal{F} = (\mathcal{O}(-n), \tau)$ is given by

$$\operatorname{tr}_{\mathcal{F}}\colon \mathbf{P}^d(\mathbf{F}_q) \to \mathbf{F}_q,\ (x_0 : \cdots : x_d) \mapsto f(x_0, \ldots, x_d).$$

Note that this is well-defined: since the degree of f is divisible by $q-1$, the value $f(x_0, \ldots, x_d)$ is invariant under scaling $(x_0, \ldots, x_d)$ by an element of $\mathbf{F}_q^\times$.

Example 1.13. Let X be a scheme of finite type and let $x \in X(\mathbf{F}_q)$. Let $\kappa(x)$ be the sky-scraper sheaf at x, with stalk $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbf{F}_q$. Then there is a canonical isomorphism $\tau\colon \sigma^\star\kappa(x) \to \kappa(x)$. The resulting coherent τ-sheaf $\kappa(x) = (\kappa(x), \tau)$ has trace function

$$\operatorname{tr}_{\kappa(x)} y = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

3. Nilpotent coherent τ-sheaves

Let X be a noetherian scheme over $\mathbf{F}_q$.

Definition 1.14. A coherent τ-sheaf $(\mathcal{F}, \tau)$ on X is called *nilpotent* if there is an $n > 0$ so that the composition

$$(\sigma^n)^\star\mathcal{F} \longrightarrow \cdots \longrightarrow (\sigma^2)^\star\mathcal{F} \xrightarrow{\sigma^\star\tau} \sigma^\star\mathcal{F} \xrightarrow{\tau} \mathcal{F}$$

is the zero map.

Equivalently $(\mathcal{F}, \tau)$ is nilpotent if there is an n such that $\tau_s^n = 0$.

Proposition 1.15. *Let $f\colon X \to Y$ be a morphism of noetherian schemes over $\mathbf{F}_q$. Let $\mathcal{N}$ be a nilpotent coherent τ-sheaf on Y. Then $f^\star\mathcal{N}$ is nilpotent.* □

Corollary 1.16. *If $\mathcal{N}$ is a nilpotent coherent τ-sheaf on X then* $\operatorname{tr}_{\mathcal{N}} = 0$. □

PROPOSITION 1.17. *Let X be a noetherian scheme over $\mathbf{F}_q$ and let*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent τ-sheaves on X. Then $\mathcal{F}_2$ is nilpotent if and only if $\mathcal{F}_1$ and $\mathcal{F}_3$ are nilpotent.

PROOF. Clearly if $\mathcal{F}_2$ is nilpotent then so are $\mathcal{F}_3$ and $\mathcal{F}_1$. So assume $\mathcal{F}_1$ and $\mathcal{F}_3$ are nilpotent. By assumption, there are n_1 and n_3 so that $\tau_{1,s}^{n_1} = \tau_{3,s}^{n_3} = 0$. We claim that $\tau_{2,s}^{n_1+n_3} = 0$. Indeed, let U be an open subset of X. We have an exact sequence

$$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$$

of abelian groups. Let $s \in \mathcal{F}_2(U)$. Then $\tau_{2,s}^{n_3} s$ vanishes in $\mathcal{F}_3(U)$, so that $\tau_{2,s}^{n_3} s \in \mathcal{F}_1(U)$ and hence $\tau_{2,s}^{n_1+n_3} s = 0$. □

We call a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of coherent τ-sheaves a *nil-isomorphism* if both kernel and cokernel of φ are nilpotent.

PROPOSITION 1.18. *Let X be a noetherian scheme over $\mathbf{F}_q$, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ and $\psi\colon \mathcal{G} \to \mathcal{H}$ be nil-isomorphisms in $\mathbf{Coh}_\tau X$. Then also the composition $\psi\varphi$ is a nil-isomorphism.*

PROOF. From the exact sequence

$$0 \to \ker \varphi \to \ker \psi\varphi \to \ker \psi$$

we see that $\ker \psi\varphi$ is an extension of a subobject of a nilpotent τ-sheaf by a nilpotent τ-sheaf, and hence a nilpotent τ-sheaf. Similarly, we learn from the exact sequence

$$\operatorname{coker} \varphi \to \operatorname{coker} \psi\varphi \to \operatorname{coker} \psi \to 0$$

that $\operatorname{coker} \psi\varphi$ is nilpotent and we conclude that $\psi\varphi$ is a nil-isomorphism. □

4. Crystals

Let X be a noetherian scheme over $\mathbf{F}_q$. Proposition 1.17 says the full subcategory $\mathbf{Nil}\, X$ of nilpotent τ-sheaves in $\mathbf{Coh}_\tau X$ is a Serre subcategory[§]. This means that one can form the quotient category

$$\mathbf{Crys}\, X := \frac{\mathbf{Coh}_\tau X}{\mathbf{Nil}\, X}.$$

[§]A Serre subcategory is also called a *thick* or *épaisse* subcategory.

We call it the category of *crystals* on X. This is a purely formal operation on abelian categories, and we refer to [**23**, Chap. III], [**51**, §10.3] and [**45**, Tag 02MN] for details and proofs of the statements below.

The objects of $\mathbf{Crys}\, X$ are the same as the objects of $\mathbf{Coh}_\tau\, X$. The category $\mathbf{Crys}\, X$ is abelian, and comes equipped with an exact functor $\mathbf{Coh}_\tau\, X \to \mathbf{Crys}\, X$, which on the level of objects is just the identity map. The maps

$$\mathrm{Hom}_{\mathbf{Coh}_\tau X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathbf{Crys}\, X}(\mathcal{F}, \mathcal{G}),$$

however, are typically not bijective.

PROPOSITION 1.19. *Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathbf{Coh}_\tau\, X$. Then f is a nil-isomorphism if and only if its image in $\mathbf{Crys}\, X$ is an isomorphism.* □

Applying this to the unique map $0 \to \mathcal{F}$ we find the following corollary.

COROLLARY 1.20. *Let $\mathcal{F}$ be a coherent τ-sheaf. Then $\mathcal{F}$ is nilpotent if and only if its image in $\mathbf{Crys}\, X$ is isomorphic to* 0. □

The functor $\mathbf{Coh}_\tau\, X \to \mathbf{Crys}\, X$ can be characterized by two universal properties, which we give in the two theorems below.

THEOREM 1.21. *Let X be a noetherian scheme over $\mathbf{F}_q$. Let $\mathcal{C}$ be a category. Let $F\colon \mathbf{Coh}_\tau\, X \to \mathcal{C}$ be a functor that maps every nil-isomorphism to an isomorphism. Then there exists a unique functor $\mathbf{Crys}\, X \to \mathcal{C}$ so that the diagram*

$$\begin{array}{ccc} \mathbf{Coh}_\tau\, X & \xrightarrow{F} & \mathcal{C} \\ \downarrow & \nearrow & \\ \mathbf{Crys}\, X & & \end{array}$$

commutes. □

THEOREM 1.22. *Let X be a noetherian scheme over $\mathbf{F}_q$. Let $\mathcal{A}$ be an abelian category and let $F\colon \mathbf{Coh}_\tau\, X \to \mathcal{A}$ be an exact functor that maps nilpotent τ-sheaves to zero. Then there exists a unique additive* *functor* $\mathbf{Crys}\, X \to \mathcal{A}$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{Coh}_\tau\, X & \xrightarrow{F} & \mathcal{A} \\ \downarrow & \nearrow & \\ \mathbf{Crys}\, X & & \end{array}$$

commutes. □

Not every map in $\mathbf{Crys}\, X$ comes from a map in $\mathbf{Coh}_\tau\, X$. The following proposition gives a way to represent arbitrary maps of crystals.

PROPOSITION 1.23. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be coherent* τ*-sheaves. Let* $\varphi\colon \mathcal{F} \to \mathcal{G}$ *be a morphism of crystals. Then there exist sub-*τ*-sheaves* $\mathcal{H} \subset \mathcal{F}$ *and* $\mathcal{N} \subset \mathcal{G}$*, and a diagram of* τ*-sheaves*

$$\mathcal{F} \overset{i}{\hookleftarrow} \mathcal{H} \xrightarrow{\tilde{\varphi}} \mathcal{G}/\mathcal{N} \overset{p}{\twoheadleftarrow} \mathcal{G} \tag{4}$$

with $\mathcal{F}/\mathcal{H}$ *and* $\mathcal{N}$ *nilpotent, so that* φ *is the composite morphism*

$$\mathcal{F} \xrightarrow{i^{-1}} \mathcal{H} \xrightarrow{\tilde{\varphi}} \mathcal{G}/\mathcal{N} \xrightarrow{p^{-1}} \mathcal{G}.$$

in $\mathbf{Crys}\, X$*.* □

PROPOSITION 1.24. *Let* $0 \to \mathcal{F} \to \mathcal{G} \to 0$ *be a short exact sequence in* $\mathbf{Crys}\, X$*. Then there exists a short exact sequence* $0 \to \mathcal{F}' \to \mathcal{G}' \to \mathcal{H}' \to 0$ *in* $\mathbf{Coh}_\tau\, X$ *that becomes isomorphic to the given one in* $\mathbf{Crys}\, X$*.*

PROOF. By the preceding proposition there are coherent τ-sheaves $\mathcal{G}'$ and $\mathcal{H}'$ nil-isomorphic to $\mathcal{G}$ and $\mathcal{H}$ respectively, and a map $\mathcal{G}' \to \mathcal{H}'$ of coherent τ-sheaves representing the given $\mathcal{G} \to \mathcal{H}$ in $\mathbf{Crys}\, X$. Replacing $\mathcal{H}'$ by the image of $\mathcal{G}' \to \mathcal{H}'$ we may even assume this map to be surjective. If we put $\mathcal{F}' := \ker(\mathcal{G}' \to \mathcal{H}')$ we find a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{G}' \to \mathcal{H}' \to 0$$

in $\mathbf{Coh}_\tau\, X$ representing the given short exact sequence in $\mathbf{Crys}\, X$. □

LEMMA 1.25. *Assume that* X *is of finite type over* $\mathbf{F}_q$*. Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be coherent* τ*-sheaves on* X *that are isomorphic in* $\mathbf{Crys}\, X$*. Then* $\mathrm{tr}_\mathcal{F} = \mathrm{tr}_\mathcal{G}$*.* □

In particular, the *trace function* $\mathrm{tr}_\mathcal{F}$ of a *crystal* $\mathcal{F}$ is well-defined. The implication in the lemma can not be reversed: There are certainly many non-isomorphic crystals giving rise to the same trace function (e.g. $X(\mathbf{F}_q)$ could be empty). See Exercise 7.3 for a positive statement.

5. Pointwise criteria

In this section we show that various properties of crystals and maps between crystals can be checked point by point.

LEMMA 1.26. *Let X be a noetherian scheme over $\mathbf{F}_q$, let K be a field containing $\mathbf{F}_q$ and let $i\colon \operatorname{Spec} K \to X$ be a map over $\operatorname{Spec} \mathbf{F}_q$. Then the functor*

$$\mathbf{Coh}_\tau X \to \mathbf{Crys}\, K,\ \mathcal{F} \mapsto i^\star \mathcal{F}$$

is exact.

PROOF. Let $x \in X$ be the image of i. The functor factors as

$$\mathbf{Coh}_\tau X \to \mathbf{Coh}_\tau \mathcal{O}_{X,x} \to \mathbf{Coh}_\tau K \to \mathbf{Crys}\, K.$$

The first and the last of these are exact, and the middle functor is right exact. So it suffices to show that monomorphisms in $\mathbf{Coh}_\tau \mathcal{O}_{X,x}$ are mapped to monomorphisms in $\mathbf{Crys}\, K$.

Let $\mathcal{F} \subset \mathcal{G}$ be an inclusion of coherent τ-sheaves on $\operatorname{Spec} \mathcal{O}_{X,x}$, corresponding to an inclusion $N \subset M$ of $\mathcal{O}_{X,x}$-modules equipped with a q-linear endomorphism τ_s. Let $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideal. Since M is a finitely generated $\mathcal{O}_{X,x}$-module we have

$$\bigcap_{n \geq 0} \mathfrak{m}^{q^n} M = 0.$$

Since $N/\mathfrak{m}N$ is finite-dimensional over the residue field $k(x)$ there is an $n > 0$ such that

$$N \cap \mathfrak{m}^{q^n} M \subset \mathfrak{m}N.$$

and hence

$$\tau_s^n(\mathfrak{m}M \cap N) \subset \mathfrak{m}N.$$

It follows that $\tau_s^n = 0$ on the kernel of $N/\mathfrak{m}N \to M/\mathfrak{m}M$. After flat base change from $k(x)$ to K, we see that $\tau_s^n = 0$ on the kernel of $i^\star\mathcal{F} \to i^\star\mathcal{G}$, so that the latter map is a monomorphism in $\mathbf{Crys}\, K$, as we had to show. □

Since the pull-back functor from $\mathbf{Coh}_\tau X$ to $\mathbf{Crys}\, K$ is exact and maps nilpotent objects to the zero crystal we obtain by the universal property of Theorem 1.22 the following corollary.

COROLLARY 1.27. *The functor* $\mathbf{Coh}_\tau X \to \mathbf{Crys}\, K$, $\mathcal{F} \mapsto i^\star \mathcal{F}$ *factors over a unique functor*

$$i^\star \colon \mathbf{Crys}\, X \to \mathbf{Crys}\, K,$$

and this functor is exact. □

In the next chapter we will generalize this to pullback along an arbitrary map of noetherian schemes over $\mathbf{F}_q$.

We denote the set of closed points of a scheme X by $|X|$.

PROPOSITION 1.28. *Let* X *be a noetherian scheme over* $\mathbf{F}_q$ *and let* $\mathcal{F}$ *be a coherent* τ*-sheaf on* X*. Then the following are equivalent.*

(1) $\mathcal{F}$ *is nilpotent;*
(2) $\iota_x^\star \mathcal{F}$ *is nilpotent for all* $x \in X$.

If moreover X *is of finite type over a field* K *with algebraic closure* $\overline{K}$ *then these are also equivalent with*

(3) $\bar{x}^\star \mathcal{F}$ *is nilpotent for all* $\bar{x} \in X(\overline{K})$;
(4) $\iota_x^\star \mathcal{F}$ *is nilpotent for all* $x \in |X|$.

PROOF. By Proposition 1.15 the first statement implies the others.

Let $\mathcal{F}$ be a coherent τ-sheaf. Let n be a positive integer and consider the subset

$$Z_n = \{x \in X \mid \iota_x^\star \mathcal{F} \text{ is nilpotent of order} \leq n\}$$

of X. We have $Z_1 \subset Z_2 \subset \cdots$. Since $\tau^n \colon (\sigma^n)^\star \mathcal{F} \to \mathcal{F}$ is a map of coherent $\mathcal{O}_X$-modules we see that Z_n is closed, and since X is noetherian the sequence stabilizes: there is an n such that $Z_n = Z_{n+1} = \cdots$.

If (2) holds then $Z_n = X$ for some n. Similarly, if X if of finite type over a field K then the closed points (resp. the images of the geometric points) are dense in X so either (3) or (4) imply that there is an n such that $Z_n = X$.

So it suffices to show that $Z_n = X$ for some n implies that $\mathcal{F}$ is nilpotent. Since X is noetherian it has a finite affine open cover, and we may reduce to the case $X = \operatorname{Spec} R$. Then the coherent τ-sheaf $\mathcal{F}$ corresponds to a pair (M, τ_s). By the assumption that $Z_n = X$, for every prime ideal $\mathfrak{p} \subset R$ we have that $\tau_s^n(M) \subset \mathfrak{p}M$. In particular, $\tau_s^n(M) \subset JM$ where J is the nilradical of R. By the q-linearity of τ_s we have $\tau_s^d(JM) \subset J^{q^d}\tau_s(M)$ for all $d \geq 0$. Since R is noetherian the nilradical J is nilpotent, and we conclude that for all d sufficiently large we have $\tau_s^{n+d}(M) = \{0\}$ so that $\mathcal{F}$ is nilpotent. □

THEOREM 1.29. *Let X be a noetherian scheme over $\mathbf{F}_q$. Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be a complex in $\mathbf{Crys}\,X$. Then the following are equivalent:*

(1) *$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact in $\mathbf{Crys}\,X$;*
(2) *$\iota_x^\star\mathcal{F}_1 \to \iota_x^\star\mathcal{F}_2 \to \iota_x^\star\mathcal{F}_3$ is exact in $\mathbf{Crys}\,k(x)$ for all $x \in X$.*

If moreover X is of finite type over a field K with algebraic closure $\overline{K}$ then these are also equivalent with

(3) *$\bar{x}^\star\mathcal{F}_1 \to \bar{x}^\star\mathcal{F}_2 \to \bar{x}^\star\mathcal{F}_3$ is exact in $\mathbf{Crys}\,\overline{K}$ for all $\bar{x} \in X(\overline{K})$;*
(4) *$\iota_x^\star\mathcal{F}_1 \to \iota_x^\star\mathcal{F}_2 \to \iota_x^\star\mathcal{F}_3$ is exact in $\mathbf{Crys}\,k(x)$ for all $x \in |X|$.*

PROOF. Consider the crystal

$$\mathcal{H} := \frac{\ker \mathcal{F}_2 \to \mathcal{F}_3}{\operatorname{im} \mathcal{F}_1 \to \mathcal{F}_2}$$

on X. In other words, $\mathcal{H}$ is the cohomology at $\mathcal{F}_2$ of the complex $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$. By Corollary 1.27 we have for every field L and every $i\colon \operatorname{Spec} L \to X$ that $i^\star\mathcal{H}$ is the cohomology of the complex $i^\star\mathcal{F}_1 \to i^\star\mathcal{F}_2 \to i^\star\mathcal{F}_3$ in $\mathbf{Crys}\,L$. Since a complex is exact if and only if its cohomology vanishes, the theorem follows immediately from Proposition 1.28. □

Notes

Let X be a noetherian scheme over $\mathbf{F}_q$. Katz [**35**, §4.1] has shown that the category of locally constant $\mathbf{F}_q$-modules with finite stalks on X_{et} (“local systems”) is equivalent with the category of coherent τ-sheaves $(\mathcal{F}, \tau)$ for which $\mathcal{F}$ is locally free and τ is an isomorphism.

The notion of crystal and most of the results in this chapter are due to Böckle and Pink [**11**]. They generalize Katz’ theorem, and show that the category $\mathbf{Crys}\,X$ is equivalent with the category of constructible $\mathbf{F}_q$-modules on X_{et}. The functor can be described quite easily. To a crystal $\mathcal{F}$ one associates the sheaf of $\mathbf{F}_q$-modules $\mathcal{F}_{\text{et}}$ on X_{et} given by mapping an étale $u\colon U \to X$ to

$$\mathcal{F}_{\text{et}}(U) := \operatorname{Hom}(\mathbf{1}, u^\star\mathcal{F}),$$

where the Hom is in $\mathbf{Crys}\,U$. Then $\mathcal{F} \mapsto \mathcal{F}_{\text{et}}$ defines an equivalence from $\mathbf{Crys}\,X$ to the category of constructible $\mathbf{F}_q$-modules on X_{et}, see [**11**, Ch. 10].

The reader familiar with the formalism of constructible étale sheaves will recognize the flavor of many of the constructions and results in the following chapters.

Exercises

EXERCISE 1.1. Let m and n be integers and consider the Serre twisting sheaves $\mathcal{O}(m)$ and $\mathcal{O}(n)$ on $\mathbf{P}^1 = \mathbf{P}^1_{\mathbf{F}_q}$. Write $n = aq + b$ with $0 \leq b < q$. Show that

$$\sigma_\star \mathcal{O}(n) \cong \mathcal{O}(a)^{b+1} \oplus \mathcal{O}(a-1)^{q-b-1}$$

as $\mathcal{O}_{\mathbf{P}^1}$-modules. Verify, without invoking the adjunction between $\sigma^\star$ and $\sigma_\star$, that

$$\operatorname{Hom}_{\mathcal{O}_{\mathbf{P}^1}}\big(\mathcal{O}(qm),\, \mathcal{O}(n)\big) \cong \operatorname{Hom}_{\mathcal{O}_{\mathbf{P}^1}}\big(\mathcal{O}(m),\, \mathcal{O}(a)^{b+1} \oplus \mathcal{O}(a-1)^{q-b-1}\big)$$

as $\mathbf{F}_q$-vector spaces.

EXERCISE 1.2. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of crystals on a scheme X of finite type over $\mathbf{F}_q$. Show that

$$\operatorname{tr}_{\mathcal{G}} = \operatorname{tr}_{\mathcal{F}} + \operatorname{tr}_{\mathcal{H}}$$

as $\mathbf{F}_q$-valued functions on $X(\mathbf{F}_q)$.

EXERCISE 1.3. Let X be a noetherian scheme over $\mathbf{F}_q$. Let $i\colon Z \rightarrowtail X$ be a closed subscheme, and let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding ideal sheaf. Let $j\colon U \hookrightarrow X$ be the open complement of Z. Let $\mathcal{F}$ be a coherent τ-sheaf on X. Show that the submodule $\mathcal{I}\mathcal{F}$ of $\mathcal{F}$ is a sub-τ-sheaf. Show that $j^\star\mathcal{I}\mathcal{F} \cong j^\star\mathcal{F}$, and that $i^\star\mathcal{I}\mathcal{F}$ is nilpotent.

EXERCISE 1.4. Let X be a noetherian scheme over $\mathbf{F}_q$ and let Z be a closed subscheme defined by a nilpotent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. (One says that X is a *nilpotent thickening* of Z). Show that the categories $\mathbf{Crys}\,X$ and $\mathbf{Crys}\,Z$ are equivalent.

Give an example where $\mathbf{Coh}_\tau\,X$ and $\mathbf{Coh}_\tau\,Z$ are not equivalent.

EXERCISE 1.5. Let X be a noetherian scheme over $\mathbf{F}_q$. Let $\mathcal{F} = (\mathcal{F}, \tau)$ be a coherent τ-sheaf on X. Show that $\mathcal{F}$ and its pullback along Frobenius $\sigma^\star\mathcal{F}$ are nil-isomorphic.

EXERCISE 1.6. Let K be a field containing $\mathbf{F}_q$ and let $X = \operatorname{Spec} K$. Show that every coherent τ-sheaf $\mathcal{F}$ on $\operatorname{Spec} K$ has a unique decomposition $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_n$ with τ_0 an isomorphism and τ_n nilpotent. Show that the category $\mathbf{Crys}\,X$ is equivalent with the full sub-category of $\mathbf{Coh}_\tau\,X$ consisting of those $(\mathcal{F}, \tau)$ with τ an isomorphism.

EXERCISE 1.7. Give an example of a noetherian scheme X over $\mathbf{F}_q$, coherent τ-sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, and a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ in $\mathbf{Crys}\,X$ which does not come from a morphism in $\mathbf{Coh}_\tau\,X$.

EXERCISE 1.8. Let X be a noetherian scheme over $\mathbf{F}_q$. Let $\mathcal{N}$ be a nilpotent τ-sheaf on X. Show that $\mathrm{Hom}(\mathbf{1}, \mathcal{N}) = 0$ in the category $\mathbf{Coh}_\tau\, X$; Show that every short exact sequence

$$0 \to \mathcal{N} \to \mathcal{F} \to \mathbf{1} \to 0$$

in $\mathbf{Coh}_\tau\, X$ has a unique splitting $\mathbf{1} \to \mathcal{F}$. Show that for every coherent τ-sheaf $\mathcal{F}$ the natural map

$$\mathrm{Hom}_{\mathbf{Coh}_\tau X}(\mathbf{1}, \mathcal{F}) \to \mathrm{Hom}_{\mathbf{Crys}\, X}(\mathbf{1}, \mathcal{F})$$

is an isomorphism.

CHAPTER 2

Functors between categories of crystals

1. Pullback

Let $f\colon X \to Y$ be a morphism of noetherian schemes over $\mathbf{F}_q$.

PROPOSITION 2.1. *The functor*

$$\mathbf{Coh}_\tau\, Y \to \mathbf{Crys}\, X,\ \mathcal{F} \mapsto f^\star \mathcal{F}$$

is exact and maps nilpotent τ-sheaves to the zero crystal.

PROOF. By Proposition 1.15 this functor maps nilpotent τ-sheaves to the zero crystal, so we only need to prove exactness.

Let x be a point of X and let $y = f(x)$ be its image in Y. Since $k(x)$ is flat over $k(y)$, the pull-back functor $\mathbf{Coh}_\tau\, k(y) \to \mathbf{Coh}_\tau\, k(x)$ is exact. It maps nilpotent τ-sheaves to nilpotent τ-sheaves, so by the universal property of Theorem 1.22 it induces an exact functor $f^\star\colon \mathbf{Crys}\, k(y) \to \mathbf{Crys}\, k(x)$. We obtain a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{Coh}_\tau\, X & \xrightarrow{\iota_x^\star} & \mathbf{Crys}\, k(x) \\ {\scriptstyle f^\star}\uparrow & & \uparrow{\scriptstyle f^\star} \\ \mathbf{Coh}_\tau\, Y & \xrightarrow{\iota_y^\star} & \mathbf{Crys}\, k(y) \end{array}$$

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence in $\mathbf{Coh}_\tau\, Y$. By Lemma 1.26 it induces a short exact sequence in $\mathbf{Crys}\, k(y)$, which under $f^\star$ induces a short exact sequence in $\mathbf{Crys}\, k(x)$. Hence

$$0 \to \iota_x^\star f^\star \mathcal{F}_1 \to \iota_x^\star f^\star \mathcal{F}_2 \to \iota_x^\star f^\star \mathcal{F}_3 \to 0$$

is exact in $\mathbf{Crys}\, k(x)$ for every $x \in X$. By the pointwise criterion for exactness (Theorem 1.29) it follows that

$$0 \to f^\star \mathcal{F}_1 \to f^\star \mathcal{F}_2 \to f^\star \mathcal{F}_3 \to 0$$

is exact in $\mathbf{Crys}\, X$ and we conclude that the functor $\mathbf{Coh}_\tau\, Y \to \mathbf{Crys}\, X$ is exact. □

By Theorem 1.22 we obtain a pullback functor on crystals:

COROLLARY 2.2. *The functor* $f^\star\colon \mathbf{Coh}_\tau\, Y \to \mathbf{Coh}_\tau\, X$ *induces a functor*

$$f^\star\colon \mathbf{Crys}\, Y \to \mathbf{Crys}\, X$$

which is exact. □

PROPOSITION 2.3. *Let* $f\colon X \to Y$ *be a morphism of* $\mathbf{F}_q$*-schemes of finite type and* $\mathcal{F}$ *a crystal on* Y*. Then*

$$\mathrm{tr}_{f^\star\mathcal{F}} = \mathrm{tr}_{\mathcal{F}} \circ f$$

as $\mathbf{F}_q$*-valued functions on* $X(\mathbf{F}_q)$*.*

PROOF. If $x \in X(\mathbf{F}_q)$ and $y = f(x) \in Y(\mathbf{F}_q)$ then $x^\star f^\star \mathcal{F} = y^\star \mathcal{F}$. □

2. Tensor product

Let X be a scheme over $\mathbf{F}_q$. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Then we have a canonical isomorphism

$$\sigma^\star(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \xrightarrow{\sim} \sigma^\star\mathcal{F} \otimes_{\mathcal{O}_X} \sigma^\star\mathcal{G}$$

of quasi-coherent $\mathcal{O}_X$-modules.

DEFINITION 2.4 (Tensor product of τ-sheaves). Let X be a scheme over $\mathbf{F}_q$. The *tensor product* of τ-sheaves $\mathcal{F}$ and $\mathcal{G}$ on X is the τ-sheaf $\mathcal{F} \otimes \mathcal{G}$ with underlying $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and with $\tau_{\mathcal{F}\otimes\mathcal{G}}$ defined as the composition

$$\sigma^\star(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \xrightarrow{\sim} \sigma^\star\mathcal{F} \otimes_{\mathcal{O}_X} \sigma^\star\mathcal{G} \xrightarrow{\tau_{\mathcal{F}}\otimes\tau_{\mathcal{G}}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}.$$

EXAMPLE 2.5. Assume $X = \operatorname{Spec} R$. Let $\mathcal{F}$ and $\mathcal{G}$ be τ-sheaves on X corresponding to R-modules M and N equipped with q-linear endomorphisms $\tau_{M,s}$ and $\tau_{N,s}$ respectively. Then the τ-sheaf $\mathcal{F} \otimes \mathcal{G}$ corresponds to the R-module $M \otimes_R N$ equipped with the q-linear map $\tau_{M\otimes N,s}$ defined by

$$\tau_{M\otimes N,s}(m \otimes n) = \tau_{M,s}(m) \otimes \tau_{N,s}(m)$$

for all $m \in M$ and $n \in N$.

Proposition 2.6. *Let X be a noetherian scheme over $\mathbf{F}_q$ and let $\mathcal{F}$ and $\mathcal{G}$ be coherent τ-sheaves on X. Then $\mathcal{F} \otimes \mathcal{G}$ is also a coherent τ-sheaf.* □

Proposition 2.7. *Let X be a scheme of finite type over $\mathbf{F}_q$. Let $\mathcal{G}$ be coherent τ-sheaf on X. Then the functor*

$$\mathbf{Coh}_\tau\, X \to \mathbf{Crys}\, X,\ \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{G}$$

is exact and maps nilpotent τ-sheaves to the zero crystal.

If $\mathcal{G}$ is not a *flat* $\mathcal{O}_X$-module then one cannot hope to replace $\mathbf{Crys}\, X$ by $\mathbf{Coh}_\tau\, X$ in the exactness statement.

Proof of Proposition 2.7. Assume $\mathcal{F}$ is a nilpotent coherent τ-sheaf on X. Then there is an n such that the map

$$\tau_{\mathcal{F}}^n\colon (\sigma^n)^\star \mathcal{F} \to \mathcal{F}$$

is the zero map. But then also $\tau^n_{\mathcal{F}\otimes\mathcal{G}} = 0$, so $\mathcal{F} \otimes \mathcal{G} = 0$ in $\mathbf{Crys}\, X$. This proves the second statement.

For the first statement, assume $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence in $\mathbf{Coh}_\tau\, X$. Let $x \in X$ be a point. Note that the functor

$$- \otimes \iota_x^\star \mathcal{G}\colon\ \mathbf{Coh}_\tau\, k(x) \to \mathbf{Coh}_\tau\, k(x)$$

is exact (since $\iota_x^\star \mathcal{G}$ is flat over $k(x)$) and maps nilpotent objects to nilpotent objects, so that it induces an exact functor $\mathbf{Crys}\, k(x) \to \mathbf{Crys}\, k(x)$. By Theorem 1.29 the sequence

$$0 \to \iota_x^\star \mathcal{F}_1 \to \iota_x^\star \mathcal{F}_2 \to \iota_x^\star \mathcal{F}_3 \to 0$$

is exact in $\mathbf{Crys}\, k(x)$. Under the above exact functor it induces an exact sequence

$$0 \to \iota_x^\star \mathcal{F}_1 \otimes \iota_x^\star \mathcal{G} \to \iota_x^\star \mathcal{F}_2 \otimes \iota_x^\star \mathcal{G} \to \iota_x^\star \mathcal{F}_3 \otimes \iota_x^\star \mathcal{G} \to 0$$

in $\mathbf{Crys}\, k(x)$. But this sequence is canonically isomorphic with the sequence

$$0 \to \iota_x^\star(\mathcal{F}_1 \otimes \mathcal{G}) \to \iota_x^\star(\mathcal{F}_2 \otimes \mathcal{G}) \to \iota_x^\star(\mathcal{F}_3 \otimes \mathcal{G}) \to 0.$$

Since x was arbitrary, we deduce using the point-wise criterion (Theorem 1.29) that the sequence

$$0 \to \mathcal{F}_1 \otimes \mathcal{G} \to \mathcal{F}_2 \otimes \mathcal{G} \to \mathcal{F}_3 \otimes \mathcal{G} \to 0$$

is exact in $\mathbf{Crys}\, X$, which finishes the proof of the proposition. □

Corollary 2.8. *The functor* $\otimes\colon \mathbf{Coh}_\tau X \times \mathbf{Coh}_\tau X \to \mathbf{Coh}_\tau X$ *induces a functor* $\otimes\colon \mathbf{Crys}\, X \times \mathbf{Crys}\, X \to \mathbf{Crys}\, X$ *which is exact in both arguments.* □

Proposition 2.9. *Let* X *be a scheme of finite type over* $\mathbf{F}_q$ *and let* $\mathcal{F}$ *and* $\mathcal{G}$ *be coherent* τ*-sheaves on* X*. Then*

$$\mathrm{tr}_{\mathcal{F}\otimes\mathcal{G}} = \mathrm{tr}_{\mathcal{F}} \cdot \mathrm{tr}_{\mathcal{G}}$$

as $\mathbf{F}_q$*-valued functions on* $X(\mathbf{F}_q)$*.*

Proof. Note that

$$x^\star\mathcal{F} \otimes_{\mathbf{F}_q} x^\star\mathcal{G} = x^\star(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

for all $x \in X(\mathbf{F}_q)$. The proposition now is a consequence of the fact that

$$\mathrm{tr}\,(\,\alpha \mid V\,) \cdot \mathrm{tr}\,(\beta \mid W) = \mathrm{tr}\,(\,\alpha \otimes \beta \mid V \otimes W\,)$$

for endomorphisms α, β of finite-dimensional vector spaces V respectively W. □

3. Direct images

Let $f\colon X \to Y$ be a morphism of schemes over $\mathbf{F}_q$ and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let i be a non-negative integer. Then $\mathrm{R}^i f_\star \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Since $f\sigma_X = \sigma_Y f$, and since $\sigma_\star$ is exact we have

$$\mathrm{R}^i f_\star(\sigma_{X,\star}\mathcal{F}) = \sigma_{Y,\star}\mathrm{R}^i f_\star \mathcal{F}.$$

In particular, if $(\mathcal{F},\tau)$ is a τ-sheaf on X, then the 'adjoint' map $\tau_a\colon \mathcal{F} \to \sigma_{X,\star}\mathcal{F}$ (see Proposition 1.2) induces a map $\tau_a\colon \mathrm{R}^i f_\star\mathcal{F} \to \sigma_{Y,\star}\mathrm{R}^i f_\star\mathcal{F}$. Hence $\mathrm{R}^i f_\star$ defines a functor $\mathbf{QCoh}_\tau X \to \mathbf{QCoh}_\tau Y$.

If $U = \operatorname{Spec} R \subset Y$ is an affine open then the restriction $(\mathrm{R}^i f_\star\mathcal{F})_U$ is given by the R-module $\mathrm{H}^i(X_U, \mathcal{F}_{|X_U})$ equipped with the induced q-linear τ_s.

Example 2.10. Let $f\colon X \to Y$ be a morphism of affine schemes over $\mathbf{F}_q$ corresponding to a map of $\mathbf{F}_q$-algebras $R \to S$. Let $\mathcal{F}$ be a τ-sheaf on X corresponding to an S-module M equipped with a q-linear endomorphism τ_s. Then the τ-sheaf $\mathrm{R}^i f_\star\mathcal{F}$ on Y is zero if $i > 0$ while $\mathrm{R}^0 f_\star\mathcal{F} = f_\star\mathcal{F}$ corresponds to the R-module M obtained by restricting scalars from S to R, equipped with the original $\tau_s\colon M \to M$.

Note that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of τ-sheaves on X then the long exact sequence of cohomology

$$0 \to f_\star \mathcal{F} \to f_\star \mathcal{G} \to f_\star \mathcal{H} \to \mathrm{R}^1 f_\star \mathcal{F} \to \cdots$$

is naturally an exact sequence of τ-sheaves on Y.

PROPOSITION 2.11. *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes over $\mathbf{F}_q$, let $\mathcal{F}$ be a coherent τ-sheaf on X, and let i be an integer. Then $\mathrm{R}^i f_\star \mathcal{F}$ is a coherent τ-sheaf on Y.*

PROOF. This follows from the fundamental fact that the derived direct images of a coherent $\mathcal{O}_X$-module along a proper map $X \to Y$ of noetherian schemes are coherent $\mathcal{O}_Y$-modules, see [**29**, 3.2.1] or [**45**, Tag 02O5]. □

PROPOSITION 2.12. *Let $f\colon X \to Y$ be a proper morphism of noetherian $\mathbf{F}_q$-schemes and $\varphi\colon \mathcal{F} \to \mathcal{G}$ a map in $\mathbf{Coh}_\tau X$. Let i be an integer. If φ is a nil-isomorphism, then so is $\mathrm{R}^i f_\star \varphi\colon \mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f_\star \mathcal{G}$.*

PROOF. If $\tau^n = 0$ then also $\mathrm{R}^i f_\star \tau^n = 0$, so $\mathrm{R}^i f_\star$ maps nilpotent τ-sheaves to nilpotent τ-sheaves.

Now, let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a nil-isomorphism in $\mathbf{Coh}_\tau X$. Let $\mathcal{H} \subset \mathcal{G}$ be the image and let $\mathcal{N}$ be the kernel of φ. Consider the short exact sequence

$$0 \to \mathcal{N} \to \mathcal{F} \xrightarrow{\varphi} \mathcal{H} \to 0.$$

The induced long exact sequence of cohomology gives a surjective map

$$\mathrm{R}^i f_\star \mathcal{N} \twoheadrightarrow \ker\left(\mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f_\star \mathcal{H}\right)$$

and an injective map

$$\operatorname{coker}\left(\mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f_\star \mathcal{H}\right) \hookrightarrow \mathrm{R}^{i+1} f_\star \mathcal{N}.$$

Since quotients and sub-objects of nilpotent τ-sheaves are nilpotent, we conclude that $\mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f_\star \mathcal{H}$ is a nil-isomorphism.

Let $\mathcal{Q}$ be the cokernel of φ. Then a similar argument for the short exact sequence $0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{Q} \to 0$ shows that $\mathrm{R}^i f_\star \mathcal{H} \to \mathrm{R}^i f_\star \mathcal{G}$ is a nil-isomorphism.

Now $\mathrm{R}^i f_\star \varphi$ is the composition

$$\mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f_\star \mathcal{H} \to \mathrm{R}^i f_\star \mathcal{G},$$

and since the composition of two nil-isomorphisms is a nil-isomorphism (Proposition 1.18) we conclude that $\mathrm{R}^i f_\star \varphi$ is a nil-isomorphism, as we had to show. □

COROLLARY 2.13. *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes over $\mathbf{F}_q$. Let i be an integer. Then the functor $\mathrm{R}^i f_\star\colon \mathbf{Coh}_\tau X \to \mathbf{Coh}_\tau Y$ induces a functor $\mathrm{R}^i f_\star\colon \mathbf{Crys}\, X \to \mathbf{Crys}\, Y$.* □

PROPOSITION 2.14. *Let $f\colon X \to Y$ be a proper map of noetherian schemes over $\mathbf{F}_q$. For every short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

there is a long exact sequence

$$0 \to f_\star\mathcal{F} \to f_\star\mathcal{G} \to f_\star\mathcal{H} \to \mathrm{R}^1 f_\star\mathcal{F} \to \cdots$$

of crystals on Y, depending functorially on the short exact sequence.

PROOF. By Proposition 1.24 we may assume that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is actually a short exact sequence in $\mathbf{Coh}_\tau X$, and the proposition then comes down to the usual long exact sequence of cohomology for higher direct images. □

THEOREM 2.15 (Leray spectral sequence). *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be proper morphisms of noetherian schemes. Let $\mathcal{F}$ be a crystal on X. There is a spectral sequence*

$$E_2^{i,j} = \mathrm{R}^i g_\star \mathrm{R}^j f_\star \mathcal{F} \Longrightarrow \mathrm{R}^{i+j}(gf)_\star\mathcal{F}$$

in $\mathbf{Crys}\, Z$.

PROOF. This follows immediately from the usual Leray spectral sequence [**45**, Tag 01F6]. □

PROPOSITION 2.16. *Let $f\colon X \to Y$ be a finite morphism of $\mathbf{F}_q$-schemes of finite type and $\mathcal{F}$ a crystal on X. Then for all $y \in Y(\mathbf{F}_q)$ we have*

$$\mathrm{tr}_{f_\star\mathcal{F}}\, y = \sum_{f(x)=y} \mathrm{tr}_{\mathcal{F}}\, x \tag{5}$$

where the sum ranges over the $x \in X(\mathbf{F}_q)$ with $f(x) = y$.

PROOF. We may pull back along $y\colon \operatorname{Spec} \mathbf{F}_q \to Y$ and reduce to $Y = \operatorname{Spec} \mathbf{F}_q$ and $X = \operatorname{Spec} R$ for a finite $\mathbf{F}_q$-algebra R. Assume $\mathcal{F}$ is given by an R-module M equipped with a q-linear $\tau_s\colon M \to M$. Write $R = R_1 \times \cdots \times R_n$ for some finite local $\mathbf{F}_q$-algebras R_i, and correspondingly $M = M_1 \times \cdots \times M_n$. Then we have

$$\mathrm{tr}_{f_\star\mathcal{F}}\, y = \mathrm{tr}_{\mathbf{F}_q}(\, \tau_s \mid M\,) = \sum_{i=1}^{n} \mathrm{tr}_{\mathbf{F}_q}(\, \tau_s \mid M_i\,)$$

and we may further reduce to the case where R is a finite local $\mathbf{F}_q$-algebra. Let $\mathfrak{m}$ be the maximal ideal of R. Note that for all $n \geq 0$ we have

$$\tau_s(\mathfrak{m}^n M) \subset \mathfrak{m}^{qn}\tau_s(M) \subset \mathfrak{m}^{n+1}M$$

so that τ_s restricts to a nilpotent $\mathbf{F}_q$-linear map $\mathfrak{m}M \to \mathfrak{m}M$. We find

$$\mathrm{tr}_{\mathbf{F}_q}(\,\tau_s \mid M\,) = \mathrm{tr}_{\mathbf{F}_q}(\,\tau_s \mid M/\mathfrak{m}M\,)$$

and therefore we may reduce to the case where R is reduced. Hence $R = \mathbf{F}_{q^d}$ for some d. If $d = 1$ then the proposition holds trivially. So we assume $d > 1$. The right-hand side of (5) is zero since $\operatorname{Spec} R$ has no $\mathbf{F}_q$-points. Let α be an element of $\mathbf{F}_{q^d}$ not in $\mathbf{F}_q$ and let $\beta \in \mathbf{F}_{q^d}$ be such that $\beta(\alpha - \alpha^q) = 1$. Then

$$\tau_s = \alpha\beta\tau_s - \beta\tau_s\alpha$$

as endomorphisms of the $\mathbf{F}_q$-vector space M. In particular

$$\mathrm{tr}_{\mathbf{F}_q}(\,\tau_s \mid M\,) = \mathrm{tr}_{\mathbf{F}_q}(\,\alpha(\beta\tau_s) \mid M\,) - \mathrm{tr}_{\mathbf{F}_q}(\,(\beta\tau_s)\alpha \mid M\,) = 0,$$

and hence also the left-hand-side of (5) vanishes. □

For a proper but not finite morphism $X \to Y$ the left-hand side of (5) must be modified to take in account higher direct images $\mathrm{R}^i f_\star \mathcal{F}$. This is the content of the Woods Hole trace formula, which will be stated and proven in the next chapter.

We now show that the higher direct images of crystals along a proper map commute with base change. Given a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is a natural 'base change' map

$$\psi\colon g^\star \mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f'_\star g'^\star \mathcal{F}$$

of $\mathcal{O}_{Y'}$-modules, see [**45**, Tag 02N6]. If $\mathcal{F}$ is a τ-sheaf on X then ψ is a map of τ-sheaves on Y'. We will show that if f is proper and $\mathcal{F}$ is a coherent τ-sheaf then ψ is a nil-isomorphism.

We first show that the fibers of the higher direct images along a proper map are the higher direct images of the fibers. We will make essential use of the theorem of formal functions.

LEMMA 2.17. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let*

$$M_1 \leftarrow M_2 \leftarrow \cdots$$

be a projective system of finitely generated R-modules. Assume that $\mathfrak{m}^n M_n = 0$ for all n. Then the natural map

$$k \otimes_R \big(\varprojlim M_i\big) \longrightarrow \varprojlim \big(k \otimes_R M_i\big)$$

is an isomorphism.

PROOF. Since R is noetherian the ideal $\mathfrak{m}$ is finitely generated, say by $r_1, \ldots, r_t$. These generators define for every R-module M an exact sequence

$$M^t \to M \to k \otimes_R M \to 0 \tag{6}$$

of R-modules. In particular, we have exact sequences

$$0 \to N_i \to M_i^t \to M_i \to k \otimes_R M_i \to 0.$$

Since the R-modules M_i and N_i have finite length, the systems (N_i), (M_i^t) and (M_i) satisfy the Mittag-Leffler condition. It follows that the sequence

$$0 \to \varprojlim N_i \to \varprojlim M_i^t \to \varprojlim M_i \to \varprojlim \big(k \otimes_R M_i\big) \to 0$$

is exact [**45**, Tag 0594]. Since $\varprojlim M_i^t = (\varprojlim M_i)^t$ we find an exact sequence

$$\big(\varprojlim M_i\big)^t \to \varprojlim M_i \to \varprojlim \big(k \otimes_R M_i\big) \to 0.$$

At the same time, the exact sequence (6) for $M = \varprojlim M_i$ gives

$$\big(\varprojlim M_i\big)^t \to \varprojlim M_i \to k \otimes_R \big(\varprojlim M_i\big) \to 0.$$

Comparing the last two exact sequences gives the desired result. □

PROPOSITION 2.18. *Let $f\colon X \to Y$ be a proper morphism of noetherian schemes over $\mathbf{F}_q$. Let $\mathcal{F}$ be a coherent τ-sheaf on X. Let $y \in Y$ be a point with residue field $k = k(y)$. Consider the cartesian square*

$$\begin{array}{ccc} X_y & \xrightarrow{\iota'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ \operatorname{Spec} k & \xrightarrow{\iota} & Y. \end{array}$$

Then for all i the natural map

$$\iota^\star \mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f'_\star \iota'^\star \mathcal{F}$$

is a nil-isomorphism of coherent τ-sheaves on $\operatorname{Spec} k$.

Proof. By flat base change [**29**, 1.4.15] [**45**, Tag 02KH] we may reduce to $Y = \operatorname{Spec} R$ and $y = \operatorname{Spec} k$ for a complete noetherian ring R with residue field k. Let $\mathfrak{m}$ be the maximal ideal of R. For every $n \geq 0$ let $\mathcal{F}_n$ be the coherent τ-sheaf $\iota'^\star(\mathcal{F} \otimes_R R/\mathfrak{m}^{n+1})$ on X_y. The theorem of formal functions [**29**, 4.1.5] [**45**, Tag 02OC] gives an isomorphism

$$\mathrm{H}^i(X, \mathcal{F}) \xrightarrow{\sim} \varprojlim_n \mathrm{H}^i(X_y, \mathcal{F}_n)$$

of coherent τ-sheaves on $\operatorname{Spec} R$. Hence we need to show that the natural map

$$\big(\varprojlim_n \mathrm{H}^i(X_y, \mathcal{F}_n)\big) \otimes_R k \longrightarrow \mathrm{H}^i(X, \mathcal{F}_0)$$

is an isomorphism. By Lemma 2.17 it suffices to show that

$$\varprojlim_n \big(\mathrm{H}^i(X_y, \mathcal{F}_n) \otimes_R k\big) \longrightarrow \mathrm{H}^i(X, \mathcal{F}_0) \tag{7}$$

of coherent τ-sheaves on $\operatorname{Spec} k$ is a nil-isomorphism.

The reduction map $\mathcal{F}_{n+1} \twoheadrightarrow \mathcal{F}_n$ is a nil-isomorphism in $\mathbf{Coh}_\tau\, X_y$, since the kernel $\mathfrak{m}^n \mathcal{F}_{n+1}$ satisfies $\tau_s(\mathfrak{m}^n \mathcal{F}_{n+1}) \subset \mathfrak{m}^{qn} \mathcal{F}_{n+1} = \{0\}$. Therefore each of the maps

$$\mathrm{H}^i(X_y, \mathcal{F}_n) \otimes_R k \to \mathrm{H}^i(X, \mathcal{F}_0)$$

is a nil-isomorphism. It follows that the map (7) is a nil-isomorphism, as we had to show. □

Theorem 2.19 (Base change). *Consider a cartesian square*

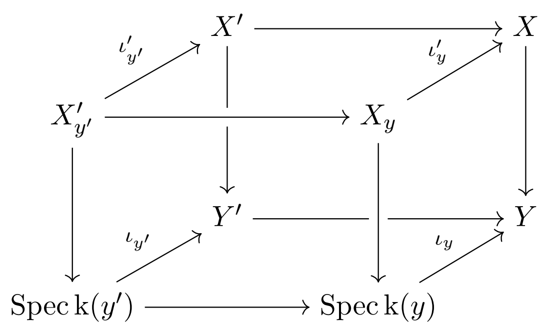
$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of noetherian schemes over $\mathbf{F}_q$. Assume that f (and hence also f') is proper. Then for every integer i and for every crystal $\mathcal{F}$ on X the natural map

$$\psi\colon g^\star \mathrm{R}^i f_\star \mathcal{F} \to \mathrm{R}^i f'_\star g'^\star \mathcal{F}$$

is an isomorphism of crystals on Y'.

Proof. By the point-wise criterion of exactness (Theorem 1.29) it suffices to check that the pullback of the map to an arbitrary $y' \in Y'$ is an isomorphism. Let $y \in Y$ be the image of y'. Consider the cube



whose six faces are cartesian squares. Base change commutes with higher direct images in the left and right squares by Proposition 2.18, and in the front square by flat base change ($\mathrm{k}(y')$ is flat over $\mathrm{k}(y)$). To see that $\iota_{y'}^{\star}\psi$ is an isomorphism, we note that it factors as

$$\begin{aligned} & \iota_{y'}^{\star} g^{\star} \mathrm{R}^i f_{\star} \mathcal{F} = g_y^{\star} \iota_y^{\star} \mathrm{R}^i f_{\star} \mathcal{F} \\ \cong_1 \quad & g_y^{\star} \mathrm{R}^i f_{y,\star} \iota_y'^{\star} \mathcal{F} \\ \cong_2 \quad & \mathrm{R}^i f'_{y',\star} g_y'^{\star} \iota_y'^{\star} \mathcal{F} = \mathrm{R}^i f'_{y',\star} \iota_{y'}'^{\star} g'^{\star} \mathcal{F} \\ \cong_3 \quad & \iota_{y'}^{\star} \mathrm{R}^i f'_{\star} g'^{\star} \mathcal{F}. \end{aligned}$$

Here $\cong_1$ and $\cong_3$ are the isomorphisms in $\mathbf{Crys}\, k(y')$ induced by the nil-isomorphisms of Proposition 2.18 for the left respectively right face of the cube, and $\cong_2$ is flat base change in the front face. □

4. Extension by zero

In this section we fix a diagram

$$U \xhookrightarrow{\ j\ } X \xleftarrow{\ i\ } Z.$$

where X is a noetherian scheme over $\mathbf{F}_q$, where i is a closed immersion where j is the open immersion of the complement. We denote the ideal sheaf of Z by $\mathcal{I} \subset \mathcal{O}_X$.

We will define a functor

$$j_! \colon \mathbf{Crys}\, U \to \mathbf{Crys}\, X,$$

called *extension by zero*. For every crystal $\mathcal{F}$ on U it satisfies $j^\star j_! \mathcal{F} \cong \mathcal{F}$ (so that $j_!\mathcal{F}$ extends $\mathcal{F}$) and $i^\star j_\star \mathcal{F} \cong 0$ (so that it extends $\mathcal{F}$ by zero).

We first collect some facts about extending coherent $\mathcal{O}_U$-modules to coherent $\mathcal{O}_X$-modules. See [**45**, Tag 01PD] for a more thorough treatment.

If $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module then $j_\star\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $(j_\star\mathcal{F})_{|U} = \mathcal{F}$. We call a *coherent extension* of $\mathcal{F}$ to X a coherent $\mathcal{O}_X$-submodule $\mathcal{G} \subset j_\star\mathcal{F}$ so that $\mathcal{G}_{|U} = \mathcal{F}$.

LEMMA 2.20. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. Then there exists a coherent extension $\mathcal{G}$ of $\mathcal{F}$ to X.*

PROOF. First assume that $X = \operatorname{Spec} R$. Choose $f_1, \ldots, f_s \in R$ so that $U = \cup_i D(f_i)$. Then there is an $N > 0$ so that for every i the $R[1/f_i]$-module $\mathcal{F}_{|D(f_i)}$ is generated by finitely elements s_{ij}/f_i^N with $s_{ij} \in \mathcal{F}(U)$. Let $M \subset \mathcal{F}(U) = j_\star\mathcal{F}(X)$ be the R-module generated by the s_{ij}. Then $\mathcal{G} := \widetilde{M}$ satisfies the requirements.

Now let X be an arbitrary noetherian scheme. Let $V_1, \ldots, V_n$ be affine opens in X so that $X = U \cup V_1 \cup \cdots \cup V_n$. Extending step by step, from U to $U \cup V_1$ to $U \cup V_1 \cup V_2$, etcetera we reduce to the case $n = 1$. So we assume $X = U \cap V$ with V affine open. By the affine case of the lemma we can extend $\mathcal{F}_{|U\cap V}$ to some $\mathcal{F}_V$ on V. Glueing this $\mathcal{F}_V$ with the given $\mathcal{F}$ on U yields a coherent extension $\mathcal{G}$ to all of X. $\square$

LEMMA 2.21. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_U$-module. If $\mathcal{G}$ is a coherent extension of $\mathcal{F}$ to X then so is $\mathcal{I}\mathcal{G}$.*

PROOF. We have $\mathcal{I}\mathcal{G} \subset \mathcal{G} \subset j_\star\mathcal{F}$ and $(\mathcal{I}\mathcal{G})_{|U} = \mathcal{G}_{IU}$. $\square$

LEMMA 2.22. *If $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ is a map of coherent $\mathcal{O}_U$-modules, and if $\mathcal{G}_1$ and $\mathcal{G}_2$ are coherent extensions of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, then there exists an $n > 0$ so that $(j_\star\varphi)(\mathcal{I}^n\mathcal{G}_1) \subset \mathcal{G}_2$ in $j_\star\mathcal{F}_2$.*

PROOF. Since X is noetherian it has a finite affine open cover and it suffices to show the existence of n for $X = \operatorname{Spec} R$. Let $I = \mathcal{I}(X) \subset R$ be the ideal defining Z and let $f_1, \ldots, f_s \in R$ be generators of I. Then $U = \cup_i D(f_i)$. Without loss of generality we may assume that U is dense in X and that the f_i are not zero-divisors.

Let $M_1 = \mathcal{G}_1(X)$ and $M_2 = \mathcal{G}_2(X)$. The R-module M_1 is finitely generated, say by $m_1, \ldots, m_t$. As in the proof of Lemma 2.20 choose $f_1, \ldots, f_s \in R$ so that $U = \cup_i D(f_i)$. There is an $n_0 > 0$ so that

$$f_j^{n_0}\varphi(m_{ij}) \in M_2 \subset M_2[1/f_j]$$

for every i and j. Take $n = sn_0$. The R-module $\varphi(M_1)$ is generated by the elements

$$f_1^{e_1} \cdots f_s^{e_s} m_i j$$

with $e_1 + \cdots + e_s = n$. In particular, one of the exponents must be $\geq n_0$ so that each of the elements $f_1^{e_1} \cdots f_s^{e_s} m_{ij} \in M_2$ lies in M_2. We conclude that $j_\star\varphi$ maps $\mathcal{I}^n\mathcal{G}_1$ into $\mathcal{G}_2$. □

We now start using these lemmata to construct the functor $j_!$.

Lemma 2.23. *Let $\mathcal{F}$ be a coherent τ-sheaf on U. Let $\mathcal{G} \subset j_\star\mathcal{F}$ be a coherent $\mathcal{O}_X$-module extending $\mathcal{F}$. Then for all n sufficiently large $\mathcal{I}^n\mathcal{G} \subset j_\star\mathcal{F}$ is a coherent sub-τ-sheaf, $j^\star\mathcal{I}^n\mathcal{G} = \mathcal{F}$ and $i^\star\mathcal{I}^n\mathcal{G}$ is nilpotent.*

Proof. $\sigma_\star\mathcal{G}$ is a coherent extension of $\sigma_\star\mathcal{F}$. By Lemma 2.22 there is an n and a map of $\mathcal{O}_X$-modules

$$\widetilde{\tau}_a \colon \mathcal{I}^n\mathcal{G} \to \sigma_\star\mathcal{G}$$

extending τ_a. By adjunction we find a map

$$\widetilde{\tau} \colon \sigma^\star(\mathcal{I}^n\mathcal{G}) \to \mathcal{G}$$

extending τ. Take $\widetilde{\mathcal{G}} = \mathcal{I}^{2n+1}\mathcal{G}$. Then $\widetilde{\tau}$ induces a map

$$\widetilde{\tau} \colon \sigma^\star\widetilde{\mathcal{G}} = \mathcal{I}^{qn+q}\sigma^\star(\mathcal{I}^n\mathcal{G}) \to \mathcal{I}^{qn+q}\mathcal{G} \subset \widetilde{\mathcal{G}},$$

where the inclusion comes from the inequality $qn + q \geq 2n + 1$. By construction $(\widetilde{\mathcal{G}}, \widetilde{\tau})$ is a coherent τ-sheaf extending $\mathcal{F}$. Moreover, since we have the strict inequality $qn + q > 2n + 1$, we have $\widetilde{\tau} \equiv 0 \bmod \mathcal{I}\widetilde{\mathcal{G}}$, so that $i^\star\widetilde{\mathcal{G}}$ is nilpotent. □

Lemma 2.24. *Let $\mathcal{G}$ be a coherent τ-sheaf on X so that $i^\star\mathcal{G}$ is nilpotent. Then the inclusion $\mathcal{I}\mathcal{G} \to \mathcal{G}$ is a nil-isomorphism.* □

Theorem 2.25. *Let $\mathcal{F}$ be a coherent τ-sheaf on U. Then there exists a pair $(\widetilde{\mathcal{F}}, \varphi)$ consisting of*

1. *a crystal $\widetilde{\mathcal{F}}$ on X such that $i^\star\widetilde{\mathcal{F}}$ is the zero crystal on Z,*
2. *an isomorphism $\varphi \colon \mathcal{F} \xrightarrow{\sim} j^\star\widetilde{\mathcal{F}}$ in* **Crys** *U.*

The pair $(\widetilde{\mathcal{F}}, \varphi)$ is unique up to unique isomorphism and depends functorially on $\mathcal{F}$.

PROOF. By Lemma 2.20 and Lemma 2.23 there is a coherent τ-sheaf $\mathcal{G}$ on X extending $\mathcal{F}$ with the desired properties.

Let $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of coherent τ-sheaves on U, and let $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ be as above. We show that there is a unique $\widetilde{\varphi}\colon \widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}_2$ in $\mathbf{Crys}\,X$ extending φ.

By Lemma 2.22 there is an n so that $j_\star\varphi$ maps $\mathcal{I}^n\widetilde{\mathcal{F}}_1$ in $\widetilde{\mathcal{F}}_2$. By Lemma 2.24 the inclusion $\mathcal{I}^n\widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}_1$ is a nil-isomorphism, so that the diagram

$$\widetilde{\mathcal{F}}_1 \longleftarrow \mathcal{I}^n\widetilde{\mathcal{F}}_1 \xrightarrow{j_\star\varphi} \widetilde{\mathcal{F}}_2$$

defines a morphism $\widetilde{\varphi}\colon \widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}_2$ in $\mathbf{Crys}\,X$, extending φ.

Now let $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ be two extensions in $\mathbf{Crys}\,X$. Consider their difference δ. For all $x \in U$ we have $\iota_x^\star\delta = 0$ since $\iota_x^\star\widetilde{\varphi}_1 = \iota_x^\star\varphi = \iota_x^\star\varphi_2$. For all $x \in Z$ we have $\iota_x^\star\delta = 0$ since it is a morphism between zero crystals. By the point-wise-criterion of Theorem 1.29 we conclude that $\delta = 0$ so that $\widetilde{\varphi}_1 = \widetilde{\varphi}_2$ in $\mathbf{Crys}\,X$.

A similar pointwise verification shows that $\varphi \mapsto \widetilde{\varphi}$ is compatible with composition. This shows functoriality, and that $(\widetilde{\mathcal{F}}, \varphi)$ is unique up to unique isomorphism. □

PROPOSITION 2.26. *The functor*

$$\mathbf{Coh}_\tau\,U \to \mathbf{Crys}\,X,\ \mathcal{F} \mapsto \widetilde{\mathcal{F}}$$

maps nilpotent τ-sheaves to the zero crystal, and is exact.

PROOF. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of coherent τ-sheaves on U. We need to show that the induced sequence

$$0 \to \widetilde{\mathcal{F}}_1 \to \widetilde{\mathcal{F}}_2 \to \widetilde{\mathcal{F}}_3 \to 0 \tag{8}$$

of crystals on X is exact. By the pointwise criterion (Theorem 1.29) it suffices to show that the sequence

$$0 \to \iota_x^\star\widetilde{\mathcal{F}}_1 \to \iota_x^\star\widetilde{\mathcal{F}}_2 \to \iota_x^\star\widetilde{\mathcal{F}}_3 \to 0$$

of crystals on $\operatorname{Spec} k(x)$ is exact for all $x \in X$. If $x \in U$ then this sequence coincides with the pullback of $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ along ι_x, and hence is exact. If $x \in X \setminus U$ then by definition of $\widetilde{\mathcal{F}}_i$ we have $\iota_x^\star\widetilde{\mathcal{F}}_i = 0$ as crystals on $\operatorname{Spec} k(x)$, so also in that case the sequence is exact.

Similarly, if $\mathcal{F}$ is a nilpotent coherent τ-sheaf on U then for $x \in U$ the pull-back $\iota_x^\star\widetilde{\mathcal{F}}$ is isomorphic to $\iota_x^\star\mathcal{F}$ and hence nilpotent, and for

$x \in X \setminus U$ the pull-back $\iota_x^\star \widetilde{\mathcal{F}}$ is nilpotent by construction. We conclude using the pointwise criterion of Proposition 1.28 that $\widetilde{\mathcal{F}}$ is nilpotent. $\square$

Corollary 2.27. *The functor $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ factors over a unique functor*

$$j_!\colon \mathbf{Crys}\, U \to \mathbf{Crys}\, X$$

which is exact. $\square$

We call the functor $j_!$ *extension by zero from U to X*.

Example 2.28. Let $U = \mathbf{A}^1 = \operatorname{Spec} \mathbf{F}_q[x]$ and $j\colon \mathbf{A}^1 \hookrightarrow \mathbf{P}^1$ the standard embedding with complement $i\colon \{\infty\} \hookrightarrow \mathbf{P}^1$. Let $\mathcal{F}$ be given by $M = \mathbf{F}_q[x]$ and

$$\tau_s\colon M \to M, s \mapsto f s^q$$

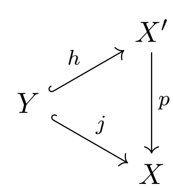
for some $f \in \mathbf{F}_q[x]$. Let m be an integer which satisfies $(q-1)m \geq \deg f$. Then f defines a global section of $\mathcal{O}_{\mathbf{P}^1}((q-1)m\infty)$. Consider the the τ-sheaf $\widetilde{\mathcal{F}} = (\mathcal{O}_{\mathbf{P}^1}(-m\infty), \tau)$ with τ the composition

$$\sigma^\star \mathcal{O}_{\mathbf{P}^1}(-m\infty) \xrightarrow{\sim} \mathcal{O}_{\mathbf{P}^1}(-qm\infty) \xrightarrow{f} \mathcal{O}_{\mathbf{P}^1}(-m\infty).$$

By construction we have $j^\star \widetilde{\mathcal{F}} \cong \mathcal{F}$. If moreover the strict inequality $(q-1)m > \deg f$ holds, then τ vanishes at the point ∞ and hence $i^\star \widetilde{\mathcal{F}}$ is nilpotent and we have $\widetilde{\mathcal{F}} = j_! \mathcal{F}$ in $\mathbf{Crys}\, \mathbf{P}^1$.

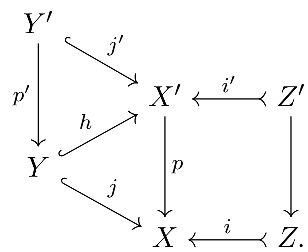
Given a crystal $\mathcal{F}$ on some Y one often chooses a compactification $j\colon Y \hookrightarrow X$ and then considers $j_! \mathcal{F}$. In such situations it is useful to be able to compare different compactifications. The main tool for doing so is the following lemma.

Lemma 2.29. *Consider a commutative diagram*



of noetherian schemes over $\mathbf{F}_q$, with j and h open immersions and p a proper map. Let $\mathcal{F}$ be a crystal on Y. Then $\mathrm{R}^n p_\star h_! \mathcal{F} = 0$ for $n > 0$ and there is a natural isomorphism $p_\star h_! \mathcal{F} = j_! \mathcal{F}$ in $\mathbf{Crys}\, X$.

Proof. Let Y' be the fibered product of Y and X' over X. Let $i\colon Z \to X$ be a closed complement and let $i'\colon Z' \to X'$ be its base change to X'. We thus have a diagram



in which the left parallelogram and the right square are cartesian.

Let us first consider the square on the right. The image of i' is disjoint with the image of h, so $i'^{\star}h_!\mathcal{F} = 0$. By the base change theorem (Theorem 2.19) we have

$$i^{\star}\mathrm{R}^n p_{\star}(h_!\mathcal{F}) = 0 \tag{9}$$

for all integers n.

Now consider the left part of the diagram. Being the base change of the open immersion j the map j' is an open immersion. Since also h is an open immersion, we see that p' is an open immersion. But being the base change of a proper map, p' is also proper. It follows that p' is both a closed and open immersion. In particular, if $\mathcal{G}$ is a crystal on X' then we have

$$\mathrm{R}^n p'_{\star} j'^{\star}\mathcal{G} = \begin{cases} h^{\star}\mathcal{G} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Taking $\mathcal{G} = h_!\mathcal{F}$ and applying the base change theorem to the left parallelogram we deduce

$$j^{\star}\mathrm{R}^n p_{\star}(h_!\mathcal{F}) = \begin{cases} \mathcal{F} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \tag{10}$$

Combining (9) and (10) we see that $\mathrm{R}^n p_{\star}(h_!\mathcal{F}) = 0$ for $n > 0$ and $\mathrm{R}^0 p_{\star}(h_!\mathcal{F}) = j_!\mathcal{F}$, which finishes the proof. □

The trace function of $j_!\mathcal{F}$ is of course the trace function of $\mathcal{F}$ extended by zero:

PROPOSITION 2.30. *Let X be a scheme of finite type over $\mathbf{F}_q$. Let $j\colon U \to X$ be an open subscheme. Let $\mathcal{F}$ be a crystal on U. Then we have*

$$\operatorname{tr}_{j_!\mathcal{F}} x = \begin{cases} \operatorname{tr}_{\mathcal{F}} x & \text{if } x \in U(\mathbf{F}_q), \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in X(\mathbf{F}_q)$. □

5. Pushforward with proper support

THEOREM 2.31 (Nagata [**43, 41**]). *Let $f\colon X \to Y$ be a morphism of noetherian schemes. Then there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \overline{X} \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \bar{f}} & \\ Y & & \end{array} \tag{11}$$

with j an open immersion and $\bar{f}$ a proper morphism if and only if f is separated and of finite type. □

We call a factorization of f into an open immersion $X \hookrightarrow \overline{X}$ and a proper morphism $\overline{X} \to Y$ a *compactification* of the map f.

PROPOSITION 2.32. *Let $f\colon X \to Y$ be a morphism of noetherian schemes over $\mathbf{F}_q$ and let*

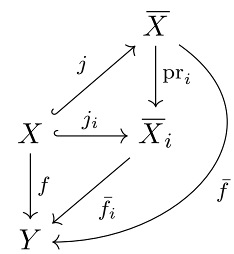
$$\begin{array}{ccc} X & \overset{j_1}{\hookrightarrow} & \overline{X}_1 \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \bar{f}_1} & \\ Y & & \end{array} \qquad \begin{array}{ccc} X & \overset{j_2}{\hookrightarrow} & \overline{X}_2 \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \bar{f}_2} & \\ Y & & \end{array}$$

be compactifications of f. Then the functors $\mathrm{R}^n\bar{f}_{1,\star}j_{1,!}$ and $\mathrm{R}^n\bar{f}_{2,\star}j_{2,!}$ from $\mathbf{Crys}\,X$ to $\mathbf{Crys}\,Y$ are isomorphic.

PROOF. Write $\overline{X}$ for the fiber product $\overline{X}_1 \times_Y \overline{X}_2$. Then we have an open immersion $j = (j_1, j_2)\colon X \to \overline{X}$ and a proper map $\bar{f}\colon \overline{X} \to Y$ giving a compactification

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \overline{X} \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \bar{f}} & \\ Y & & \end{array}$$

of $f\colon X \to Y$. For each $i \in \{1, 2\}$ we have a commutative diagram



and it suffices to show that

$$\mathrm{R}^n\bar{f}_{i,\star} \circ j_{i,!} \cong \mathrm{R}^n\bar{f} \circ j_!$$

as functors from $\mathbf{Crys}\, X$ to $\mathbf{Crys}\, Y$. Let $\mathcal{F}$ be a crystal on X. Then by Lemma 2.29 and by the Leray spectral sequence (Theorem 2.15) applied to the map $\bar{f} = \bar{f}_i \circ \mathrm{pr}_i$ and the crystal $j_!\mathcal{F}$ we obtain an isomorphism

$$\mathrm{R}^n\bar{f}_{i,\star} \circ j_{i,!} \cong \mathrm{R}^n\bar{f} \circ j_!$$

of crystals on Y, functorial in $\mathcal{F}$. $\square$

For every separated finite type morphism $f\colon X \to Y$ of schemes over $\mathbf{F}_q$, and for every integer n, we obtain a functor

$$\mathrm{R}^n f_! := \mathrm{R}^n\bar{f}_\star \circ j_!\colon\ \mathbf{Crys}\, X \to \mathbf{Crys}\, Y,$$

well-defined up to isomorphism. This functor is called the n-th *direct image with proper support*.

THEOREM 2.33. *Let $f\colon X \to Y$ be a separated morphism of finite type between noetherian schemes over $\mathbf{F}_q$. For every short exact sequence*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \tag{12}$$

in $\mathbf{Crys}\, X$ there is a long exact sequence

$$0 \to \mathrm{R}^0 f_!\mathcal{F}_1 \to \mathrm{R}^0 f_!\mathcal{F}_2 \to \mathrm{R}^0 f_!\mathcal{F}_3 \to \mathrm{R}^1 f_!\mathcal{F}_1 \to \cdots$$

in $\mathbf{Crys}\, Y$, depending functorially on (12).

PROOF. Choose a compactification $X \overset{j}{\hookrightarrow} \overline{X} \overset{\bar{f}}{\to} Y$ of f. The desired exact sequence is the long exact sequence for the $\mathrm{R}^i\bar{f}_\star$ (Proposition 2.14) applied to the short exact sequence

$$0 \to j_!\mathcal{F}_1 \to j_!\mathcal{F}_2 \to j_!\mathcal{F}_3 \to 0.$$

in $\mathbf{Crys}\,\overline{X}$. □

THEOREM 2.34 (Leray spectral sequence). *Let X, Y and Z be noetherian schemes over $\mathbf{F}_q$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be separated morphisms of finite type. Then for every crystal $\mathcal{F}$ on X there is a spectral sequence in $\mathbf{Crys}\,Z$ with*

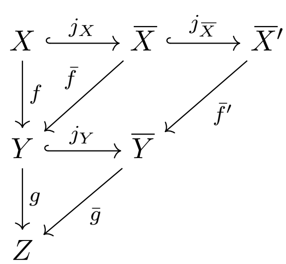
$$E_2^{p,q} = \mathrm{R}^p g_! \mathrm{R}^q f_! \mathcal{F},$$

and converging to $\mathrm{R}^{p+q}(gf)_!\mathcal{F}$, depending functorially on $\mathcal{F}$.

PROOF. Choose compactifications $X \hookrightarrow \overline{X} \to Y$ and $Y \hookrightarrow \overline{Y} \to Z$ of f and g respectively. Let

$$\overline{X} \hookrightarrow \overline{X}' \to \overline{Y}$$

be a compactification of the composite map $\overline{X} \to \overline{Y}$. We obtain a commutative diagram



in which the three j's are open immersions, and $\bar{f}$, $\bar{f}'$ and $\bar{g}$ are proper maps. We will now modify this construction slightly, to obtain the additional property that the parallelogram is cartesian. Consider the fibre product $Y \times_{\overline{Y}} \overline{X}'$. In the sequence of maps

$$\overline{X} \to Y \times_{\overline{Y}} \overline{X}' \hookrightarrow \overline{X}'$$

the second map is an open immersion (as it is the base change of the open immersion $Y \hookrightarrow \overline{Y}$), and the composite is an open immersion, so also the first map is an open immersion. At the same time, the map $Y \times_{\overline{Y}} \overline{X}' \to Y$ is proper, as it is the base change of the proper map $\overline{X}' \to \overline{Y}$. We conclude that replacing $\overline{X}$ by the fibre product $Y \times_{\overline{Y}} \overline{X}'$ we may assume that in the commutative diagram the parallelogram is cartesian, with j_X, $j_{\overline{X}}$, j_Y still being open immersions, and $\bar{f}$, $\bar{f}'$ and $\bar{g}$ still being proper maps.

We claim that for every crystal $\mathcal{G}$ on $\overline{X}$ and for every n we there is a natural isomorphism

$$\psi\colon j_{Y,!}\left(\mathrm{R}^n\bar{f}_\star\mathcal{G}\right) \to \mathrm{R}^n\bar{f}'_\star\left(j_{\bar{X},!}\mathcal{G}\right) \tag{13}$$

of crystals on $\overline{Y}$. Indeed, $\mathrm{R}^n\bar{f}_\star$ may be computed locally on the target, which shows that both sides are isomorphic on $Y \subset \overline{Y}$. By the defining property of $j_{Y,!}$ this isomorphism extends to a natural map (13). Let $i\colon Z \to \overline{Y}$ be a closed immersion whose image is the complement of Y. It suffices to show that $i^\star\psi$ is an isomorphism, or, which is the same, that $i^\star\mathrm{R}^n\bar{f}'_\star\left(j_{\bar{X},!}\mathcal{G}\right)$ is an isomorphism. But the latter follows from the base change theorem (Theorem 2.19).

Now taking $\mathcal{G} = j_{X,!}\mathcal{F}$ in the isomorphism (13) we find

$$\mathrm{R}^p g_! \mathrm{R}^q f_! \mathcal{F} = \mathrm{R}^p \bar{g}_\star j_{Y,!} \mathrm{R}^q f_! \mathcal{F} = \mathrm{R}^p \bar{g}_\star \mathrm{R}^q \bar{f}'_\star j_! \mathcal{F}$$

where j is the inclusion of X in $\overline{X}'$. The theorem now follows from the Leray spectral sequence for $\bar{g} \circ \bar{f}'$ applied to the the crystal $j_!\mathcal{F}$ on $\bar{X}'$. □

THEOREM 2.35 (Proper base change). *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of noetherian schemes over $\mathbf{F}_q$. *Assume that* f *is separated and of finite type. Then* f' *is separated and of finite type, and for every integer* i *and crystal* $\mathcal{F}$ *on* X *there is a natural isomorphism*

$$g^\star \mathrm{R}^i f_! \mathcal{F} \xrightarrow{\sim} \mathrm{R}^i f'_! g'^\star \mathcal{F}$$

of crystals on Y'.

PROOF. Let

$$\begin{array}{ccc} X & \stackrel{j}{\hookrightarrow} & \overline{X} \\ \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \bar{f}} & \\ Y & & \end{array}$$

be a compactification of $f\colon X \to Y$. Then its base change

$$\begin{array}{ccc} X' & \xrightarrow{j'} & \overline{X}' \\ \downarrow{\scriptstyle f'} & \swarrow{\scriptstyle \bar{f}'} & \\ Y' & & \end{array}$$

along $Y' \to Y$ is a compactification of $f'\colon X' \to Y'$.

Now let $\mathcal{F}$ be a crystal on X. Denote the base change map $\overline{X}' \to \overline{X}$ by $\bar{g}'$. By construction we have an isomorphism $\bar{g}'^{\star} j_! \mathcal{F} \cong j'_! g'^{\star} \mathcal{F}$ in $\mathbf{Crys}\, \overline{X}'$. Applying $\mathrm{R}^i \bar{f}'$ yields an isomorphism

$$\mathrm{R}^i \bar{f}' \bar{g}'^{\star} j_! \mathcal{F} \cong \mathrm{R}^i \bar{f}' j'_! g'^{\star} \mathcal{F}$$

in $\mathbf{Crys}\, Y'$. Applying corollary 2.19 to the crystal $j_! \mathcal{F}$ we find an isomorphism

$$\bar{g}^{\star} \mathrm{R}^i f_{\star} j_! \mathcal{F} \cong \mathrm{R}^i \bar{f}'_{\star} \bar{g}'^{\star} j_! \mathcal{F}$$

of crystals on Y'. Combining the above two isomorphisms yields

$$\bar{g}^{\star} \mathrm{R}^i f_! \mathcal{F} = \bar{g}^{\star} \mathrm{R}^i f_{\star} j_! \mathcal{F} \cong \mathrm{R}^i \bar{f}' j'_! g'^{\star} \mathcal{F} = \mathrm{R}^i f'_! g'^{\star} \mathcal{F},$$

the desired isomorphism. □

Notes

Most of the material in the chapter is due to Böckle and Pink [**11**], in particular the crucial observation that inverting nil-isomorphisms leads to a well-defined extension-by-zero functor. Because we work with $\mathbf{F}_q$-coefficients, a number of things become significantly easier. In particular, tensor product is exact. Also, the restriction to $\mathbf{F}_q$-coefficients allows us to give an easier proof of the base-change theorems, based on the theorem of formal functions.

One can similarly define functors $\otimes$, $f^{\star}$ and $\mathrm{R}f_!$ on the derived categories $\mathcal{D}^b(\mathbf{Crys}\, X)$. The functor $\mathrm{R}f_!$ is defined as $\mathrm{R}\bar{f}_{\star} \circ j_!$ for any compactification $f = \bar{f} \circ j$.

The extension-by-zero functor of Böckle and Pink that we described in this chapter is closely related to Deligne's approach to defining $j_!$ on quasi-coherent sheaves [**15**]. He defines extension by zero $j_! \mathcal{F}$ of a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$ as the projective system consisting of all possible extensions $\widetilde{\mathcal{F}}$ to X. If $\mathcal{F}$ is a coherent τ-sheaf, then τ will extend to any 'small enough' extension, and the projective system will stabilize in the sense that eventually all maps become nil-isomorphisms.

Exercises

EXERCISE 2.1. Let X be a noetherian scheme over $\mathbf{F}_q$. Let $j\colon U \hookrightarrow X$ be an open immersion. Show that for all crystals $\mathcal{F}$ on U and $\mathcal{G}$ on X we have a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, j^\star\mathcal{G}) = \mathrm{Hom}(j_!\mathcal{F}, \mathcal{G}).$$

In other words, show that the functor $j_!$ is a left adjoint of $j^\star$.

EXERCISE 2.2. Let X be a noetherian scheme over $\mathbf{F}_q$. Let $i\colon Z \rightarrowtail X$ be a closed immersion and let $j\colon U \hookrightarrow X$ be the open complement. Let $\mathcal{F}$ be a crystal on X. Show that there is an exact sequence

$$0 \to j_!j^\star\mathcal{F} \to \mathcal{F} \to i_\star i^\star\mathcal{F} \to 0$$

of crystals on X.

EXERCISE 2.3 (Projection formula). Let $f\colon X \to Y$ be a compactifiable morphism of noetherian schemes over $\mathbf{F}_q$. Let $\mathcal{F}$ be a crystal on X and $\mathcal{G}$ a crystal on Y. Let i be an integer. Show that there is a natural isomorphism

$$\mathcal{G} \otimes \mathrm{R}^i f_! \mathcal{F} \xrightarrow{\sim} \mathrm{R}^i f_!(f^\star\mathcal{G} \otimes \mathcal{F})$$

of crystals on X.

EXERCISE 2.4 (Künneth formula ($\star$)). Let $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ be morphisms of schemes of finite type over $\mathbf{F}_q$. Let $f\colon X_1 \times_Y X_2 \to Y$ be their fiber product, and let $p_i\colon X_1 \times_Y X_2 \to X_i$ denote the projections. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be crystals on X_1 and X_2 respectively. Show that there is a natural isomorphism

$$\bigoplus_{i+j=n} \mathrm{R}^i f_{1,!}\mathcal{F}_1 \otimes \mathrm{R}^j f_{2,!}\mathcal{F}_2 \xrightarrow{\sim} \mathrm{R}^n f_!(p_1^\star\mathcal{F}_1 \otimes p_2^\star\mathcal{F}_2)$$

of crystals on Y.

CHAPTER 3

The Woods Hole trace formula

1. The Grothendieck group of crystals

Let X be a noetherian scheme over $\mathbf{F}_q$. Denote by $\mathrm{K}_0(X)$ the Grothendieck group of the abelian category $\mathbf{Crys}\, X$. This is the abelian group generated by isomorphism classes $[\mathcal{F}]$ of crystals, modulo relations $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ for every short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ in $\mathbf{Crys}\, X$.

LEMMA 3.1. *Let $\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n = 0$ be crystals on X. Then we have $[\mathcal{F}] = \sum_i [\mathcal{F}^i/\mathcal{F}^{i+1}]$ in $\mathrm{K}_0(X)$.*

PROOF. For every i the exact sequence $0 \to \mathcal{F}^{i+1} \to \mathcal{F}^i \to \mathcal{F}^i/\mathcal{F}^{i+1} \to 0$ gives $[\mathcal{F}^i] - [\mathcal{F}^{i+1}] = [\mathcal{F}^i/\mathcal{F}^{i+1}]$ in $\mathrm{K}_0(X)$ which summing over all $i \in \{0, \ldots, n-1\}$ gives the claimed identity. □

LEMMA 3.2. *Let $\mathcal{F}^\bullet$ be a bounded complex of crystals on X. Then*

$$\sum_i (-1)^i [\mathcal{F}^i] = \sum_i (-1)^i [\mathrm{H}^i(\mathcal{F}^\bullet)]$$

in $\mathrm{K}_0(X)$.

It follows that if $0 \to \mathcal{F}_1 \to \cdots \to \mathcal{F}_n \to 0$ is exact in $\mathbf{Crys}\, X$ then we have $\sum_i (-1)^i [\mathcal{F}_i] = 0$ in $\mathrm{K}_0(X)$.

PROOF OF LEMMA 3.2. On the one hand, the complex $\mathcal{F}^\bullet$ splits into short exact sequences

$$0 \to \ker(\mathcal{F}^i \to \mathcal{F}^{i+1}) \to \mathcal{F}^i \to \mathrm{im}(\mathcal{F}^i \to \mathcal{F}^{i+1}) \to 0$$

while on the other hand the cohomology crystals sit in short exact sequences

$$0 \to \mathrm{im}(\mathcal{F}^{i-1} \to \mathcal{F}^i) \to \ker(\mathcal{F}^i \to \mathcal{F}^{i+1}) \to \mathrm{H}^i(\mathcal{F}^\bullet) \to 0.$$

Taking the alternating sum over i of the resulting identities in $\mathrm{K}_0(X)$ and comparing the terms yields the desired identity $\sum_i (-1)^i [\mathcal{F}^i] = \sum_i (-1)^i [\mathrm{H}^i(\mathcal{F}^\bullet)]$. $\square$

Since the functor $-\otimes-$ on crystals is exact in both arguments (Corollary 2.8), it induces a bi-additive map

$$\mathrm{K}_0(X) \times \mathrm{K}_0(X) \to \mathrm{K}_0(X),\ \big([\mathcal{F}],[\mathcal{G}]\big) \mapsto [\mathcal{F}\otimes\mathcal{G}]$$

which gives $\mathrm{K}_0(X)$ the structure of a commutative ring.

Similarly, for a map $f\colon X \to Y$ of noetherian schemes over $\mathbf{F}_q$ the functor $f^\star\colon \mathbf{Crys}\,Y \to \mathbf{Crys}\,X$ is exact (Corollary 2.2), so it induces a map

$$f^\star\colon \mathrm{K}_0(Y) \to \mathrm{K}_0(X),\ [\mathcal{F}] \mapsto [f^\star\mathcal{F}].$$

This map is a ring homomorphism.

Finally, if $f\colon X \to Y$ is a separated map of finite type between noetherian schemes over $\mathbf{F}_q$ then the long exact sequence of Theorem 2.33 combined with Lemma 3.2 shows that the map

$$\mathrm{R}f_!\colon \mathrm{K}_0(X) \to \mathrm{K}_0(Y),\ [\mathcal{F}] \mapsto \sum_{i\geq 0}(-1)^i[\mathrm{R}^i f_!\mathcal{F}]$$

is well-defined. It is additive, but in general not a ring homomorphism. It has the following important transitivity property.

Proposition 3.3. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be separated morphisms of finite type between noetherian schemes over $\mathbf{F}_q$. Then $\mathrm{R}(gf)_! = \mathrm{R}g_! \circ \mathrm{R}f_!$ as maps from $\mathrm{K}_0(X)$ to $\mathrm{K}_0(Z)$.*

Proof. Let $\mathcal{F}$ be a crystal on X. Consider the associated Leray spectral sequence (Theorem 2.34)

$$E_2^{s,t} = \mathrm{R}^s g_! \mathrm{R}^t f_! \mathcal{F} \Longrightarrow \mathrm{R}^{s+t}(gf)_!\mathcal{F}.$$

We have in $\mathrm{K}_0(Z)$ the identity

$$\mathrm{R}g_!\mathrm{R}f_![\mathcal{F}] = \sum_s\sum_t (-1)^{s+t}[\mathrm{R}^s g_!\mathrm{R}^t f_!\mathcal{F}] = \sum_{s,t}(-1)^{s+t}[E_2^{s,t}].$$

Consider one of the slanted complexes

$$\cdots \to E_i^{s-i,t+i-1} \to E_n^{s,t} \to E_n^{s+i,t-i+1} \to \cdots$$

on page i. By the definition of a spectral sequence we have that the cohomology at $E_i^{s,t}$ of this complex is isomorphic to $E_{i+1}^{s,t}$. With Lemma

3.2 we find

$$\sum_{s,t}(-1)^{s+t}[E_i^{s,t}] = \sum_{s,t}(-1)^{s+t}[E_{i+1}^{s,t}]$$

in $\mathrm{K}_0(Z)$. Applying this successively with $i = 2, 3, \dots$ we find

$$\mathrm{R}g_!\mathrm{R}f_![\mathcal{F}] = \sum_{s,t}(-1)^{s+t}[E_\infty^{s,t}].$$

On the other hand, applying Lemma 3.1 to the spectral sequence filtration on $\mathrm{R}^n(gf)_!\mathcal{F}$ we obtain

$$(-1)^n[\mathrm{R}^n(gf)_!\mathcal{F}] = \sum_{s+t=n}(-1)^{s+t}[E_\infty^{s,t}]$$

and summing over all n gives

$$\mathrm{R}(gf)_![\mathcal{F}] = \sum_{s,t}(-1)^{s+t}[E_\infty^{s,t}],$$

which completes the proof. □

2. The sheaves-functions dictionary

Let X be a scheme of finite type over $\mathbf{F}_q$. Denote by $\mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q)$ the algebra of $\mathbf{F}_q$-valued functions on $X(\mathbf{F}_q)$. We have a homomorphism

$$\mathrm{tr}\colon \mathrm{K}_0(X) \to \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q),\ [\mathcal{F}] \mapsto \mathrm{tr}_{\mathcal{F}}$$

which is well-defined by Exercise 1.2. It is a ring homomorphism by Proposition 2.9. If $f\colon X \to Y$ is a morphism of schemes of finite type over $\mathbf{F}_q$ then the diagram

$$\begin{array}{ccc} \mathrm{K}_0(Y) & \longrightarrow & \mathrm{Map}(Y(\mathbf{F}_q), \mathbf{F}_q) \\ \downarrow f^\star & & \downarrow -\circ f \\ \mathrm{K}_0(X) & \longrightarrow & \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q) \end{array}$$

commutes by Proposition 2.3.

THEOREM 3.4 (Woods Hole trace formula). *Let $f\colon X \to Y$ be a separated morphism of schemes of finite type over $\mathbf{F}_q$ and let $K \in \mathrm{K}_0(X)$. Then have*

$$\mathrm{tr}_{\mathrm{R}f_!K}\, y \ = \sum_{x \in f^{-1}(y)(\mathbf{F}_q)} \mathrm{tr}_K\, x \tag{14}$$

for every $y \in Y(\mathbf{F}_q)$.

Equivalently, the diagram

$$\begin{array}{ccc} \mathrm{K}_0(X) & \longrightarrow & \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q) \\ \Big\downarrow{\scriptstyle \mathrm{R}f_!} & & \Big\downarrow{\scriptstyle f_!} \\ \mathrm{K}_0(Y) & \longrightarrow & \mathrm{Map}(Y(\mathbf{F}_q), \mathbf{F}_q) \end{array}$$

commutes. Here the right-hand map is "integration along the fibers":

$$f_!\colon \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q) \to \mathrm{Map}(Y(\mathbf{F}_q), \mathbf{F}_q),\ g \mapsto \Big(y \mapsto \sum_{\substack{x\in X(\mathbf{F}_q) \\ f(x)=y}} g(x)\Big).$$

In the coming two sections we will prove Theorem 3.4, following the elegant approach of Fulton [**22**].

3. A quotient of the Grothendieck group of crystals

Let X be a noetherian scheme over $\mathbf{F}_q$. We denote by $G(X)$ the quotient of $\mathrm{K}_0(X)$ by the relations

$$[(\mathcal{F}, \tau_1)] + [(\mathcal{F}, \tau_2)] = [(\mathcal{F}, \tau_1 + \tau_2)] \tag{15}$$

for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and maps $\tau_i\colon \sigma^\star\mathcal{F} \to \mathcal{F}$. We will denote the class of a coherent τ-sheaf $\mathcal{F}$ in $\mathrm{K}_0(X)$ and in $G(X)$ both by $[\mathcal{F}]$. When necessary we will specify in which group we are working.

For all $\lambda \in \mathbf{F}_q$ the construction $(\mathcal{F}, \tau) \mapsto (\mathcal{F}, \lambda\tau)$ defines an endomorphism of the groups $\mathrm{K}_0(X)$ and $G(X)$. Thanks to the relations (15) the map $\mathbf{F}_q \to \mathrm{End}\, G(X)$ is a ring homomorphism. This makes $G(X)$ into an $\mathbf{F}_q$-vector space.

The tensor product of coherent τ-sheaves is compatible with the relations (15) and makes $G(X)$ into an $\mathbf{F}_q$-algebra. The quotient map $\mathrm{K}_0(X) \to G(X)$ is a ring homomorphism.

For every morphism $f\colon X \to Y$ of noetherian schemes over $\mathbf{F}_q$ the ring homomorphism $f^\star\colon \mathrm{K}_0(Y) \to \mathrm{K}_0(X)$ induces an $\mathbf{F}_q$-algebra homomorphism $f^\star\colon G(Y) \to G(X)$.

PROPOSITION 3.5. *Let $f\colon X \to Y$ be a separated morphism of finite type between noetherian schemes over $\mathbf{F}_q$. Then $\mathrm{R}f_!\colon \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$ induces an $\mathbf{F}_q$-linear map $\mathrm{R}f_!\colon G(X) \to G(Y)$.*

PROOF. Choose a compactification

$$X \xrightarrow{j} \overline{X} \xrightarrow{\bar f} Y$$

of $f\colon X \to Y$. Consider coherent τ-sheaves $(\mathcal{F}, \tau_1)$ and $(\mathcal{F}, \tau_2)$ on U with the same underlying coherent $\mathcal{O}_U$-module $\mathcal{F}$. Let $\mathcal{G} \subset j_\star \mathcal{F}$ be an arbitrary coherent $\mathcal{O}_{\overline{X}}$-module extending $\mathcal{F}$. By Lemma 2.23 we have, for n large enough, that $(\mathcal{I}^n\mathcal{G}, \tau_1)$, $(\mathcal{I}^n\mathcal{G}, \tau_2)$ and $(\mathcal{I}^n\mathcal{G}, \tau_1+\tau_2)$ define extensions by zero of $(\mathcal{F}, \tau_1)$, $(\mathcal{F}, \tau_2)$ and $(\mathcal{F}, \tau_1+\tau_2)$ respectively. We find isomorphisms of crystals

$$\mathrm{R}^s f_!\,(\mathcal{F}, \tau) \cong (\mathrm{R}^s \bar{f}_\star \mathcal{I}^n \mathcal{G}, \tau)$$

for $\tau \in \{\tau_1, \tau_2, \tau_1+\tau_2\}$ and for all s, and conclude that the map $\mathrm{R}f_!\colon \mathrm{K}_0(X) \to \mathrm{K}_0(Y) \twoheadrightarrow G(Y)$ factors over $G(X)$, as we had to show. $\square$

Now assume that X is a scheme of finite type over $\mathbf{F}_q$. The trace of an endomorphism τ of a finite-dimensional $\mathbf{F}_q$-vector space is linear in τ, hence the map $\mathrm{tr}\colon \mathrm{K}_0(X) \to \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q)$ factors over an $\mathbf{F}_q$-algebra homomorphism $G(X) \to \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q)$.

THEOREM 3.6 (Localization Theorem). *Let X be a scheme of finite type over $\mathbf{F}_q$. Then the $\mathbf{F}_q$-linear maps*

$$\mathrm{tr}\colon G(X) \to \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q),\ K \mapsto \mathrm{tr}_K$$

and

$$\ell\colon \mathrm{Map}(X(\mathbf{F}_q), \mathbf{F}_q) \to G(X),\ f \mapsto \sum_{x \in X(\mathbf{F}_q)} f(x)\,\big[i_{x,\star}\mathbf{1}_{\mathrm{Spec}\,\mathbf{F}_q}\big]$$

are mutually inverse isomorphisms.

Of course we have $\mathrm{tr} \circ \ell = \mathrm{id}$. In the next section we will show $\ell \circ \mathrm{tr} = \mathrm{id}$. Theorem 3.6 almost immediately implies the Woods Hole trace formula, as we now show.

PROOF OF THEOREM 3.4, ASSUMING THEOREM 3.6. Let $f\colon X \to Y$ be a separated morphism of schemes of finite type over $\mathbf{F}_q$. We need to show

$$\mathrm{tr}_{\mathrm{R}f_! K}\, y = \sum_{\substack{x \in X(\mathbf{F}_q) \\ f(x)=y}} \mathrm{tr}_K\, x \tag{16}$$

for all $K \in \mathrm{K}_0(X)$. Both sides depend only on the class of K in $G(X)$, and depend linearly on this class. By Theorem 3.6 we may restrict to

$K = [i_{x_0,\star}\mathbf{1}]$ for $x_0 \in X(\mathbf{F}_q)$. Let $y_0 \in Y(\mathbf{F}_q)$ be the image of x_0. Then we have

$$\mathrm{R}^n f_! i_{x_0,\star}\mathbf{1} = \begin{cases} i_{y_0,\star}\mathbf{1} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

so that $\mathrm{R}f_! K = [i_{y_0,\star}\mathbf{1}]$. Both the left-hand-side and right-hand-side of (16) are 1 for $y = y_0$ and 0 for $y \neq y_0$, and the theorem follows. □

4. Proof of the Localization Theorem

The proof of the Localization Theorem (Theorem 3.6) goes by reduction to the case $X = \mathbf{P}^n$, which we treat first.

Let $R = \mathbf{F}_q[x_0, \ldots, x_n]$ and $\mathbf{P}^n = \mathbf{P}^n_{\mathbf{F}_q} = \operatorname{Proj} R$. Let M be a graded R-module. We call a map $\tau_s\colon M \to M$ *graded q-linear* if

(1) $\tau_s(rm) = r^q \tau_s(m)$ for all $r \in R$ and $m \in M$, and,
(2) $\tau_s(M_i) \subset M_{qi}$ for all i.

Let $\widetilde{M}$ be the quasi-coherent $\mathcal{O}_{\mathbf{P}^n}$-module associated with the graded R-module M. A graded q-linear map $\tau_s\colon M \to M$ induces a q-linear map $\tau_s\colon \widetilde{M} \to \widetilde{M}$, making $\widetilde{M}$ into a quasi-coherent τ-sheaf. It is coherent if M is finitely generated.

PROPOSITION 3.7. *The functor $M \to \widetilde{\mathcal{M}}$ from the category of pairs (M, τ_s) with M a finitely generated R-module and $\tau_s\colon M \to M$ graded q-linear to the category $\mathbf{Coh}_\tau\,\mathbf{P}^n$ is exact and essentially surjective.*

See exercise 3.3 for a refinement of this Proposition.

PROOF OF PROPOSITION 3.7. The exactness follows immediately from the corresponding statements on finitely generated graded R-modules and and coherent $\mathcal{O}_{\mathbf{P}^n}$-modules. For the essential surjectivity, let $(\mathcal{F}, \tau)$ be a coherent τ-sheaf and set

$$M := \bigoplus_{d \geq 0} M_d, \quad M_d := \Gamma(\mathbf{P}^n, \mathcal{F}(d)),$$

so that $\widetilde{M} = \mathcal{F}$ as coherent $\mathcal{O}_{\mathbf{P}^n}$-modules. The map $\tau\colon \sigma^\star\mathcal{F} \to \mathcal{F}$ induces maps $\sigma^\star(\mathcal{F}(d)) \to \mathcal{F}(qd)$, which induce q-linear maps $\tau_s\colon M_d \to M_{qd}$, and the functor in the proposition maps the pair (M, τ_s) to $(\mathcal{F}, \tau)$. □

LEMMA 3.8. *The group $\mathrm{K}_0(\mathbf{Crys}\,\mathbf{P}^n)$ is generated by the classes of coherent τ-sheaves $(\mathcal{F}, \tau)$ with $\mathcal{F} \cong \oplus_i \mathcal{O}(d_i)$ for some $d_i \leq 0$.*

PROOF. Let $\mathcal{F}$ be a coherent τ-sheaf on $\mathbf{P}^n$. Let (M, τ_s) be a pair representing $\mathcal{F}$ as in Proposition 3.7. The graded R-module M has a finite free graded resolution

$$0 \to M_n \to \cdots \to M_0 \to M \to 0.$$

Since M_0 is free, and since a q-linear map is determined by the images of a set of generators, there exists a q-linear $\tau_{s,0} \colon M_0 \to M_0$ such that the square

$$\begin{array}{ccc} M_0 & \twoheadrightarrow & M \\ \downarrow{\scriptstyle \tau_{s,0}} & & \downarrow{\scriptstyle \tau_s} \\ M_0 & \twoheadrightarrow & M \end{array}$$

commutes, and such that τ_s maps $(M_0)_i$ to $(M_0)_{qi}$ for all i. Repeating this argument with the kernel of $M_0 \twoheadrightarrow M$ we find a diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M_n & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_{s,n}} & & & & \downarrow{\scriptstyle \tau_{s,0}} & & \downarrow{\scriptstyle \tau_s} & & \\ 0 & \longrightarrow & M_n & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and hence a resolution

$$0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0$$

of $\mathcal{F}$ by coherent τ-sheaves of the form $(\oplus_i \mathcal{O}(d_i), \tau)$. In particular we have

$$[\mathcal{F}] = [\mathcal{F}_0] - [\mathcal{F}_1] + [\mathcal{F}_2] - \cdots$$

in $\mathrm{K}_0(\mathbf{Crys}\,\mathbf{P}^n)$, and conclude that $\mathrm{K}_0(\mathbf{Crys}\,\mathbf{P}^n)$ is generated by the classes of coherent τ-sheaves $(\mathcal{F}, \tau)$ with $\mathcal{F} \cong \oplus_i \mathcal{O}(d_i)$ with $d_i \in \mathbf{Z}$.

Now let $\mathcal{F}$ be a coherent τ-sheaf of the form $(\oplus_i \mathcal{O}(d_i), \tau)$. Let d be the maximum of the d_i and assume that $d > 0$. Consider the sub-$\mathcal{O}_{\mathbf{P}^n}$-module

$$\mathcal{G} = \bigoplus_{\{i \mid d_i = d\}} \mathcal{O}(d_i) \subset \mathcal{F}.$$

We have $\sigma^\star \mathcal{O}(d) = \mathcal{O}(qd)$. Since $qd > d_i$ for all i, the map τ vanishes on $\sigma^\star \mathcal{G}$. In particular, $\mathcal{G}$ is a nilpotent sub-τ-sheaf of $\mathcal{F}$ and $[\mathcal{F}] = [\mathcal{F}/\mathcal{G}]$ in $\mathrm{K}_0(\mathbf{Crys}\,\mathbf{P}^n)$. The coherent τ-sheaf $\mathcal{F}/\mathcal{G}$ is again of the form $(\oplus_i \mathcal{O}(d_i), \tau)$, but with a strictly smaller $d = \max_i d_i$. We may repeat this argument until d is less than or equal to 0 and we conclude that $\mathrm{K}_0(\mathbf{Crys}\,\mathbf{P}^n)$ is indeed generated by the classes of the coherent τ-sheaves $(\oplus_i \mathcal{O}(d_i), \tau)$ with $d_i \leq 0$ for all i. □

LEMMA 3.9. *The group $G(\mathbf{P}^n)$ is generated by the $[(\mathcal{O}(d), \tau)]$, with $d \leq 0$ and τ arbitrary.*

PROOF. If $\mathcal{F} = (\oplus_i \mathcal{O}(d_i), \tau)$ then τ is given by a matrix (τ_{ij}) with

$$\tau_{ij} \colon \sigma^\star \mathcal{O}(d_i) \to \sigma^\star \mathcal{O}(d_j).$$

Let us denote by $\hat{\tau}_{ij}$ the matrix of which all entries are zero, except for the (i,j)-th which equals τ_{ij}. We have $(\tau_{ij}) = \sum_{i,j} \hat{\tau}_{ij}$ so that in $G(\mathbf{P}^n)$ we have

$$[\mathcal{F}] = \sum_{i,j} [(\oplus_k \mathcal{O}(d_k), \hat{\tau}_{ij})].$$

If $i \neq j$ then $\hat{\tau}_{ij}$ is nilpotent, hence in $G(\mathbf{P}^n)$ we find the identity

$$[\mathcal{F}] = \sum_{i} [(\oplus_k \mathcal{O}(d_k), \hat{\tau}_{ii})].$$

Decomposing into rank one τ-sheaves we find

$$[\mathcal{F}] = \sum_{k} [(\mathcal{O}(d_k), \tau_{kk})]$$

and we see that G is indeed generated by the classes of τ-sheaves of the desired form. □

LEMMA 3.10. *The $\mathbf{F}_q$-vector space $G(\mathbf{P}^n)$ is generated by the classes $[(\mathcal{O}(d), \tau)]$, where $\tau \in \Gamma(\mathbf{P}^n, \mathcal{O}((1-q)d))$ is given by a monomial*

$$x_0^{e_0} \cdots x_n^{e_n}$$

with $\sum e_i = (1-q)d$ and $e_i \leq q-1$ for all i.

In particular, the Lemma shows that the dimension of $G(\mathbf{P}^n)$ is finite.

PROOF OF LEMMA 3.10. By the previous lemma, $G(\mathbf{P}^n)$ is generated by the classes of the pairs $(\mathcal{O}(d), \tau)$ with $d \leq 0$ and τ given by an $f \in \mathbf{F}_q[x_0, \ldots, x_n]$ which is homogenous of degree $(1-q)d$. Clearly we can restrict the f to monomials in the x_i.

Now let $f = x_0^{e_0} \cdots x_n^{e_n}$ be a monomial of degree $(1-q)d$ with $e_i \geq q$ for some i, so that $f = x_i^{q-1} f'$ with f' a monomial that is divisible by

x_i. Let $\iota\colon H \rightarrowtail \mathbf{P}^n$ be the hyperplane given by $x_i = 0$. Then we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}(qd) & \xrightarrow{x_i^q} & \mathcal{O}(q(d+1)) & \longrightarrow & \iota_\star \mathcal{O}_H \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle 0} \\ \mathcal{O}(d) & \xrightarrow{x_i} & \mathcal{O}(d+1) & \longrightarrow & \iota_\star \mathcal{O}_H \end{array}$$

This gives a short exact sequence

$$0 \to (\mathcal{O}(d), \tau) \to (\mathcal{O}(d+1), \tau') \to (\iota_\star \mathcal{O}_H, 0) \to 0$$

of coherent τ-sheaves on $\mathbf{P}^n$ (with τ' given by f'), and since $(\iota_\star \mathcal{O}_H, 0)$ is nilpotent we have

$$[(\mathcal{O}(d), \tau)] = [(\mathcal{O}(d+1), \tau')]$$

in $G(\mathbf{P}^n)$. We can repeat this argument until f is replaced by a monomial with all exponents $\leq q-1$. $\square$

In fact we can exclude $[(\mathcal{O}, 1)]$ from the set of generators of the previous Lemma.

LEMMA 3.11. *As an $\mathbf{F}_q$-vector space $G(\mathbf{P}^n)$ is generated by the classes $[(\mathcal{O}(d), \tau)]$, where $\tau \in \Gamma(\mathbf{P}^n, \mathcal{O}((1-q)d))$ is given by a monomial*

$$x_0^{e_0} \cdots x_n^{e_n}$$

with $\sum e_i = (1-q)d$ and $e_i \leq q-1$ for all i and $e_i > 0$ for some i.

PROOF. Given $i_0 < i_1 < \cdots < i_m$ in $\{0, ..., n\}$ we consider the τ-sheaf

$$\mathcal{F}_{i_0 \cdots i_m} = \left(\mathcal{O}(-m-1), x_{i_0}^{q-1} \cdots x_{i_m}^{q-1}\right)$$

on $\mathbf{P}^n$. Note that it is one of the generators of Lemma 3.10. As usual, we denote by $i_0 \cdots \widehat{i_k} \cdots i_m$ the index where i_k has been removed. For every k there is a map $\mathcal{F}_{i_0 \cdots i_m} \to \mathcal{F}_{i_0 \cdots \widehat{i_k} \cdots i_m}$ given by multiplication with x_{i_k}. This is a map of τ-sheaves since the square

$$\begin{array}{ccc} \mathcal{O}(-q(m+1)) & \xrightarrow{x_{i_k}^q} & \mathcal{O}(-qm) \\ {\scriptstyle x_{i_0}^{q-1} \cdots x_{i_m}^{q-1}} \downarrow & & \downarrow {\scriptstyle x_{i_0}^{q-1} \cdots x_{i_m}^{q-1} / x_{i_k}^{q-1}} \\ \mathcal{O}(-(m+1)) & \xrightarrow{x_{i_k}} & \mathcal{O}(-m) \end{array}$$

commutes. Now consider the sequence

$$0 \to \mathcal{F}_{01\cdots n} \to \cdots \to \prod_{i_0<i_1} \mathcal{F}_{i_0 i_1} \to \prod_{i_0} \mathcal{F}_{i_0} \to \mathbf{1}_{\mathbf{P}^n} \to 0 \tag{17}$$

of coherent τ-sheaves on $\mathbf{P}^n$, where a section α of $\mathcal{F}_{i_0\cdots i_m}$ gets mapped to

$$\sum_k (-1)^k \, (x_{i_k}\alpha)_{i_0\cdots\widehat{i_k}\cdots i_m}\,.$$

This is the Koszul complex of the surjective map

$$(x_0,\ldots,x_n)\colon \mathcal{O}(-1)^{n+1} \to \mathcal{O}.$$

In particular the sequence (17) is exact. The exact sequence gives an expression of the class of $\mathbf{1}_{\mathbf{P}^n}$ as linear combination of the other $[\mathcal{F}_{i_0\cdots i_m}]$, and we conclude using Lemma 3.10. □

The identity in $\mathrm{K}_0(\mathbf{Crys}\,\mathbf{P}^n)$ established in the proof of the lemma, induces an identity of trace functions. For example, if $n=1$ this is the identity

$$x^{q-1}y^{q-1} - x^{q-1} - y^{q-1} + 1 = 0$$

for all $(x:y) \in \mathbf{P}^1(\mathbf{F}_q)$.

Corollary 3.12 (Localization Theorem for $\mathbf{P}^n$). *The maps*

$$\mathrm{tr}\colon G(\mathbf{P}^n) \to \mathrm{Map}(\mathbf{P}^n(\mathbf{F}_q), \mathbf{F}_q)$$

and

$$\ell\colon\ \mathrm{Map}(\mathbf{P}^n(\mathbf{F}_q), \mathbf{F}_q) \to G(\mathbf{P}^n)$$

are mutually inverse isomorphisms of $\mathbf{F}_q$-vector spaces.

Proof. We already know that $\mathrm{tr}\circ\ell = \mathrm{id}$, so it suffices to establish the inequality

$$\dim G(\mathbf{P}^n) \le \dim \mathrm{Map}(\mathbf{P}^n(\mathbf{F}_q), \mathbf{F}_q) = q^n + \cdots + q + 1.$$

By the preceding lemma we have that $\dim G(\mathbf{P}^n)$ is at most the number of nonzero tuples $(e_0,\ldots,e_n)$ with $0 \le e_i \le q-1$ and $\sum e_i$ divisible by $q-1$. Let S be the set of such tuples. For an integer m denote by S_m the set of tuples $(e_0,\ldots,e_n) \in S$ for which $e_m \neq 0$ and $e_i = 0$ for all $i > m$. For every choice of $e_0,\ldots,e_{m-1}$ in $\{0,\ldots,q-1\}$ there is a unique e_m such that $(e_0,\ldots,e_m,0,\ldots,0) \in S_n$. Therefore $\#S_m = q^m$. Since S is the disjoint union of the S_m we find $\#S = 1 + q + \cdots + q^n$, which yields the desired upper bound for $\dim G(\mathbf{P}^n)$. □

To deduce the full Localization Theorem, we use the following proposition.

PROPOSITION 3.13. *Let X be a scheme of finite type over $\mathbf{F}_q$. Let $i\colon Z \rightarrowtail X$ be a closed subscheme and $j\colon U \hookrightarrow X$ its open complement. Then the Localization Theorem holds for X if and only if it holds for Z and for U.*

PROOF. We claim that the maps

$$G(X) \to G(U) \oplus G(Z),\ [\mathcal{F}] \mapsto ([j^\star\mathcal{F}], [i^\star\mathcal{F}]) \tag{18}$$

and

$$G(Z) \oplus G(U) \to G(Z),\ ([\mathcal{F}_1], [\mathcal{F}_2]) \mapsto [j_!\mathcal{F}_1 \oplus i_\star\mathcal{F}_2] \tag{19}$$

are mutually inverse isomorphisms. Indeed, for crystals $\mathcal{F}_1$ and $\mathcal{F}_2$ on U and Z respectively we have $j^\star(j_!\mathcal{F}_1 \oplus i_\star\mathcal{F}_2) = \mathcal{F}_1$ and $i^\star(j_!\mathcal{F}_1 \oplus i_\star\mathcal{F}_2) = \mathcal{F}_2$. Conversely, if $\mathcal{F}$ is a crystal on X then the short exact sequence

$$0 \to j_!j^\star\mathcal{F} \to \mathcal{F} \to i_\star i^\star\mathcal{F} \to 0$$

shows that (19) maps $([j^\star\mathcal{F}], [i^\star\mathcal{F}])$ to $[\mathcal{F}]$.

Now for any scheme S of finite type over $\mathbf{F}_q$ consider the endomorphism $\ell \circ \mathrm{tr}$ of $G(S)$. Note that S satisfies the Localization Theorem if and only if $\ell\circ\mathrm{tr} = \mathrm{id}$ on $G(S)$. The decomposition $G(X) = G(U) \oplus G(Z)$ respects the action of $\ell \circ \mathrm{tr}$ on $G(X)$, $G(U)$ and $G(Z)$ so that $G(X)$ satisfies the Localization Theorem if and only if $G(U)$ and $G(Z)$ do. □

We can now finish the proof of the Localization Theorem.

PROOF OF THEOREM 3.6. We first show that the theorem holds for affine schemes X of finite type over $\mathbf{F}_q$. Indeed, choose a closed immersion $X \rightarrowtail \mathbf{A}^n$ and an open immersion $\mathbf{A}^n \hookrightarrow \mathbf{P}^n$. By Corollary 3.12 the theorem holds for $\mathbf{P}^n$, hence by the preceding proposition also for $\mathbf{A}^n$ and X.

Now let X be an arbitrary scheme of finite type over $\mathbf{F}_q$. Choose an affine open subset $U_0 \subset X_0 := X$ with closed complement X_1. Then choose an affine open subset $U_1 \subset X_1$ with closed complement X_2, and so on, leading to a finite decomposition

$$X = U_0 \amalg U_1 \amalg \cdots \amalg U_d.$$

Each of the U_i satisfies the localization theorem, and repeatedly applying the preceding proposition shows that X_d, X_{d-1}, and finally $X_0 = X$ satisfy the localization theorem. □

Notes

There are several closely related theorems that go by the name Woods Hole trace formula or Woods Hole fixed point theorem.

Let X be a compact complex manifold, $\sigma\colon X \to X$ an endomorphism with simple fixed points, $\mathcal{V}$ a vector bundle on X and $\varphi\colon \sigma^\star\mathcal{V} \to \mathcal{V}$ a map of vector bundles. Then one has

$$\sum_i (-1)^i \operatorname{tr}_{\mathbf{C}}\big(\varphi \mid \mathrm{H}^i(X,\mathcal{V})\big) = \sum_{\substack{x\in X\\ f(x)=x}} \frac{\operatorname{tr}_{\mathbf{C}}(\varphi_x \mid \mathcal{V}_x)}{\det_{\mathbf{C}}(1-\mathrm{d}\sigma_x \mid \mathrm{T}_{X,x})} \tag{20}$$

This was conjectured by Shimura and proven in 1964 at a seminar at a conference in Woods Hole [**49**, Introduction]. For a published proof of this formula, and of a much more general theorem in differential geometry, see Atiyah and Bott [**6**, Theorem 2] and [**7**].

A purely algebraic statement and proof can be found in SGA 5 [**27**, Exp. III, 6.12], or in Appendix A of these notes. It applies to a proper smooth scheme X over a field K, and an endomorphism $\sigma\colon X \to X$ with isolated transversal fixed points. Taking $K = \mathbf{F}_q$ and σ the Frobenius endomorhpism one recovers Theorem 3.4 for $Y = \operatorname{Spec} \mathbf{F}_q$ and $X \to Y$ proper smooth. Note that in this case $\mathrm{d}\sigma = 0$, so that the determinants in the denominators in (20) disappear. The more general Theorem 3.4 could easily be deduced from this special case.

The proof we have given, including the Localization Theorem, is due to Fulton [**22**] (except for the addition of Lemma 3.11, which corrects a mistake in the "simple count" after Lemma 3 in *loc. cit.*).

In their book Böckle and Pink [**11**] take an approach following Anderson [**3**]. They reduce to the case where $X(\mathbf{F}_q)$ is empty, so that the right-hand side of the trace formula (14) vanishes, and then use Serre duality and a trick similar to the proof of Proposition 2.16 to deduce that also the left-hand side vanishes. In [**46**] the trace formula is proven for curves using a variation of this trick that avoids the use of Serre duality. Using the Leray spectral sequence, the full Theorem 3.4 could also be deduced from this by factoring the map $X \to Y$ as a composition of relative curves.

Exercises

Exercise 3.1. Let X be a scheme and let Z be a closed subscheme with complement U. Show that the quasi-coherent $\mathcal{O}_X$-modules which are supported on Z form a Serre subcategory of the category of all quasi-coherent $\mathcal{O}_X$-modules. Show that the quotient category is equivalent with the category of quasi-coherent $\mathcal{O}_U$-modules.

EXERCISE 3.2. Under the sheaves-functions dictionary, to which identities between $\mathbf{F}_q$-valued functions on $\mathbf{F}_q$-points do the projection formula (Exercise 2.3) and the proper base change theorem (Theorem 2.35) correspond?

EXERCISE 3.3. This exercise refines Proposition 3.7. Let $\mathcal{C}$ be the category of pairs (M, τ_s) with M a finitely generated $\mathbf{F}_q[x_0, \ldots, x_n]$-module and $\tau_s \colon M \to M$ a graded q-linear map. Let $\mathcal{C}_0$ be the full subcategory consisting of those pairs for which there is a d_0 with $M_d = 0$ for all $d \geq d_0$. Show that $\mathcal{C}_0$ is a Serre subcategory of $\mathcal{C}$ and that the quotient $\mathcal{C}/\mathcal{C}_0$ is equivalent with the category $\mathbf{Coh}_\tau\, \mathbf{P}^n$.

CHAPTER 4

Elementary applications

In this chapter we illustrate the use of the sheaves-functions dictionary by means of a few well-known, classical applications. Several of these can also be shown by elementary combinatorial arguments.

1. Congruences between number of points

PROPOSITION 4.1. *Let $f\colon X \to Y$ be a morphism of proper schemes over $\mathbf{F}_q$. Assume that for every i the induced map*

$$f^\star\colon \mathrm{H}^i(Y, \mathcal{O}_Y) \to \mathrm{H}^i(X, \mathcal{O}_X)$$

is an isomorphism. Then $\#X(\mathbf{F}_q) \equiv \#Y(\mathbf{F}_q) \pmod p$.

PROOF. Denote the structure maps by $g_X\colon X \to \operatorname{Spec}\mathbf{F}_q$ and $g_Y\colon Y \to \operatorname{Spec}\mathbf{F}_q$. We have $f^\star\mathbf{1}_Y = \mathbf{1}_X$, inducing morphisms

$$\mathrm{R}^i g_{Y,\star}\mathbf{1}_Y \to \mathrm{R}^i g_{X,\star}\mathbf{1}_X \tag{21}$$

of crystals on $\operatorname{Spec}\mathbf{F}_q$, which by the hypothesis are isomorphisms. By the trace formula applied to the crystal $\mathbf{1}_X$ and the proper map g_X we have

$$\#X(\mathbf{F}_q) \bmod p = \sum_{i\geq 0}(-1)^i \operatorname{tr}_{\mathrm{R}^i g_{X,\star}\mathbf{1}_X} \star$$

and similarly for $\#Y(\mathbf{F}_q) \bmod p$. By the isomorphisms (21) we find that $\#X(\mathbf{F}_q) \equiv \#Y(\mathbf{F}_q) \pmod p$. □

EXAMPLE 4.2 (Birational maps). Let $f\colon X \to Y$ be a birational map between proper smooth schemes over $\operatorname{Spec}\mathbf{F}_q$. Then by [**12**, Theorem 1] the induced maps

$$f^\star\colon \mathrm{H}^i(Y, \mathcal{O}_Y) \to \mathrm{H}^i(X, \mathcal{O}_X)$$

are isomorphisms, and hence $\#X(\mathbf{F}_q) \equiv \#Y(\mathbf{F}_q) \pmod p$.

If f is the blow-up of a smooth closed subscheme Z, then this is obvious, since the fiber $f^{-1}(z)$ at any point $z \in Z(\mathbf{F}_q)$ will be a projective space of some dimension d and $\#\mathbf{P}^d(\mathbf{F}_q) \equiv 1 \pmod p$. By the *weak factorization theorem* [**1, 52**] every birational map between proper smooth schemes over a field of characteristic zero can be factored as a sequence of blow-ups and inverses of blow-ups along smooth centers. However, it is not known if this is the case in characteristic p.

2. Chevalley-Warning

The standard example of an application of coherent cohomology to the existence of rational points over finite fields is the following.

Proposition 4.3 (Chevalley-Warning). *Let $f_1, \dots, f_r \in \mathbf{F}_q[x_0, \dots, x_n]$ be homogeneous polynomials of degree $d_1, \dots, d_r$ respectively. Let $X \rightarrowtail \mathbf{P}^n_{\mathbf{F}_q}$ be the closed subscheme given by the homogeneous ideal $(f_1, \dots, f_r)$. Assume that $d_1 + \cdots + d_r \leq n$. Then $\#X(\mathbf{F}_q) \equiv 1$ modulo p.*

In particular, $X(\mathbf{F}_q)$ is non-empty.

Proof of Proposition 4.3. We first show that the cohomology of $\mathcal{O}_X$ is as follows

$$\mathrm{H}^i(X, \mathcal{O}_X) = \begin{cases} \mathbf{F}_q & i = 0 \\ 0 & i > 0 \end{cases} \tag{22}$$

For a closed subscheme $i\colon Z \to \mathbf{P}^n$ we write $\mathcal{O}_Z$ for the $\mathcal{O}_{\mathbf{P}^n}$-module $i_\star \mathcal{O}_Z$. Note that $\mathrm{H}^i(Z, \mathcal{O}_Z) = \mathrm{H}^i(\mathbf{P}^n, \mathcal{O}_Z)$. For a non-empty subset $S \subset \{1, \dots, r\}$ we denote by Z_S the closed subscheme $Z(\{f_s \colon s \in S\})$ of $\mathbf{P}^n$, so that that $Z_{\{1,\dots,r\}} = X$. We denote by Z the closed subscheme $Z(f_1 \cdots f_s)$ of X.

We prove (22) by induction on the number of homogeneous polynomials r. For $r = 1$ we have a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-d_1) \xrightarrow{f_1} \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{Z_1} \to 0.$$

Since $1 \leq d_1 \leq n$ we have $\mathrm{H}^i(\mathbf{P}^n, \mathcal{O}(-d_1)) = 0$ for all i, so that the long exact sequence gives the claimed cohomology groups for $\mathcal{O}_{Z_1}$.

For arbitrary r, we use the Koszul resolution

$$0 \to \mathcal{O}_{Z_\emptyset} \to \prod_{\#S=1} \mathcal{O}_{Z_S} \to \cdots \to \prod_{\#S=r-1} \mathcal{O}_{Z_S} \to \mathcal{O}_X \to 0.$$

This is an exact sequence, and the induction hypothesis shows that all except for possibly the last term has trivial higher cohomology, so that this is an acyclic resolution of $\mathcal{O}_X$. The cohomology of $\mathcal{O}_X$ therefore agrees with the cohomology of the complex of global sections, which by the induction hypothesis equals

$$0 \to \mathbf{F}_q \to \prod_{\#S=1} \mathbf{F}_q \to \cdots \to \prod_{\#S=r-1} \mathbf{F}_q.$$

This is exact except for in degree 0, where the cohomology is $\mathbf{F}_q$, as we had to show.

Now consider the number of $\mathbf{F}_q$-points on X. Modulo p this number equals

$$\sum_{x\in X(\mathbf{F}_q)} 1 = \sum_{x\in X(\mathbf{F}_q)} \operatorname{tr}_{\mathbf{1}} x \in \mathbf{F}_q.$$

By the Woods Hole trace formula applied to $X \to \operatorname{Spec} \mathbf{F}_q$ this equals

$$\sum_i (-1)^i \operatorname{tr}_{\mathbf{F}_q}\bigl(\tau \mid \mathrm{H}^i(X, \mathcal{O}_X)\bigr).$$

Since Frobenius acts as the identity on $\mathrm{H}^0(X, \mathcal{O}_X)$ we find that $\#X(\mathbf{F}_q) \equiv 1$, as we had to show. Alternatively, for the last part of the proof one may observe that $i\colon X \to \mathbf{P}^n$ induces an isomorphism on coherent cohomology and apply Proposition 4.1. $\square$

3. Polynomial sums

PROPOSITION 4.4. *Let n be a positive integer. Let $f \in \mathbf{F}_q[x_1, \cdots, x_n]$ be a polynomial of degree less than $(q-1)n$. Then*

$$\sum_{x\in\mathbf{F}_q^n} f(x) = 0.$$

PROOF. Consider on $\mathbf{A}^n$ over $\mathbf{F}_q$ the τ-sheaf $\mathcal{F} = (\mathcal{F}, \tau)$ with $\mathcal{F} = \mathcal{O}_{\mathbf{A}^n}$ and τ the composition

$$\sigma^\star \mathcal{O}_{\mathbf{A}^n} \xrightarrow{\sim} \mathcal{O}_{\mathbf{A}^n} \xrightarrow{f} \mathcal{O}_{\mathbf{A}^n}.$$

We have

$$\sum_{x\in\mathbf{F}_q^n} f(x) = \sum_{x\in\mathbf{A}^n(\mathbf{F}_q)} \operatorname{tr}_{\mathcal{F}} x.$$

(See also example 1.11). Let $\pi\colon \mathbf{A}^n \to \operatorname{Spec} \mathbf{F}_q$ be the structure map. Then by the trace formula we have

$$\sum_{x \in \mathbf{A}^n(\mathbf{F}_q)} \operatorname{tr}_{\mathcal{F}} x = \sum_{i \geq 0} (-1)^i \operatorname{tr}_{\mathrm{R}^i \pi_! \mathcal{F}} \star.$$

To compute the $\mathrm{R}^i\pi_!\mathcal{F}$ we work with the standard embedding $j\colon \mathbf{A}^n \hookrightarrow \mathbf{P}^n$. Let H be the (reduced) hyperplane at infinity.

$$\begin{array}{ccccc} \mathbf{A}^n & \overset{j}{\hookrightarrow} & \mathbf{P}^n & \hookleftarrow & H \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\bar{\pi}} & \swarrow & \\ & & \operatorname{Spec} \mathbf{F}_q & & \end{array}$$

Let m be a positive integer with $(q-1)m \geq \deg f$. Consider on $\mathbf{P}^n$ the τ-sheaf $\widetilde{\mathcal{F}}$ with $\widetilde{\mathcal{F}} = \mathcal{O}_{\mathbf{P}^n}(-mH)$ and τ the composition

$$\sigma^\star \mathcal{O}_{\mathbf{P}^n}(-mH) \xrightarrow{\sim} \mathcal{O}_{\mathbf{P}^n}(-qmH) \xrightarrow{f} \mathcal{O}_{\mathbf{P}^n}(-mH)$$

(see also Example 1.12). Clearly $j^\star\widetilde{\mathcal{F}} \cong \mathcal{F}$. Now assume that the strict inequality $(q-1)m > \deg f$ holds. Then τ vanishes along H so that $\widetilde{\mathcal{F}} \cong j_!\mathcal{F}$ as crystals on $\mathbf{P}^n$. Finally, we have $\mathrm{R}^i\pi_!\mathcal{F} = \mathrm{R}^i\bar{\pi}_\star\widetilde{\mathcal{F}}$. But note that for $0 < m \leq n$ the cohomology of $\mathcal{O}_{\mathbf{P}^n}(-mH)$ vanishes. So we conclude that if $\deg f < (q-1)n$ then $\mathrm{R}^i\pi_!\mathcal{F} = 0$ for all i, and the proposition follows. □

4. Hasse invariant

PROPOSITION 4.5. *Let q be a power of an odd prime p and let $f \in \mathbf{F}_q[x]$ be a polynomial of degree 3. Then the number of $(x, y) \in \mathbf{F}_q^2$ such that $y^2 = f(x)$ is congruent modulo p to the coefficient of x^{q-1} in $-f^{(q-1)/2}$.*

In particular, the coefficient of x^{q-1} lies in the subfield $\mathbf{F}_p$ of $\mathbf{F}_q$.

PROOF. Let $X = \operatorname{Spec} \mathbf{F}_q[x, y]/(y^2 - f(x))$ and let $\pi\colon X \to \mathbf{A}^1$ be the map given by x. The algebra $\pi_\star\mathcal{O}_X$ is free of rank 2 and can be written as

$$\pi_\star\mathcal{O}_X = \mathcal{O}_{\mathbf{A}^1} \cdot 1 \oplus \mathcal{O}_{\mathbf{A}^1} \cdot \epsilon$$

with multiplication given by $\epsilon^2 = f \cdot 1$. It follows that the q-th power map on $\pi_\star\mathcal{O}_X$ maps ϵ to $f^{(q-1)/2}\epsilon$. Let $\mathbf{1}_X$ be the unit crystal on X.

Then the above shows that $\pi_\star \mathbf{1}_X \cong \mathbf{1}_{\mathbf{A}^1} \oplus \mathcal{F}$ where $\mathcal{F} = (\mathcal{O}_{\mathbf{A}^1},\ f^{(q-1)/2})$. We have

$$\#X(\mathbf{F}_q) \bmod p \;=\; \sum_{x\in\mathbf{A}^1(\mathbf{F}_q)} \operatorname{tr}_{\mathbf{1}} x \;+\; \sum_{x\in\mathbf{A}^1(\mathbf{F}_q)} \operatorname{tr}_{\mathcal{F}} x \;=\; \sum_{x\in\mathbf{A}^1(\mathbf{F}_q)} \operatorname{tr}_{\mathcal{F}} x.$$

We will compute the right-hand side using the trace formula. Let $j\colon \mathbf{A}^1 \hookrightarrow \mathbf{P}^1$ be the usual inclusion. Consider the τ-sheaf $\widetilde{\mathcal{F}} = ((\mathcal{O}(-2\infty),\ f^{(q-1)/2})$ on $\mathbf{P}^1$. Since $\deg f^{(q-1)/2} < 2(q-1)$ we have $j_!\mathcal{F} \cong \widetilde{\mathcal{F}}$ in $\mathbf{Crys}\,\mathbf{P}^1$. We have $\mathrm{H}^0(\mathbf{P}^1, \mathcal{O}(-2\infty)) = 0$. A Čech computation shows that

$$\mathrm{H}^1(\mathbf{P}^1, \mathcal{O}(-2\infty)) = \frac{\mathbf{F}_q[x,x^{-1}]}{\mathbf{F}_q[x] \oplus x^{-2}\mathbf{F}_q[x^{-1}]}.$$

It is one-dimensional with basis x^{-1}, and the map induced by τ sends x^{-1} to the coefficient of x^{-1} in $x^{-q}f^{(q-1)/2} \in \mathbf{F}_q[x,x^{-1}]$. Now the trace formula tells us that

$$\#X(\mathbf{F}_q) \bmod p \;=\; \sum_{x\in\mathbf{A}^1(\mathbf{F}_q)} \operatorname{tr}_{\mathcal{F}} = -\operatorname{tr}_{\mathbf{F}_q}\big(\,\tau \mid \mathrm{H}^1(\mathbf{P}^1,\, j_!\mathcal{F})\,\big)\,.$$

By the preceding calculation the trace in the right-hand side equals the coefficient of x^{q-1} in $f^{(q-1)/2}$, which proves the proposition. □

Notes

The theorem of Chevalley-Warning was shown by Chevalley [**13**] (existence of a solution) and Warning [**50**] (congruence on the number of solutions) in 1936. It has since been strengthened and generalized in several ways. Ax [**8**] and Katz [**34**] showed that one can replace the congruence modulo p by a congruence modulo q, and even modulo a certain power of q depending on the d_i. Of course one cannot obtain such congruences from the Woods Hole trace formula. Berthelot, Bloch and Esnault [**10**] have shown that the congruence modulo q can be obtained using Witt vector cohomology.

Deuring [**17**] has shown that an elliptic curve E given by a Weierstrass equation $y^2 = f(x)$ over a field of odd characteristic p is supersingular if and only if the coefficient of x^{p-1} in $f^{(p-1)/2}$ vanishes. A computation similar to the one in the proof of Proposition 4.5 shows that this is equivalent with the vanishing of the action of τ on $\mathrm{H}^1(E, \mathcal{O}_E)$.

Exercises

EXERCISE 4.1. Assume p is odd. Express the number of points modulo p on a hyperelliptic curve of the form $y^2 = f(x)$ over $\mathbf{F}_q$ in terms of the coefficients of $f^{(q-1)/2}$.

EXERCISE 4.2. Let $f \in \mathbf{F}_q[x_0, \dots, x_n]$ be nonzero and homogenous of degree $n + 1$. Let $X = Z(f)$. Let $a \in \mathbf{F}_q$ be the coefficient of $(x_0 \cdots x_n)^{q-1}$ in f^{q-1}. Show that $a \in \mathbf{F}_p$ and that $\#X(\mathbf{F}_q) \equiv 1 + (-1)^n a$ modulo p.

CHAPTER 5

Crystals with coefficients

1. τ-sheaves and crystals with coefficients

In this chapter C will always denote the spectrum of a commutative $\mathbf{F}_q$-algebra A (on which we will impose various conditions). The C stands for *coefficients*. We will be considering objects on which the algebra A acts linearly.

Let X be a scheme over $\mathbf{F}_q$. A *τ-sheaf on X with coefficients in* A is a pair $\mathcal{F} = (\mathcal{F}, \tau)$ consisting of a quasi-coherent $\mathcal{O}_{C\times X}$-module $\mathcal{F}$ and an $\mathcal{O}_{C\times X}$-linear map

$$\tau\colon (\mathrm{id}_C \times \sigma_X)^\star \mathcal{F} \to \mathcal{F}.$$

Morphisms are defined in the obvious way. We denote the category of such objects by $\mathbf{QCoh}_\tau(X, A)$. By adjunction, specifying τ is equivalent to either giving an $\mathcal{O}_{C\times X}$-linear map

$$\tau_a\colon \mathcal{F} \to (\mathrm{id}_C \times \sigma_X)_\star \mathcal{F}$$

or an additive map

$$\tau_s\colon \mathcal{F} \to \mathcal{F}$$

satisfying $\tau_s((a\otimes r)s) = (a\otimes r^q)\tau_s(s)$ for all $a \in A$ and all local sections r and s of $\mathcal{O}_X$ and $\mathcal{F}$ respectively.

Now assume that $C\times X$ is noetherian. This is the case, for example, if X is noetherian and A of finite type over $\mathbf{F}_q$. We say that a τ-sheaf on X with coefficients in A is *coherent* if the underlying $\mathcal{O}_{C\times X}$-module is coherent. The category of such objects is denoted $\mathbf{Coh}_\tau(X, A)$.

An object $\mathcal{F} = (\mathcal{F}, \tau)$ in $\mathbf{Coh}_\tau(X, A)$ is said to be nilpotent if

$$\tau \circ \cdots \circ (\mathrm{id} \times \sigma^{n-1})^\star \tau\colon (\mathrm{id} \times \sigma^n)^\star \mathcal{F} \to \mathcal{F}$$

is the zero map for some $n > 0$, or equivalently, if $\tau_s^n = 0$ for some $n > 0$.

PROPOSITION 5.1. *Assume that $C \times X$ is noetherian. Let*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence in $\mathbf{Coh}_\tau(X, A)$. Then $\mathcal{F}_2$ is nilpotent if and only if both $\mathcal{F}_1$ and $\mathcal{F}_3$ are nilpotent.

PROOF. The proof is identical to that of Proposition 1.17. □

In other words, the full subcategory of nilpotent objects of $\mathbf{Coh}_\tau(X, A)$ is a thick subcategory. We define the category of A-crystals on X as the quotient category of $\mathbf{Coh}_\tau(X, A)$ by the thick subcategory of nilpotent objects. It is denoted $\mathbf{Crys}(X, A)$.

The categories $\mathbf{QCoh}_\tau(X, A)$, $\mathbf{Coh}_\tau(X, A)$ and $\mathbf{Crys}(X, A)$ are A-linear categories. This means that the Hom-sets in these categories are naturally A-modules, and that composition is bilinear.

If $A = \mathbf{F}_q$ then these categories coincide with the categories $\mathbf{QCoh}_\tau X$, $\mathbf{Coh}_\tau X$ and $\mathbf{Crys}\, X$ of Chapter 1.

2. Crystals with finite coefficients

We now assume that X is noetherian and that A is a *finite* $\mathbf{F}_q$-algebra (that is, of finite cardinality). We will show that the category $\mathbf{Crys}(X, A)$ of A-crystals can be described as a category of 'A-modules' in the category $\mathbf{Crys}\, X$. This will allow us to inherit constructions and results obtained for $A = \mathbf{F}_q$ to the more general setting of finite coefficients A.

An *A-module* in $\mathbf{Coh}_\tau X$ is a pair $(\mathcal{F}, \alpha)$ consisting of a coherent τ-sheaf $\mathcal{F}$ and an $\mathbf{F}_q$-algebra homomorphism $\alpha\colon A \to \operatorname{End} \mathcal{F}$. A morphism of A-modules in $\mathbf{Coh}_\tau X$ is a morphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$ of coherent τ-sheaves such that $\varphi \circ \alpha(a) = \alpha'(a) \circ \varphi$ for all $a \in A$. We denote the category of A-modules in $\mathbf{Coh}_\tau X$ by $(\mathbf{Coh}_\tau X)_A$. We similarly define the category $(\mathbf{Crys}\, X)_A$ of A-modules in $\mathbf{Crys}\, X$. The functor

$$\mathbf{Coh}_\tau(X, A) \to (\mathbf{Coh}_\tau X)_A,\ \mathcal{F} \mapsto \mathrm{pr}_{X,*} \mathcal{F}$$

is an equivalence of categories. It is exact, and maps nilpotent τ-sheaves to nilpotent τ-sheaves, so it induces a functor

$$F\colon \mathbf{Crys}(X, A) \to (\mathbf{Crys}\, X)_A. \tag{23}$$

We will show that also this induced functor is an equivalence.

PROPOSITION 5.2. *Let X be a noetherian scheme over $\mathbf{F}_q$ and let A be a finite $\mathbf{F}_q$-algebra. Then the functor F is an equivalence of categories.*

PROOF. We first show that F is essentially surjective. Consider an object $(\mathcal{F},\alpha)$ of $(\mathbf{Crys}\,X)_A$. By the equivalence $\mathbf{Coh}_\tau(X,A) = (\mathbf{Coh}_\tau\,X)_A$, it suffices to construct a coherent τ-sheaf $\widetilde{\mathcal{F}}$ and an action $\widetilde{\alpha}\colon A \to \operatorname{End}\widetilde{\mathcal{F}}$ in the category $\mathbf{Coh}_\tau\,X$, so that $(\widetilde{\mathcal{F}},\widetilde{\alpha})$ in $\mathbf{Coh}_\tau(X,A)$ maps to $(\mathcal{F},\alpha)$ in $(\mathbf{Crys}\,X)_A$.

For every $a \in A$ we are given a morphism of crystals $\alpha_a\colon \mathcal{F} \to \mathcal{F}$. By Proposition 1.23 we can represent every α_a by a diagram

$$\mathcal{F} \hookleftarrow \mathcal{H}_a \xrightarrow{\widetilde{\alpha}_a} \mathcal{F}/\mathcal{N}_a \twoheadleftarrow \mathcal{F}$$

with $\mathcal{F}/\mathcal{H}_a$ and $\mathcal{N}_a$ nilpotent. Since A is finite, we may replacing the $\mathcal{H}_a$ by their intersection $\mathcal{H}$ and the $\mathcal{N}_a$ by their sum $\mathcal{N}$, so that every α_a is represented by a diagram

$$\mathcal{F} \hookleftarrow \mathcal{H} \xrightarrow{\widetilde{\alpha}_a} \mathcal{F}/\mathcal{N} \twoheadleftarrow \mathcal{F}$$

with $\mathcal{F}/\mathcal{H}$ and $\mathcal{N}$ nilpotent. Replacing $\mathcal{F}$ by the nil-isomorphic $\mathcal{F}' = \mathcal{H}$ and $\mathcal{N}$ by $\mathcal{N}' = \mathcal{N} \cap \mathcal{H}$ we may represent the collection $(\alpha_a)_a$ by diagrams

$$\mathcal{F}' \xrightarrow{\widetilde{\alpha}_a} \mathcal{F}'/\mathcal{N}' \twoheadleftarrow \mathcal{F}'.$$

Now let $\mathcal{N}'' \subset \mathcal{F}'$ be the τ-sheaf generated by $\mathcal{N}'$ and the inverse images of the $\alpha_a^n(\mathcal{N})$ for all $n \geq 0$ and $a \in A$. Then $\mathcal{N}''$ is nilpotent and replacing $\mathcal{F}'$ by the nil-isomorphic $\mathcal{F}'' = \mathcal{F}'/\mathcal{N}''$ we find a collection of maps

$$\widetilde{\alpha}_a\colon \mathcal{F}'' \to \mathcal{F}''$$

in $\mathbf{Coh}_\tau\,X$, representing the α_a. Finally, let $\mathcal{N}'''$ be the sub-τ-sheaf of $\mathcal{F}''$ generated by the kernels of $\widetilde{\alpha}_{ab} - \widetilde{\alpha}_a\widetilde{\alpha}_b$ for all a, b in A. Since $\alpha_{ab} = \alpha_a\alpha_b$ in $\mathbf{Crys}\,X$, the τ-sheaf $\mathcal{N}'''$ is nilpotent. Let $\mathcal{N}''''$ be generated by the $\widetilde{\alpha}_a^n(\mathcal{N})$ for all $n \geq 0$ and $a \in A$. Replacing $\mathcal{F}''$ by the nil-isomorphic $\widetilde{\mathcal{F}} := \mathcal{F}''/\mathcal{N}''''$ we obtain a genuine action

$$\widetilde{\alpha}\colon A \to \operatorname{End}\widetilde{\mathcal{F}}$$

in $\mathbf{Coh}_\tau\,X$, lifting the given action α in $\mathbf{Crys}\,X$. This shows that the functor F of (23) is essentially surjective.

A similar argument shows that every morphism in $(\mathbf{Crys}\,X)_A$ is represented by a morphism in $\mathbf{Coh}_\tau(X,A)$. In particular, F is full. To see that it is faithful, let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathbf{Crys}(X, A)$ which becomes zero in $(\mathbf{Crys}\, X)_A$. By what we have just seen, we may assume φ to be a morphism $\mathcal{F} \to \mathcal{G}$ in $\mathbf{Coh}_\tau(X, A) = (\mathbf{Coh}_\tau\, X)_A$. Since φ becomes zero in $(\mathbf{Crys}\, X)_A$, the coherent τ-sheaves $\mathcal{F}/\ker\varphi$ and $\operatorname{im}\varphi$ are nilpotent, and hence φ already becomes zero in $\mathbf{Crys}(X, A)$. □

If $f\colon X \to Y$ be a morphism of noetherian schemes over $\mathbf{F}_q$ then the functor $f^\star\colon\ \mathbf{Crys}\, Y \to \mathbf{Crys}\, X$ induces an exact functor $(\mathbf{Crys}\, Y)_A \to (\mathbf{Crys}\, X)_A$, and hence by Proposition 5.2 an exact functor $f^\star\colon\ \mathbf{Crys}(Y, A) \to \mathbf{Crys}(X, A)$. If moreover f is compactifiable then we also have induced functors $\mathrm{R}^i f_!\colon\ \mathbf{Crys}(X, A) \to \mathbf{Crys}(Y, A)$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are crystals in $\mathbf{Crys}\, X$ equipped with an A-action, then $\mathcal{F}_1 \otimes \mathcal{F}_2$ is naturally equipped with an $A \otimes_{\mathbf{F}_q} A$-action. Let I be the kernel of the map

$$A \otimes_{\mathbf{F}_q} A \to A,\ a \otimes b \mapsto ab.$$

We denote by $\mathcal{F}_1 \otimes_A \mathcal{F}_2$ the quotient of $\mathcal{F}_1 \otimes \mathcal{F}_2$ by $I \cdot (\mathcal{F}_1 \otimes \mathcal{F}_2)$. In other words, $\mathcal{F}_1 \otimes_A \mathcal{F}_2$ is the largest quotient of $\mathcal{F}_1 \otimes \mathcal{F}_2$ on which the actions of $a \otimes 1$ and $1 \otimes a$ agree for all $a \in A$. This construction defines a functor

$$- \otimes_A -\colon\ \mathbf{Crys}(X, A) \times \mathbf{Crys}(X, A) \to \mathbf{Crys}(X, A).$$

This functor will not be exact, unless A is reduced or X is empty.

Proposition 5.3. *Let X be a noetherian scheme over $\mathbf{F}_q$ and let A be a finite and reduced $\mathbf{F}_q$-algebra. Then the functor $- \otimes_A -$ on $\mathbf{Crys}(X, A)$ is exact in both arguments.*

Proof. Under the identification $\mathbf{Crys}(X, A) = (\mathbf{Crys}\, X)_A$, the functor is the composite of

$$(\mathbf{Crys}\, X)_A \times (\mathbf{Crys}\, X)_A \to (\mathbf{Crys}\, X)_{A\otimes A},\ (\mathcal{F}_1, \mathcal{F}_2) \mapsto \mathcal{F}_1 \otimes \mathcal{F}_2$$

and

$$(\mathbf{Crys}\, X)_{A\otimes A} \to (\mathbf{Crys}\, X)_A,\ \mathcal{G} \mapsto \mathcal{G}/I\mathcal{G}.$$

The former is exact in both arguments by Corollary 2.8. Since $A \otimes_{\mathbf{F}_q} A$ is a product of finite fields, the ideal I is generated by an idempotent e. Every $\mathcal{G} \in (\mathbf{Crys}\, X)_{A\otimes A}$ decomposes canonically as

$$\mathcal{G} = e\mathcal{G} \oplus (1 - e)\mathcal{G}$$

with $(1 - e)\mathcal{G} = \mathcal{G}/I\mathcal{G}$, so that also the functor $\mathcal{G} \mapsto \mathcal{G}/I\mathcal{G}$ is exact. □

3. Traces of crystals with finite field coefficients

Assume that A is a finite field extension of $\mathbf{F}_q$. Let X be a scheme of finite type over $\mathbf{F}_q$. Let $\mathcal{F}$ be an A-crystal on X. For every $x \in X(\mathbf{F}_q)$ the pull-back $x^\star\mathcal{F}$ is a finite-dimensional A-vector space equipped with an A-linear endomorphism τ. We define

$$\operatorname{tr}_{A,\mathcal{F}} x := \operatorname{tr}_A (\tau \mid x^\star\mathcal{F}) \in A.$$

This is independent of the choice of coherent τ-sheaf $(\mathcal{F}, \tau)$ representing the crystal $\mathcal{F}$. We obtain an additive map

$$\mathrm{K}_0(X, A) \to \operatorname{Map}(X(\mathbf{F}_q), A),\ \mathcal{F} \mapsto \operatorname{tr}_{A,\mathcal{F}}.$$

The following is the main result of this section, it generalizes the trace formula from the special case $A = \mathbf{F}_q$ of Chapter 3.

Theorem 5.4. *Let A be a finite field extension of $\mathbf{F}_q$. Let $f\colon X \to Y$ be a separated morphism of schemes of finite type over $\mathbf{F}_q$. Let $\mathcal{F}$ be a crystal on X with coefficients in A. Then for every $y \in Y(\mathbf{F}_q)$ we have*

$$\sum_{n\geq 0} (-1)^n \operatorname{tr}_{A,\mathrm{R}^n f_! \mathcal{F}} y = \sum_{\substack{x \in X(\mathbf{F}_q)\\ f(x)=y}} \operatorname{tr}_{A,\mathcal{F}} x$$

in A.

We will show this by reducing it to the special case $A = \mathbf{F}_q$. The following lemma is crucial to the reduction.

Lemma 5.5. *Let A be a finite field extension of $\mathbf{F}_q$. Then the $\mathbf{F}_q$-linear map*

$$A \to \operatorname{Hom}_{\mathbf{F}_q}(A, \mathbf{F}_q)$$

given by

$$a \mapsto \big(\lambda \mapsto \operatorname{tr}_{A/\mathbf{F}_q} \lambda a\big)$$

is injective.

Proof. This is a restatement of the separability of $A/\mathbf{F}_q$. □

Proof of Theorem 5.4. Given $\lambda \in A$, a scheme S of finite type over $\mathbf{F}_q$ and an A-crystal $\mathcal{G}$ on S we denote by $\mathcal{G}(\lambda)$ the A-crystal

$$\mathcal{G}(\lambda) = \big(\mathcal{G},\ (\lambda \otimes 1) \cdot \tau_{\mathcal{G}}\big).$$

We have $\operatorname{tr}_{A,\mathcal{G}(\lambda)} = \lambda \cdot \operatorname{tr}_{A,\mathcal{G}}$ as A-valued functions on $S(\mathbf{F}_q)$.

We now apply this to our $\mathcal{F}$ on X. Note that

$$\mathrm{R}^n f_!(\mathcal{F}(\lambda)) = (\mathrm{R}^n f_! \mathcal{F})(\lambda).$$

Let $y \in Y(\mathbf{F}_q)$. Forgetting the A-action we can interpret $\mathcal{F}(\lambda)$ as a crystal with $\mathbf{F}_q$-coefficients, and the trace formula with $\mathbf{F}_q$-coefficients (Theorem 3.4) guarantees

$$\sum_{n\geq 0}(-1)^n \operatorname{tr}_{\mathbf{F}_q, \mathrm{R}^n f_! \mathcal{F}(\lambda)} y = \sum_{\substack{x \in X(\mathbf{F}_q)\\ f(x)=y}} \operatorname{tr}_{\mathbf{F}_q, \mathcal{F}(\lambda)} x$$

in $\mathbf{F}_q$, for every $\lambda \in A$. By the transitivity of the trace, this can be rewritten as

$$\operatorname{tr}_{A/\mathbf{F}_q}\Big(\sum_{n\geq 0}(-1)^n \operatorname{tr}_{A, \mathrm{R}^n f_! \mathcal{F}(\lambda)} y\Big) = \operatorname{tr}_{A/\mathbf{F}_q}\Big(\sum_{\substack{x \in X(\mathbf{F}_q)\\ f(x)=y}} \operatorname{tr}_{A, \mathcal{F}(\lambda)} x\Big)$$

or equivalently, as

$$\operatorname{tr}_{A/\mathbf{F}_q}\Big(\lambda \sum_{n\geq 0}(-1)^n \operatorname{tr}_{A, \mathrm{R}^n f_! \mathcal{F}} y\Big) = \operatorname{tr}_{A/\mathbf{F}_q}\Big(\lambda \sum_{\substack{x \in X(\mathbf{F}_q)\\ f(x)=y}} \operatorname{tr}_{A, \mathcal{F}} x\Big)$$

for all $\lambda \in A$. By Lemma 5.5 we conclude that

$$\sum_{n\geq 0}(-1)^n \operatorname{tr}_{A, \mathrm{R}^n f_! \mathcal{F}} y = \sum_{\substack{x \in X(\mathbf{F}_q)\\ f(x)=y}} \operatorname{tr}_{A, \mathcal{F}} x$$

holds in A, as we had to prove. □

Notes

In their monograph Böckle and Pink [**11**] define functors $f^\star$, $f_!$ and $\otimes_A$ between categories $\mathbf{Crys}(X, A)$ in much larger generality: assuming only that X is noetherian and that $C = \operatorname{Spec} A$ is the localization of a scheme of finite type over $\mathbf{F}_q$. They also show that the pointwise criteria of section 5 extend to this more general setting.

There are two obstacles against extending the trace formula to this more general setting. First, the trace of an endomorphism of a finitely generated A-module need not be well-defined. Second, if A is not reduced then the trace of a nilpotent endomorphism need not vanish, so that one cannot readily pass to crystals. The first obstacle can be overcome by working with perfect complexes. The second is more serious.

In fact, an example due to Deligne [**16**, p. 127] shows that at least in the context of étale cohomology the trace formula with non-reduced coefficients A fails to hold.

For *finite* A one can use Proposition 5.2 to deduce from the equivalence between $\mathbf{Crys}\, X$ and the category of constructible $\mathbf{F}_q$-modules on X_{et} (see the notes to Chapter 1) an equivalence between $\mathbf{Crys}(X, A)$ and the category of constructible A-modules on X_{et}. Crystals with *infinite* coefficient rings A (such as $A = \mathbf{F}_q[t]$) play an important role in the theory of Drinfeld modules, see for example [**2**] or [**11**, 3.5]. See also Chapter 8, where they will be used to study special values of L-functions.

CHAPTER 6

Cohomology of symmetric powers of curves

In the first three sections of this chapter K denotes an arbitrary field, possibly of characteristic zero. These sections do not depend on the previous chapters. In the fourth and last section, we apply the results to crystals.

In this chapter, all unspecified tensor products and products are over K and $\operatorname{Spec} K$ respectively.

1. Symmetric tensors and exterior tensor powers

This section contains a number of results from multi-linear algebra.

Let K be a field and let V be a K-vector space. Let n be a non-negative integer. The *space of symmetric tensors of degree* n of V is the vector space

$$\Gamma^n V := (V \otimes \cdots \otimes V)^{\mathfrak{S}_n}$$

of elements of $V^{\otimes n}$ that are invariant under the action of the symmetric group $\mathfrak{S}_n$. This is also sometimes denoted $\operatorname{Sym}_n V$. It should not be confused with $\operatorname{Sym}^n V$, which is a quotient of $V^{\otimes n}$, in stead of a subspace (see also exercise 6.1).

LEMMA 6.1. *Let $(v_i)_{i \in I}$ be a basis of V. Let B be the set of $\mathfrak{S}_n$-orbits in I^n. Then the vectors*

$$\sum_{(i_1,\ldots,i_n) \in b} v_{i_1} \otimes \cdots \otimes v_{i_n}$$

with $b \in B$ form a basis of $\Gamma^n V$ indexed by B. □

The *n-th exterior tensor power* of V is the quotient space

$$\bigwedge^n V = \frac{V \otimes \cdots \otimes V}{\langle v_1 \otimes \cdots \otimes v_n \mid v_i = v_j \text{ for some } i \neq j \rangle}$$

The image of $v_1 \otimes \cdots \otimes v_n$ in $\bigwedge^n V$ is denoted $v_1 \wedge \cdots \wedge v_n$.

LEMMA 6.2. *Let $(v_i)_{i\in I}$ be a basis of V and $<$ a total ordering on I. Then the vectors $v_{i_1} \wedge \cdots \wedge v_{i_n}$ with $i_1 < \cdots < i_n$ form a basis for $\bigwedge^n V$.* □

Note that $\bigwedge^0 V$ and $\Gamma^0 V$ are canonically isomorphic with K.

Let V_0 be a subspace of V. For $0 \leq i \leq n$ we define $F^i\Gamma^n V \subset \Gamma^n V$ as

$$F^i\Gamma^n V = \Gamma^n V \bigcap \sum_{\sigma\in\mathfrak{S}_n} \sigma(V_0^{\otimes i} \otimes V^{\otimes n-i}),$$

where the intersection is taken in $V^{\otimes n}$. We put $F^i\Gamma^n V = 0$ for $i > n$. These subspaces define a descending filtration

$$\Gamma^n V = F^0\Gamma^n V \supset F^1\Gamma^n V \supset \cdots \supset F^{n+1}\Gamma^n V = 0$$

which we will use extensively. We first describe its intermediate quotients.

LEMMA 6.3. *If $0 \to V_0 \to V \to V_1 \to 0$ is a short exact sequence of K-vector spaces then we have isomorphisms*

$$\operatorname{gr}^i \Gamma^n V := F^i\Gamma^n V/F^{i+1}\Gamma^n V = \Gamma^i V_0 \otimes \Gamma^{n-i} V_1,$$

functorial in the short exact sequence. If $V = V_0 \oplus V_1$ then

$$\Gamma^n V = \bigoplus_{i+j=n} \Gamma^i V_0 \otimes \Gamma^j V_1,$$

functorially in V_0 and V_1.

PROOF. Consider the map

$$\Gamma^i V_0 \otimes \Gamma^{n-i} V \to F^i\Gamma^n V,\, x \mapsto \sum_\sigma \sigma(x),$$

where x is seen as an element of $V^{\otimes n}$ and σ runs over any set of representatives for $\mathfrak{S}_n/(\mathfrak{S}_i \times \mathfrak{S}_{n-i})$. The composition with the quotient map $F^i\Gamma^n V \to \operatorname{gr}^i \Gamma^n V$ factors over a map

$$\Gamma^i V_0 \otimes \Gamma^{n-i} V_1 \to \operatorname{gr}^i \Gamma^n V.$$

Choosing a basis for V_0 and extending it to a basis for V one verifies using Lemma 6.1 that this map is an isomorphism. The second claim can be shown along the same lines. □

Similarly, if V_0 is a subspace of V we define

$$F^i{\textstyle\bigwedge}^n V := \operatorname{im}\left(V_0^{\otimes i} \otimes V^{\otimes n-i} \to {\textstyle\bigwedge}^n V\right)$$

fir $0 \le i \le n$ and $F^i {\textstyle\bigwedge}^n V = 0$ for $i > n$. Again, this defines a descending filtration

$$\textstyle\bigwedge^n V = F^0 \bigwedge^n V \supset F^1 \bigwedge^n V \supset \cdots \supset F^{n+1} \bigwedge^n V = 0$$

whose intermediate quotients are described in the following lemma.

LEMMA 6.4. *If $0 \to V_0 \to V \to V_1 \to 0$ is a short exact sequence of K-vector spaces then we have isomorphisms*

$$\textstyle\operatorname{gr}^i \bigwedge^n V := F^i \bigwedge^n V / F^{i+1} \bigwedge^n V = \bigwedge^i V_0 \otimes \bigwedge^{n-i} V_1,$$

functorial in the short exact sequence. If $V = V_0 \oplus V_1$ then

$$\textstyle\bigwedge^n V = \bigoplus_{i+j=n} \bigwedge^i V_0 \otimes \bigwedge^j V_1,$$

functorially in V_0 and V_1.

PROOF. We omit the proof, which is completely analogous to the proof of Lemma 6.3. □

Let $\delta \colon V \to W$ be a map of K-vector spaces. Then for all i and j we have an induced map

$$\textstyle\Gamma^{i+1} V \otimes \bigwedge^j W \to \Gamma^i V \otimes \bigwedge^{j+1} W$$

defined as the composition

$$\textstyle\Gamma^{i+1} V \otimes \bigwedge^j W \hookrightarrow (\Gamma^i V \otimes V) \otimes \bigwedge^j W \to \Gamma^i V \otimes (W \otimes \bigwedge^j W) \twoheadrightarrow \Gamma^i V \otimes \bigwedge^{j+1} W,$$

where the middle map is $\mathrm{id} \otimes \delta \otimes \mathrm{id}$. For each n we obtain a sequence

$$\textstyle 0 \to \Gamma^n V \to \cdots \to \Gamma^{n-i} V \otimes \bigwedge^i W \to \cdots \to \bigwedge^n W \to 0 \tag{24}$$

of linear maps which is a complex, with $\Gamma^{n-i} V \otimes \bigwedge^i W$ in degree i. We call it the *degree n Koszul complex* of $\delta \colon V \to W$ and denote it by $\operatorname{Kosz}_n \delta$. Note that $\operatorname{Kosz}_0 \delta$ is the complex consisting of the vector space K placed in degree 0.

The cohomology of this complex is given by the following proposition. The remainder of this section will be devoted to its proof.

PROPOSITION 6.5. *The cohomology of $\operatorname{Kosz}_n \delta$ is given by isomorphisms*

$$\textstyle\mathrm{H}^i(\operatorname{Kosz}_n \delta) = \Gamma^{n-i}(\ker \delta) \otimes \bigwedge^i(\operatorname{coker} \delta), \tag{25}$$

functorial in $\delta \colon V \to W$.

First of all, observe that $\operatorname{Kosz}_n \delta$ is functorial in δ, that is, a commutative square

$$\begin{array}{ccc} V & \xrightarrow{\delta} & W \\ \downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle f_W} \\ V' & \xrightarrow{\delta'} & W' \end{array}$$

(which we shall denote by $f\colon \delta \to \delta'$), induces a morphism

$$\operatorname{Kosz}_n f\colon \operatorname{Kosz}_n \delta \to \operatorname{Kosz}_n \delta'$$

in the obvious way. In particular, $f\colon \delta \to \delta'$ induces maps $\mathrm{H}^\bullet(\operatorname{Kosz}_n \delta) \to \mathrm{H}^\bullet(\operatorname{Kosz}_n \delta')$.

Proof of Proposition 6.5. Observe that $V \to W$ is the direct limit of all the sub-objects $V' \to W'$ with V' and W' finite-dimensional. Since direct limits are exact, and the formation of $\wedge^n$ and Γ^n commutes with direct limits, it suffices to prove the proposition for V and W of finite dimension.

We proceed by induction on the rank of δ. If δ has rank 0, then $\ker \delta = V$, $\operatorname{coker} \delta = W$, and all the maps in $\operatorname{Kosz}_n \delta$ vanish, so that the proposition is immidiate.

Assume we have obtained functorial isomorphisms (25) for all δ of rank $\leq N$. Let $\delta\colon V \to W$ be a morphism of rank $\leq N+1$. Choose decompositions

$$V = L \oplus V', \quad W = L' \oplus W'$$

with L and L' one-dimensional, and such that δ restricts to an isomorphism $L \to L'$, and to a map $\delta'\colon V' \to W'$. Note that δ' has rank $\leq N$. One verifies that $\operatorname{Kosz}_n \delta$ has a decomposition

$$\operatorname{Kosz}_n \delta = \operatorname{Kosz}_n \delta' \oplus C^\bullet$$

for an *exact* complex $C^\bullet$. In particular, using the induction hypothesis we find isomorphisms

$$\mathrm{H}^i(\operatorname{Kosz}_n \delta) = \mathrm{H}^i(\operatorname{Kosz}_n \delta') = \Gamma^{n-i}(\ker \delta') \otimes \bigwedge^i(\operatorname{coker} \delta').$$

Moreover, since δ' has the same kernel and cokernel as δ, we have an isomorphism

$$\Gamma^{n-i}(\ker \delta') \otimes \bigwedge^i(\operatorname{coker} \delta') \xrightarrow{\sim} \Gamma^{n-i}(\ker \delta) \otimes \bigwedge^i(\operatorname{coker} \delta).$$

To see that the resulting isomorphism $\mathrm{H}^i(\operatorname{Kosz}_n \delta) \xrightarrow{\sim} \Gamma^{n-i}(\ker \delta) \otimes \bigwedge^i(\operatorname{coker} \delta)$ is independent of the choice of decomposition observe that any two decompositions are related by an automorphism s of $\delta\colon V \to W$. A similar argument, using that the automorphism s can be taken to induce the identity on $\ker\delta$ and $\operatorname{coker}\delta$, establishes the functoriality in δ. $\square$

2. Some sheaves on the symmetric powers of a scheme

Let K be a field. If R is a commutative K-algebra, then so is $\Gamma^n R$. If M is an R-module, then $\Gamma^n M$ and $\bigwedge^n M$ are naturally $\Gamma^n R$-modules. Note that the tensor products in the definition of $\Gamma^n M$ and $\bigwedge^n M$ are over K, not R.

If M_0 is a submodule of M then the resulting filtrations on $\Gamma^n M$ and $\bigwedge^n M$ become filtrations by $\Gamma^n R$-modules. The isomorphisms of Lemmas 6.3 and 6.4 are then isomorphisms of $\Gamma^n R$-modules. Similarly, if $\delta\colon M \to N$ is a map of R-modules, then $\operatorname{Kosz}_n \delta$ is a complex of $\Gamma^n R$-modules, and Proposition 6.5 describes the cohomology groups of this complex as $\Gamma^n R$-modules.

The construction of $\Gamma^n M$ and $\bigwedge^n M$ commutes with localization on R, in the following sense.

LEMMA 6.6. *Let R be a commutative K-algebra. Let S be a multiplicative subset of R. Denote by $\tilde{S}$ the multiplicative subset $\{s \otimes s \otimes \cdots \otimes s \mid s \in S\}$ of $\Gamma^n R$. Then the localized rings $\tilde{S}^{-1}\Gamma^n_K R$ and $\Gamma^n(S^{-1}R)$ are naturally isomorphic. Moreover, if M is an R-module then we have natural isomorphisms*

$$\tilde{S}^{-1}(\Gamma^n M) = \Gamma^n S^{-1} M$$

and

$$\tilde{S}^{-1}\textstyle\bigwedge^n M = \bigwedge^n S^{-1} M.$$

of $\Gamma^n(S^{-1}R)$-modules.

PROOF. If M is an R-module then clearly

$$\tilde{S}^{-1}\left(M^{\otimes n}\right) = \left(S^{-1}M\right)^{\otimes n}.$$

Also, if B is a K-algebra, $\tilde{S} \subset B$ is a multiplicative subset and N an B-module equipped with an action of a group G then

$$\tilde{S}^{-1}(N^G) = (\tilde{S}^{-1}N)^G.$$

In particular, taking $B = \Gamma^n_K R$, $N = M^{\otimes n}$ and $\tilde{S}$ as in the proposition yields

$$\tilde{S}^{-1}\Gamma^n M = \Gamma^n(S^{-1}M)$$

with $\tilde{S}^{-1}\Gamma^n R = \Gamma^n(S^{-1}R)$ as the special case $M = R$.

For the last statement it suffices to observe that the submodules

$$\tilde{S}^{-1}\langle m_1 \otimes \cdots \otimes m_n \in M^{\otimes n} \mid m_i = m_j \text{ for some } i \neq j\rangle$$

and

$$\langle m_1 \otimes \cdots \otimes m_n \in (S^{-1}M)^{\otimes n} \mid m_i = m_j \text{ for some } i \neq j\rangle$$

of $(S^{-1}M)^{\otimes n}$ coincide. □

Now let X be a quasi-projective scheme over K and let n be a positive integer.

Lemma 6.7. *For all $x \in X^n$ there is an affine open $U \subset X$ so that $x \in U^n$.*

Proof. Embed X in some projective scheme $\overline{X}$, and embed $\overline{X}$ into some projective space $\mathbf{P}^d_K$. Let L be the residue field of x and $(x_1, \ldots, x_n) \in X^n(L)$ be the corresponding L-point corresponding to x. In order to prove the lemma, it suffices to find an affine open $U \subset X$ so that $x_1, \ldots, x_n \in U$. Let f be any homogenous polynomial in $K[X_0, \ldots, X_d]$ so that $f(x_i) \neq 0$ for all i and $f(x) = 0$ for all x in the closed subset $\overline{X} \setminus X$ of $\mathbf{P}^d$. Then $D_+(f) \cap \overline{X}$ is an affine open contained in X and containing all the x_i. □

This lemma guarantees that the n-th symmetric power of X over K, defined as

$$\mathrm{Sym}^n X := (X \times \cdots \times X)\,/\mathfrak{S}_n$$

exists as a scheme. It has a cover by affine open subsets of the form $\mathrm{Sym}^n U$. In the affine case, we have,

$$\mathrm{Sym}^n \operatorname{Spec} R = \operatorname{Spec} \Gamma^n R.$$

If $n = 0$ then $\mathrm{Sym}^n X = \operatorname{Spec} K$.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Because of Lemmas 6.6 and 6.7 there are uniquely defined quasi-coherent $\mathcal{O}_{\mathrm{Sym}^n X}$-modules $\Gamma^n \mathcal{F}$ and $\bigwedge^n \mathcal{F}$ so that for all affine open $U \subset X$ we have

$$(\Gamma^n \mathcal{F})(\mathrm{Sym}^n U) = \Gamma^n(\mathcal{F}(U))$$

and

$$(\textstyle\bigwedge^n \mathcal{F})(\mathrm{Sym}^n U) = \textstyle\bigwedge^n(\mathcal{F}(U))$$

as $\Gamma^n(\mathcal{O}_X(U))$-modules. Again, note that the tensor products are over K. In particular, $\bigwedge^n \mathcal{F}$ does *not* denote the exterior power of $\mathcal{F}$ over $\mathcal{O}_X$. If $\mathcal{F}$ is coherent then so are $\bigwedge^n \mathcal{F}$ and $\Gamma^n \mathcal{F}$.

3. Cohomology of exterior symmetric tensor powers

Let K be a field. The main result of this chapter is the following theorem, which is a special case of a more general result of Deligne. We will give an independent, elementary proof.

THEOREM 6.8. *Let X be a quasi-projective scheme over K. Assume that X can be covered by two affine opens. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then for all i there are natural isomorphisms*

$$\mathrm{H}^i(\operatorname{Sym}^n X,\, \Gamma^n \mathcal{F}) = \Gamma^{n-i}\mathrm{H}^0(X,\, \mathcal{F}) \otimes {\textstyle\bigwedge}^i \mathrm{H}^1(X,\, \mathcal{F}).$$

of K-vector spaces.

Of course the typical example of a quasi-projective scheme covered by two affine opens is a curve, and this is precisely the case that we will use in the next chapter.

We give a proof using Čech cohomology. We will make use of two distinct affine open covers of X.

First, we choose an affine open cover $\mathfrak{U} = (U_i)_i$ of X so that $X^n = \cup_i U_i^n$. For every partition $n = n_1 + \cdots + n_k$ we have that

$$(\operatorname{Sym}^{n_1} U_i \times \cdots \times \operatorname{Sym}^{n_k} U_i)_i$$

is an affine open cover of $\operatorname{Sym}^{n_1} X \times \cdots \times \operatorname{Sym}^{n_k} X$. In particular the $\operatorname{Sym}^n U_i$ cover $\operatorname{Sym}^n X$. As usual, we write U_{ij} for the intersection of U_i and U_j, and similarly for higher intersections.

Second, we choose a two-element affine open cover $\mathfrak{V} = \{V_0, V_1\}$ of X. Since X is quasi-projective the intersection $V_{01} = V_0 \cap V_1$ is also affine open.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Denote by j_0, j_1 and j_{01} the inclusions of V_0, V_1 and $V_0 \cap V_1$ into X. The Čech resolution of $\mathcal{F}$ with respect to (V_0, V_1) is the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to j_{0,\star} j_0^\star \mathcal{F} \oplus j_{1,\star} j_1^\star \mathcal{F} \to j_{01,\star} j_{01}^\star \mathcal{F} \to 0,$$

which we abbreviate as

$$0 \to \mathcal{F} \to \mathcal{F}_0 \oplus \mathcal{F}_1 \to \mathcal{F}_{01} \to 0.$$

We have $\mathrm{H}^0(X, \mathcal{F}_i) = \mathcal{F}(V_i)$ and $\mathrm{H}^1(X, \mathcal{F}_i) = 0$ for all $i \in \{0, 1, 01\}$ so that the long exact sequence of cohomology associated to the above short exact sequence is

$$0 \to \mathrm{H}^0(X, \mathcal{F}) \to \mathcal{F}(V_0) \oplus \mathcal{F}(V_1) \to \mathcal{F}(V_{01}) \to \mathrm{H}^1(X, \mathcal{F}) \to 0,$$

the usual Čech sequence.

For every s and t consider the K-vector space

$$C^{s,t} = \bigoplus_{i_0<\cdots<i_s} \Gamma^{n-t}\big(\mathcal{F}_0(U_{i_0\cdots i_s}) \oplus \mathcal{F}_1(U_{i_0\cdots i_s})\big) \otimes \textstyle\bigwedge^t \mathcal{F}_{01}(U_{i_0\cdots i_s}).$$

We make $(C^{s,t})_{s,t}$ into a double complex, compute the cohomology of its columns and rows, and compare these to prove Theorem 6.8.

First we make the columns into complexes. We introduce differentials $d\colon C^{s,t} \to C^{s,t+1}$ so that for every s the row $C^{s,\bullet}$ is the sum over all $i_0 < \cdots < i_s$ of the degree n Koszul complex (see Proposition 6.5) of the map

$$\delta\colon \mathcal{F}_0(U_{i_0\cdots i_s}) \oplus \mathcal{F}_1(U_{i_0\cdots i_s}) \to \mathcal{F}_{01}(U_{i_0\cdots i_s}).$$

The cohomology of these column complexes is as follows.

LEMMA 6.9 (Cohomology of the columns). *For every s and for every t we have*

$$\mathrm{H}^t(C^{s,\bullet}) = \begin{cases} \bigoplus_{i_0<\cdots<i_s} \Gamma^n \mathcal{F}(U_{i_0\cdots i_s}) & t = 0 \\ 0 & t \neq 0. \end{cases}$$

PROOF. Fix s and a sequence of indices $i_0 < \cdots < i_s$. Since $U_{i_0\cdots i_s}$ is affine, the sequence

$$(26) \quad 0 \to \mathcal{F}(U_{i_0\cdots i_s}) \to \mathcal{F}_0(U_{i_0\cdots i_s}) \oplus \mathcal{F}_1(U_{i_0\cdots i_s}) \xrightarrow{\delta} \mathcal{F}_{01}(U_{i_0\cdots i_s}) \to 0$$

is exact. By Proposition 6.5 the cohomology of the degree n Koszul complex of the map δ in (26) is concentrated in degree 0, with value $\Gamma^n \mathcal{F}(U_{i_0\cdots i_s})$.

Summing over all indices now yields the result. □

Now we treat the rows. By Lemma 6.3 we have

$$C^{s,t} = \bigoplus_{a+b+t=n} \bigoplus_{i_0<\cdots<i_s} \Gamma^a \mathcal{F}_0(U_{i_0\cdots i_s}) \otimes \Gamma^b \mathcal{F}_1(U_{i_0\cdots i_s}) \otimes \textstyle\bigwedge^t \mathcal{F}_{01}(U_{i_0\cdots i_s}).$$

We introduce differentials $d'\colon C^{s,t} \to C^{s+1,t}$ so that for every t the row $C^{\bullet,t}$ is the sum over all a, b with $a + b = n - t$ of the Čech complex of the sheaf

$$\Gamma^a \mathcal{F}_0 \otimes_K \Gamma^b \mathcal{F}_1 \otimes_K \textstyle\bigwedge^t \mathcal{F}_{01}$$

on $\mathrm{Sym}^a V_0 \times \mathrm{Sym}^b V_1 \times \mathrm{Sym}^t V_{01}$, with respect to the affine open cover

$$(27) \qquad \Big(\mathrm{Sym}^a(V_0 \cap U_i) \times \mathrm{Sym}^b(V_1 \cap U_i) \times \mathrm{Sym}^t(V_{01} \cap U_i)\Big)_i .$$

LEMMA 6.10 (Cohomology of the rows). *For every t and for every s we have*

$$\mathrm{H}^s(C^{\bullet,t}) = \begin{cases} \Gamma^{n-t}\big(\mathcal{F}_0(X) \oplus \mathcal{F}_1(X)\big) \otimes \bigwedge^t \mathcal{F}_{01}(X) & s = 0 \\ 0 & s \neq 0. \end{cases}$$

PROOF. First fix a, b, and t with $a+b+t=n$. The complex $C^{\bullet,t}_{a,b}$ with

$$C^{s,t}_{a,b} = \bigoplus_{i_0<\cdots<i_s} \Gamma^a \mathcal{F}_0(U_{i_0\cdots i_s}) \otimes \Gamma^b \mathcal{F}_1(U_{i_0\cdots i_s}) \otimes \textstyle\bigwedge^t \mathcal{F}_{01}(U_{i_0\cdots i_s})$$

is by construction the Čech complex of $\Gamma^a_K \mathcal{F}_0 \otimes_K \Gamma^b_K \mathcal{F}_1 \otimes_K \bigwedge^t_K \mathcal{F}_{01}$ on $\operatorname{Sym}^a V_0 \times \operatorname{Sym}^b V_1 \times \operatorname{Sym}^t V_{01}$, with respect to the affine open cover (27), so that the cohomology of $C^{\bullet,t}_{a,b}$ coincides with the sheaf cohomology of $\Gamma^a_K \mathcal{F}_0 \otimes_K \Gamma^b_K \mathcal{F}_1 \otimes_K \wedge^t_K \mathcal{F}_{01}$. Since $\operatorname{Sym}^a V_0 \times \operatorname{Sym}^b V_1 \times \operatorname{Sym}^t V_{01}$ is affine the higher cohomology of $C^{\bullet,t}_{a,b}$ vanishes and we have

$$\mathrm{H}^0(C^{\bullet,t}_{a,b}) = \Gamma^a \mathcal{F}(V_0) \otimes \Gamma^b \mathcal{F}(V_1) \otimes \textstyle\bigwedge^t \mathcal{F}(V_{01}).$$

Summing over all a, b with $a+b+t=n$, and using Lemma 6.3 we find that the higher cohomology of $C^{\bullet,t}$ vanishes, and that

$$\mathrm{H}^0(C^{\bullet,t}) = \Gamma^{n-t}\big(\mathcal{F}(V_0) \oplus \mathcal{F}(V_1)\big) \otimes \textstyle\bigwedge^t \mathcal{F}(V_{01}),$$

as claimed. □

So far we have been vague about the signs in the differentials in the various Čech and Koszul complexes. For a suitable choice of signs $C = (C^{\bullet,\bullet}, d, d')$ becomes a double complex. This means that for every s, t the square

$$\begin{array}{ccc} C^{s,t+1} & \xrightarrow{d'} & C^{s+1,t+1} \\ {\scriptstyle d}\big\uparrow & & {\scriptstyle d}\big\uparrow \\ C^{s,t} & \xrightarrow{d'} & C^{s+1,t} \end{array}$$

is anti-commutative, so that $dd' + d'd = 0$. This in turn implies that the *total complex* defined as

$$(\operatorname{Tot} C)^n = \bigoplus_{s+t=n} C^{s,t},$$

with differential $d + d'$, is indeed a complex.

For a general double complex, there are spectral sequences relating the cohomology of the rows or the columns to the cohomology of the total complex. Since we are dealing with a rather degenerate case, we will only need the following relatively elementary lemma.

LEMMA 6.11. *Let C be a double complex so that $C^{s,t} = 0$ if $s < 0$ or $t < 0$, and so that the cohomology of the rows is concentrated in degree 0, i.e.* $\mathrm{H}^s(C^{\bullet,t}) = 0$ *for all t and all $s \neq 0$. Then the cohomology of the complex*

$$\cdots \to \mathrm{H}^0(C^{\bullet,-1}) \to \mathrm{H}^0(C^{\bullet,0}) \to \mathrm{H}^0(C^{\bullet,1}) \to \cdots \tag{28}$$

is naturally isomorphic with the cohomology of $\operatorname{Tot} C$.

PROOF. (See also [**45**, Tag 0133].) Let H be the double complex which has the complex (28) as the 0-th column, and which vanishes along all other columns. By the assumption that $C^{s,t} = 0$ for all $s < 0$ we have a natural injective map $H \to C$ of double complexes. Let Q be the quotient. This induces a short exact sequence

$$0 \to \operatorname{Tot} H \to \operatorname{Tot} C \to \operatorname{Tot} Q \to 0$$

of complexes. By the assumption that the higher cohomology of the rows of C vanishes, the rows of Q are exact. Since for every n there are only finitely many s, t with $s + t = n$ and $Q^{s,t} \neq 0$ this implies that also $\operatorname{Tot} Q$ is exact. It follows that $\operatorname{Tot} C$ has the same cohomology as $\operatorname{Tot} H$, which by construction coincides with (28). □

PROOF OF THEOREM 6.8. By Lemma 6.9 the double complex C satisfies the conditions of Lemma 6.11. The complex $\mathrm{H}^0(C^{s,\bullet})$ (indexed by s) is the Čech complex for the sheaf $\Gamma^n \mathcal{F}$ with respect to the cover $(\operatorname{Sym}^n \mathcal{U}_i)_i$, and hence

$$\mathrm{H}^i(\operatorname{Tot} C) = \mathrm{H}^i(\operatorname{Sym}^n X, \Gamma^n \mathcal{F}).$$

By Lemma 6.10 also the "transpose" of the double complex C satisfies the conditions of Lemma 6.11. The complex $\mathrm{H}^0(C^{\bullet,t})$ (indexed by t) is the Koszul complex of the map

$$\delta\colon \mathcal{F}_0(X) \oplus \mathcal{F}_1(X) \to \mathcal{F}_{01}(X).$$

By the Čech computation of $\mathrm{H}^\bullet(X, \mathcal{F})$ we have $\ker \delta = \mathrm{H}^0(X, \mathcal{F})$ and $\operatorname{coker} \delta = \mathrm{H}^1(X, \mathcal{F})$, so with Proposition 6.5 we find

$$\mathrm{H}^i(\operatorname{Tot} C) = \Gamma^{n-i}\mathrm{H}^0(X, \mathcal{F}) \otimes \textstyle\bigwedge^i \mathrm{H}^1(X, \mathcal{F}).$$

Comparing both expressions for $\mathrm{H}^i(\operatorname{Tot} C)$ yields the theorem. □

4. Crystals on symmetric powers

Let K be a field containing $\mathbf{F}_q$, and let A be a finite field extension of $\mathbf{F}_q$. For an $A \otimes_{\mathbf{F}_q} K$-module V we denote by $\Gamma^n_A V$ the $A \otimes_{\mathbf{F}_q} K$-module

$$\Gamma^n_A V := \left(V \bigotimes_{A\otimes_{\mathbf{F}_q} K} \cdots \bigotimes_{A\otimes_{\mathbf{F}_q} K} V \right)^{\mathfrak{S}_n}.$$

Since $A \otimes_{\mathbf{F}_q} K \cong L^d$ for some d and some separable extension $K \subset L$, this space of invariant tensors can be computed coordinate-wise. Therefore, the constructions and results of the previous sections extend from a base field K to the base ring $A \otimes_{\mathbf{F}_q} K$.

Let X be a quasi-projective scheme over K. We denote its symmetric powers (over K) by $\mathrm{Sym}^n X$. Let $(\mathcal{F}, \tau)$ be a coherent τ-sheaf on X with coefficients in A. Then the coherent sheaf $\Gamma^n_A \mathcal{F}$ on $C \times \mathrm{Sym}^n X$ carries a natural structure of τ-sheaf with coefficients in A, so that we obtain a functor Γ^n_A from $\mathbf{Coh}_\tau(X, A)$ to $\mathbf{Coh}_\tau(\mathrm{Sym}^n X, A)$. Similarly we have a functor $\bigwedge^n_A$. These induce functors

$$\Gamma^n_A \colon \mathbf{Crys}(X, A) \to \mathbf{Crys}(\mathrm{Sym}^n X, A).$$

and

$$\textstyle\bigwedge^n_A \colon \mathbf{Crys}(X, A) \to \mathbf{Crys}(\mathrm{Sym}^n X, A)$$

by the universal property of Theorem 1.21 and the following lemma.

LEMMA 6.12. *The functors Γ^n_A and $\bigwedge^n_A$ map nil-isomorphisms in $\mathbf{Coh}_\tau(X, A)$ to nil-isomorphisms in $\mathbf{Coh}_\tau(\mathrm{Sym}^n X, A)$.*

PROOF. First observe that if $\mathcal{N}$ is nilpotent, then so are $\Gamma^n_A \mathcal{N}$ and $\bigwedge^n_A \mathcal{N}$. We use this to show that Γ^n_A maps nil-isomorphisms to nil-isomorphisms. The argument for $\bigwedge^n_A$ is entirely analogous.

Without loss of generality we may assume that X is affine. Assume that $\varphi \colon \mathcal{F} \to \mathcal{G}$ is an *surjective* nil-isomorphism, so that we have a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{F} \to \mathcal{G} \to 0$$

in $\mathbf{Coh}_\tau(X, A)$, with $\mathcal{N}$ nilpotent. Then $\Gamma^n_A \mathcal{F}$ has a filtration $F^\bullet \Gamma^n_A \mathcal{F}$ as in Lemma 6.3, with intermediate quotients

$$\mathrm{gr}^i \, \Gamma^n_A \mathcal{F} = \Gamma^i_A \mathcal{N} \otimes_A \Gamma^{n-i}_A \mathcal{G}$$

These are all nilpotent, except possibly $\mathrm{gr}^0 = \Gamma^n_A \mathcal{G}$, so that the natural map $\Gamma^n_A \mathcal{F} \to \Gamma^n_A \mathcal{G}$ is a nil-isomorphism. Similarly, one shows that

Γ^n_A maps injective nil-isomorphisms to nil-isomorphisms. Since every nil-isomorphism is the composition of a surjective and an injective nil-isomorphism, the lemma follows. $\square$

Theorem 6.8 now has the following immediate consequence.

THEOREM 6.13. *Let A be a finite field extension of $\mathbf{F}_q$. Let K be a field containing $\mathbf{F}_q$. Let X be a quasi-projective scheme over K that can be covered by two affine opens. Let $\mathcal{F}$ be a crystal on X with coefficients in A. Then we have natural isomorphisms*

$$\mathrm{H}^i(C \times \mathrm{Sym}^n X, \Gamma^n_A \mathcal{F}) = \Gamma^{n-i}_A \mathrm{H}^0(C \times X, \mathcal{F}) \otimes_A \textstyle\bigwedge^i_A \mathrm{H}^1(C \times X, \mathcal{F})$$

in $\mathbf{Crys}(K, A)$. $\square$

Finally, for later use we record an important property of symmetric tensor powers of crystals.

PROPOSITION 6.14. *Let $i\colon Z \rightarrowtail X$ be a closed immersion of quasi-projective schemes over K. Let $\mathcal{F}$ be a coherent τ-sheaf on X. Then there is a natural isomorphism*

$$\Gamma^n i^\star \mathcal{F} \xrightarrow{\sim} (\mathrm{Sym}^n i)^\star \Gamma^n \mathcal{F}$$

of crystals on $\mathrm{Sym}^n Z$.

In general the isomorphism is not an isomorphism of coherent τ-sheaves. Also, the condition that i be a closed immersion cannot be dropped, see exercise 6.3.

PROOF OF PROPOSITION 6.14. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Consider the short exact sequence

$$0 \to \mathcal{I}\mathcal{F} \to \mathcal{F} \to i_\star i^\star \mathcal{F} \to 0$$

of coherent τ-sheaves on X. It induces a filtration on $\Gamma^n \mathcal{F}$, as in Lemma 6.3. We have a natural surjective map

$$\Gamma^n \mathcal{F} \to \mathrm{gr}^0 \mathcal{F} = \Gamma^n i_\star i^\star \mathcal{F} = i_\star \Gamma^n i^\star \mathcal{F}.$$

So to prove the proposition, it suffices to show that $F^1 \Gamma^n \mathcal{F}$, the kernel of this map, vanishes along $\mathrm{Sym}^n Z$. Consider the quotient maps

$$\pi_{i,j}\colon \mathrm{Sym}^i X \times \mathrm{Sym}^j X \to \mathrm{Sym}^{i+j} X.$$

The filtration $F^1 \Gamma^n \mathcal{F}$ on the kernel has intermediate quotients

$$\pi_{i,j,\star}\left(\Gamma^i \mathcal{I}\mathcal{F} \otimes \Gamma^j i_\star i^\star \mathcal{F}\right)$$

with $i+j=n$ and $i>0$. To show that $F^1\Gamma^n\mathcal{F}$ vanishes along $\operatorname{Sym}^n Z$ it therefore suffices to show that $\Gamma^i\mathcal{I}\mathcal{F}$ vanishes along $\operatorname{Sym}^i Z$ for all $i>0$.

Since $i^\star\mathcal{I}\mathcal{F}=0$ in $\mathbf{Crys}\,Z$, we are left with showing the following: if $\mathcal{G}$ is a crystal on X so that $i^\star\mathcal{G}=0$ in $\mathbf{Crys}\,Z$, then $(\operatorname{Sym}^n i)^\star\Gamma^n\mathcal{F}\cong 0$ in $\mathbf{Crys}\operatorname{Sym}^n Z$ for all $n>0$. In fact, this holds without the assumption that i is a closed immersion. Consider the commutative square

$$\begin{array}{ccc} Z^n & \xrightarrow{\;i^n\;} & X^n \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Sym}^n Z & \xrightarrow{\;\operatorname{Sym}^n i\;} & \operatorname{Sym}^n X \end{array}$$

(This square is in general *not* cartesian). Assume that $\mathcal{G}$ is a crystal in X that vanishes along Z. Consider the crystal

$$\mathcal{G}^{\boxtimes n} := \operatorname{pr}_1^\star\mathcal{G}\otimes\cdots\otimes\operatorname{pr}_n^\star\mathcal{G}$$

on X^n. We have

$$(i^n)^\star\pi^\star\pi_\star\mathcal{G}^{\boxtimes n} = (i^n)^\star\left(\mathcal{G}^{\boxtimes n}\otimes\pi^\star\mathbf{1}_{\operatorname{Sym}^n X}\right)=0,$$

by the assumption that $i^\star\mathcal{G}=0$. Because the above square commutes, we also find

$$\pi'^\star(\operatorname{Sym}^n i)^\star\pi_\star\mathcal{G}^{\boxtimes n}=0.$$

As the map π' is surjective, we conclude (for example using the pointwise criterion) that $(\operatorname{Sym}^n i)^\star\pi_\star\mathcal{G}^{\boxtimes n}=0$ in $\mathbf{Crys}\operatorname{Sym}^n Z$. Now by construction we have an injective map $\Gamma^n\mathcal{G}\to\pi_\star\mathcal{G}^{\boxtimes n}$ in $\mathbf{Crys}\operatorname{Sym}^n X$. Since pullback is exact, it induces an injective map

$$(\operatorname{Sym}^n i)^\star\Gamma^n\mathcal{G}\hookrightarrow(\operatorname{Sym}^n i)^\star\pi_\star\mathcal{G}^{\boxtimes n}=0,$$

and we conclude that $(\operatorname{Sym}^n i)^\star\Gamma^n\mathcal{G}=0$ in $\mathbf{Crys}\operatorname{Sym}^n Z$, as we had to show. □

Notes

When $n!$ is invertible in K the functor $V\to V^{\mathfrak{S}_n}$ on $K[\mathfrak{S}_n]$-modules is exact, from which one deduces

$$\mathrm{H}^i(\operatorname{Sym}^n X,\,\Gamma^n\mathcal{F})=\mathrm{H}^i(X^n,\,\operatorname{pr}_1^\star\mathcal{F}\otimes\cdots\otimes\operatorname{pr}_n^\star\mathcal{F})^{\mathfrak{S}_n}. \tag{29}$$

The Künneth formula computes $\mathrm{H}^i(X^n,\,\operatorname{pr}_1^\star\mathcal{F}\otimes\cdots\otimes\operatorname{pr}_n^\star\mathcal{F})$, and taking $\mathfrak{S}_n$-invariants yields the cohomology groups of Theorem 6.8 (in taking invariants, some care is needed with the signs in the Künneth formula). In small characteristics the above argument breaks down since taking $\mathfrak{S}_n$-invariants is no longer exact. In fact the isomorphism (29) does not hold in small characteristics.

A more conceptual approach to computing the cohomology of external symmetric powers is due to Deligne [**5**, XVII, §5.5]. The (non-additive!) functor $\Gamma^n\colon \mathbf{Vec}\,K \to \mathbf{Vec}\,K$ has a total derived functor $\mathrm{L}\Gamma^n$ in the sense of Dold and Puppe [**14**]. These are defined using simplicial methods. Given a map $f\colon X \to Y$ of quasi-projective schemes over $\operatorname{Spec} K$ and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, Deligne shows that $\mathrm{R}(\mathrm{Sym}^n f)_\star \Gamma_K \mathcal{F}$ coincides with $\mathrm{L}\Gamma^n \mathrm{R} f_\star \mathcal{F}$. In the special case where $Y = \operatorname{Spec} K$ and $\mathrm{H}^\bullet(X, \mathcal{F})$ is concentrated in degrees 0 and 1 one can explicitly compute $\mathrm{L}\Gamma^n \mathrm{R} f_\star \mathcal{F}$ and one recovers Theorem 6.8 from Deligne's result. In general, it is hard to explicitly compute the functor $\mathrm{L}\Gamma^n_K$.

Deligne's theorem induces a similar result for crystals, generalizing Theorem 6.13. It has an interesting and entirely explicit corollary on the level of Grothendieck groups. Let X be a quasi-projective scheme over a field K containing $\mathbf{F}_q$, let A be a finite reduced $\mathbf{F}_q$-algebra. Let n be a positive integer. The Koszul complex (Proposition 6.5) implies that the function

$$\mathrm{L}\Gamma^n_A\colon \mathrm{K}_0(X, A) \to \mathrm{K}_0(\mathrm{Sym}^n X, A)$$

given by

$$[\mathcal{F}] - [\mathcal{G}] \mapsto \sum_i (-1)^i [\Gamma^{n-i}_A \mathcal{F} \otimes \wedge^i_A \mathcal{G}]$$

is well-defined (see also exercise 6.4). Let $f\colon X \to Y$ be a morphism between quasi-projective schemes over a field K containing $\mathbf{F}_q$. Then Deligne's theorem implies that the square

$$\begin{array}{ccc}
\mathrm{K}_0(X, A) & \xrightarrow{\mathrm{L}\Gamma^n_A} & \mathrm{K}_0(\mathrm{Sym}^n X, A) \\
\Big\downarrow{\scriptstyle \mathrm{R}f_!} & & \Big\downarrow{\scriptstyle \mathrm{R}(\mathrm{Sym}^n f)_!} \\
\mathrm{K}_0(Y, A) & \xrightarrow{\mathrm{L}\Gamma^n_A} & \mathrm{K}_0(\mathrm{Sym}^n Y, A)
\end{array}$$

is commutative.

Exercises

EXERCISE 6.1. Let K be a field of characteristic $p > 0$, and let n be an integer $\geq p$. Show that the functors $V \mapsto \Gamma^n V$ and $V \mapsto \operatorname{Sym}^n V$ on K-vector spaces are not isomorphic.

EXERCISE 6.2. Show that $\operatorname{Sym}^n \mathbf{P}^1 \cong \mathbf{P}^n$ and that $\Gamma^n(\mathcal{O}_{\mathbf{P}^1}(e)) \cong \mathcal{O}_{\mathbf{P}^n}(e)$.

EXERCISE 6.3. Show that the condition that i be a closed immersion can not be dropped from Proposition 6.14. (Hint, consider $\operatorname{Spec} K \coprod \operatorname{Spec} K \to \operatorname{Spec} K$.)

EXERCISE 6.4. Let A be a finite reduced $\mathbf{F}_q$-algebra, and X a quasi-projective scheme over a field K containing $\mathbf{F}_q$. Verify in detail that the function

$$\mathrm{L}\Gamma^n_A \colon \mathrm{K}_0(X, A) \to \mathrm{K}_0(\operatorname{Sym}^n X, A)$$

given by

$$[\mathcal{F}] - [\mathcal{G}] \mapsto \sum_i (-1)^i [\Gamma^{n-i}_A \mathcal{F} \otimes \textstyle\bigwedge^i_A \mathcal{G}]$$

is well-defined. (Note that $\mathrm{L}\Gamma^n_A$ is not a group homomorphism if $n \neq 1$.)

EXERCISE 6.5 (*). Let K be a field and X a quasiprojective scheme over $\operatorname{Spec} K$. Assume that X admits a cover by two affine opens. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that

$$\mathrm{H}^i(\operatorname{Sym}^n X, \textstyle\bigwedge^n \mathcal{F}) \cong \bigwedge^{n-i} \mathrm{H}^0(X, \mathcal{F}) \otimes \operatorname{Sym}^i \mathrm{H}^1(X, \mathcal{F}),$$

where $\operatorname{Sym}^d V$ denotes the largest quotient of $V^{\otimes d}$ invariant under $\mathfrak{S}_d$.

CHAPTER 7

Trace formula for L-functions

1. L-functions of τ-sheaves and crystals

Let X be a scheme of finite type over $\mathbf{F}_q$. Let A be an $\mathbf{F}_q$-algebra of finite type, and, as before, denote $\operatorname{Spec} A$ by C.

Let $\mathcal{F}$ be a coherent τ-sheaf on X with coefficients in A. We will assume that one of the following two hypotheses hold:

(H1) A is a finite field;

(H2) there is an $\mathcal{O}_X$-module $\mathcal{F}_0$ so that $\mathcal{F} \cong \operatorname{pr}_X^\star \mathcal{F}_0$ as $\mathcal{O}_{C\times X}$-modules.

This assumption guarantees that for every closed point $x \in X$ the A-module $i_x^\star \mathcal{F}$ is free of finite rank. We then define the L-function of $\mathcal{F}$ by

$$L(X,\mathcal{F},T) := \prod_{x\in|X|} \det{}_A\,(1-\tau_s T \mid i_x^\star\mathcal{F})^{-1} \in 1+TA[[T]].$$

This infinite product converges because by the following lemma the factor at a point x of degree d is 1 modulo T^d.

Lemma 7.1. *Let k be an extension of $\mathbf{F}_q$ of degree d, and let $\mathcal{F}$ be an object of $\mathbf{Coh}_\tau(k,A)$. Assume that* (H1) *or* (H2) *are satisfied. Then*

$$\det{}_A(\,1-\tau_s T \mid \mathcal{F}\,) = \det{}_{A\otimes k}(\,1-\tau_s^d T^d \mid \mathcal{F}\,)$$

in $1+TA[[T]]$.

Note that τ_s^d is indeed an $A\otimes k$-linear endomorphism of $\mathcal{F}$.

Proof of Lemma 7.1. Consider the map

$$\psi\colon k\otimes_{\mathbf{F}_q}\mathcal{F} \to \mathcal{F}^d,\ \lambda\otimes s \mapsto (\lambda s, \lambda^q s, \ldots, \lambda^{q^{d-1}} s).$$

If we give $\mathcal{F}^d$ the structure of a k-module via

$$\lambda\cdot(s_1,\cdots,s_d) := (\lambda s_1,\ldots,\lambda^{q^{d-1}} s_d),$$

then the map ψ is an isomorphism of $A\otimes k$-modules. The endomorphism $\mathrm{id}\otimes\tau_s$ of $k\otimes_{\mathbf{F}_q}\mathcal{F}$ corresponds under ψ to the endomorphism

$$\mu\colon \mathcal{F}^d \to \mathcal{F}^d,\ (s_1,\ldots,s_d)\mapsto \big(\tau_s(s_d),\tau_s(s_1),\ldots,\tau_s(s_{d-1})\big).$$

We deduce that

$$\det_{A\otimes k}(\,1-(\mathrm{id}\otimes\tau_s)T\mid k\otimes_{\mathbf{F}_q}\mathcal{F}\,)=\det_{A\otimes k}(\,1-\mu T\mid \mathcal{F}^d\,).$$

By extension of scalars we have

$$\det_{A\otimes k}(\,1-(\mathrm{id}\otimes\tau_s)T\mid k\otimes_{\mathbf{F}_q}\mathcal{F}\,)=\det_{A}(\,1-\tau_s T\mid \mathcal{F}\,),$$

whereas for μ one computes using Lemma 7.2 below

$$\det_{A\otimes k}(\,1-\mu T\mid \mathcal{F}^d\,)=\det_{A\otimes k}(\,1-\tau_s^d T^d\mid \mathcal{F}\,).$$

Comparing both expressions, we see that the lemma indeed holds. □

LEMMA 7.2. *Let K be a field. Let $V_1,\ldots,V_d$ be finite-dimensional K-vector spaces. Consider a sequence of linear maps*

$$V_1\xrightarrow{\alpha_1}V_2\xrightarrow{\alpha_2}\cdots\xrightarrow{\alpha_{d-1}}V_d\xrightarrow{\alpha_d}V_1$$

and let $\alpha\colon V_1\to V_1$ be their composition. Let $V=\oplus V_i$ and consider the endomorphism

$$\mu\colon V\to V,\ \big(v_1,\ldots,v_d\big)\mapsto\big(\alpha_d(v_d),\alpha_1(v_1),\ldots,\alpha_{d-1}(v_{d-1})\big).$$

Then

$$\det_K(\,1-T\mu\mid V\,)=\det_K(\,1-T^d\alpha\mid V_1\,)$$

in $K[T]$. □

Now assume that A is a finite field, so that we are in case (H1). If $\mathcal{N}$ in $\mathbf{Coh}_\tau(X,A)$ is nilpotent then $L(X,\mathcal{N},T)=1$. Also, if

$$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$$

is a short exact sequence in $\mathbf{Coh}_\tau(X,A)$ then we have

$$L(X,\mathcal{F}_2,T)=L(X,\mathcal{F}_1,T)\cdot L(X,\mathcal{F}_3,T)$$

in $1+TA[[T]]$. These two facts imply that $L(X,\mathcal{F},T)$ is well-defined for an A-crystal $\mathcal{F}$ on X, and that we have a group homomorphism

$$\mathrm{K}_0(X,A)\to 1+TA[[T]],\ [\mathcal{F}]\mapsto L(X,\mathcal{F},T).$$

The main result of this chapter is the following theorem. It shows that taking L-functions is compatible with pushforward with proper support.

THEOREM 7.3 (Trace formula for L-functions). *Let A be a finite field extension of $\mathbf{F}_q$. Let $f\colon X \to Y$ be a separated morphism of schemes of finite type over $\mathbf{F}_q$. Let $\mathcal{F}$ be an A-crystal on X. Then*

$$L(X, \mathcal{F}, T) = \prod_{n \geq 0} L(Y, \mathrm{R}^n f_! \mathcal{F}, T)^{(-1)^n}$$

in $1 + TA[[T]]$.

Recall from Chapter 2 that we have defined $f_!$ only for separated morphisms f.

Applying the theorem to the structure map $X \to \operatorname{Spec} \mathbf{F}_q$ we see that it implies the rationality of the L-function of a crystal:

COROLLARY 7.4. *Let X be a scheme of finite type over $\mathbf{F}_q$, let A be a finite field extension of $\mathbf{F}_q$, and let $\mathcal{F}$ be a crystal on X. Then $L(X, \mathcal{F}, T)$ lies in $A(T) \cap (1 + TA[[T]])$.* □

In case (H2) we have the following variant.

THEOREM 7.5. *Let A be a reduced $\mathbf{F}_q$-algebra of finite type. Let X be a proper scheme over* $\operatorname{Spec} \mathbf{F}_q$. *Let $\mathcal{F}$ in $\mathbf{Coh}_\tau(X, A)$ be such that there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}_0$ such that $\mathcal{F} \cong \mathrm{pr}_2^\star \mathcal{F}$ as $\mathcal{O}_{C \times X}$-modules. Then*

$$L(X, \mathcal{F}, T) = \prod_{n \geq 0} \det{}_A \left(1 - \tau_s T \mid \mathrm{H}^n(C \times X, \mathcal{F})\right)^{(-1)^{n+1}} \tag{30}$$

in $1 + TA[[T]]$.

Note that, as an A-module, $\mathrm{H}^n(C \times X, \mathcal{F})$ is isomorphic with $A \otimes_{\mathbf{F}_q} \mathrm{H}^n(X, \mathcal{F}_0)$. In particular it is a free A-module of finite rank, so that the determinants in (30) are defined.

As before we deduce the rationality of $L(X, \mathcal{F}, T)$:

COROLLARY 7.6. *Let X, A and $\mathcal{F}$ be as in Theorem 7.5. Let Q be the total quotient ring of A. Then $L(X, \mathcal{F}, T) \in Q(T) \cap (1 + TA[[T]])$.* □

PROOF OF THEOREM 7.5, ASSUMING THEOREM 7.3. Let $\mathfrak{m}$ be a maximal ideal of A. Since A is of finite type, $A/\mathfrak{m}$ is a finite field. Consider the object $\mathcal{F}/\mathfrak{m}\mathcal{F}$ of $\mathbf{Coh}_\tau(X, A/\mathfrak{m})$. Let $f\colon X \to \operatorname{Spec} \mathbf{F}_q$ be the structure morphism. From the definitions, we see

$$L(X, \mathcal{F}/\mathfrak{m}\mathcal{F}, T) = L(X, \mathcal{F}, T) \bmod \mathfrak{m}$$

and

$$L(\operatorname{Spec}\mathbf{F}_q, \mathrm{R}^n f_\star(\mathcal{F}/\mathfrak{m}\mathcal{F}), T) = \det_A\big(1-\tau_s T \mid \mathrm{H}^n(C\times X,\mathcal{F})\big) \bmod \mathfrak{m},$$

in $1+T(A/\mathfrak{m})[[T]]$. Comparing these with Theorem 7.3 shows that (30) holds modulo every maximal ideal $\mathfrak{m}$. Since A is reduced, the map $A \to \prod_{\mathfrak{m}} A/\mathfrak{m}$ is injective and Theorem 7.5 follows. □

In the following sections we will prove Theorem 7.3, deducing it from the trace formula (Theorem 3.4) using the results on symmetric powers of Chapter 6.

2. Computation of L-functions via symmetric powers

L-functions are related to traces on symmetric powers by the following theorem.

THEOREM 7.7. *Let X be a quasi-projective scheme over $\mathbf{F}_q$. Let A be a finite field extension of $\mathbf{F}_q$. Let $\mathcal{F}$ be an A-crystal on X. Then*

$$L(X,\mathcal{F},T) = \sum_{n=0}^{\infty}\sum_{x}(\operatorname{tr}_{A,\Gamma^n_A\mathcal{F}}\, x)T^n \tag{31}$$

in $1+TA[[T]]$, where x in the inner sum ranges over the $\mathbf{F}_q$-points of $\operatorname{Sym}^n X$.

In this section we will prove Theorem 7.7.

LEMMA 7.8. *Let K be a field and let V be a finite-dimensional vector space over K. Let α be an endomorphism of V. Then we have*

$$\det_K(\,1-\alpha T\mid V\,) = \sum_{n\ge 0}(-1)^n \operatorname{tr}_K(\,\alpha \mid \wedge^n V\,)\,T^n \tag{32}$$

and

$$\det_K(\,1-\alpha T\mid V\,)^{-1} = \sum_{n\ge 0}\operatorname{tr}_K(\,\alpha \mid \Gamma^n V\,)\,T^n \tag{33}$$

in $1+TK[[T]]$.

PROOF. Clearly the formulas hold if V has dimension ≤ 1. The left-hand-sides of (32) and (33) are multiplicative in short exact sequences

$$0\to V_1\to V_2\to V_3\to 0$$

of finite-dimensional vector spaces equipped with endomorphisms α_i. By the filtrations of Lemmas 6.3 and 6.4 also the right-hand-sides are multiplicative.

Now, without loss of generality we may assume that K is algebraically closed. Since every (V, α) of positive dimension has a nonzero eigenvector, we can always find a one-dimensional subspace V_1 preserved by α, and the formulas follow by induction on the dimension of V. $\square$

LEMMA 7.9. *Let X_1 and X_2 be quasi-projective schemes over K. Let $X = X_1 \amalg X_2$ be their disjoint sum. Let n be a non-negative integer. Then we have a natural isomorphism*

$$\operatorname{Sym}^n X = \coprod_{i+j=n} (\operatorname{Sym}^i X_1) \times (\operatorname{Sym}^j X_2).$$

Moreover, let $\mathcal{F}$ be an A-crystal on X and denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ its restrictions to X_1 and X_2. Then the restriction of $\Gamma^n_A \mathcal{F}$ to $(\operatorname{Sym}^i X_1) \times (\operatorname{Sym}^j X_2)$ is the A-crystal $\operatorname{pr}_1^\star \Gamma^i_A \mathcal{F}_1 \otimes_A \operatorname{pr}_2^\star \Gamma^j_A \mathcal{F}_2$. $\square$

PROOF OF THEOREM 7.7. It suffices to prove the identity modulo T^N for arbitrary N. Let $i\colon X_0 \to X$ be the inclusion of the disjoint union of all the closed points of degree $< N$ in X. Then

$$L(X, \mathcal{F}, T) \equiv L(X_0, i^\star \mathcal{F}, T) \pmod{T^N}.$$

On the other hand, i induces a bijection

$$\operatorname{Sym}^n i\colon (\operatorname{Sym}^n X_0)(\mathbf{F}_q) \xrightarrow{\sim} (\operatorname{Sym}^n X)(\mathbf{F}_q)$$

for all $n < N$. By Proposition 6.14 we have for all $x \in (\operatorname{Sym}^n X_0)(\mathbf{F}_q)$ an isomorphism

$$x^\star \Gamma^n i^\star \mathcal{F} \cong x^\star (\operatorname{Sym}^n i)^\star \Gamma^n \mathcal{F}$$

of A-crystals on $\operatorname{Spec} \mathbf{F}_q$. We see that the theorem holds modulo T^N for X if and only if it holds modulo T^N for X_0. So we may assume $X = \coprod \operatorname{Spec} k_i$ for finitely many finite extensions $k_i/\mathbf{F}_q$.

By Lemma 7.9 the right-hand side of (31) is multiplicative in disjoint decompositions $X = X_1 \coprod X_2$. By definition of the L-function the same holds for the left-hand side, so we may further reduce and assume $X = \operatorname{Spec} k$ for a finite extension $k/\mathbf{F}_q$.

For $X = \operatorname{Spec} k$, we will show that for all n the coefficient of T^n in both sides of (31) agree. Let d be the degree of k over $\mathbf{F}_q$.

If n is not divisible by d then the coefficient of T^n of the left-hand side vanishes. Since in that case $(\operatorname{Sym}^n X)(\mathbf{F}_q)$ is empty, the same holds for the corresponding coefficient of the right-hand side.

Assume n is divisible by d. Then $\operatorname{Sym}^n X$ has a unique $\mathbf{F}_q$-point, say x. We have

$$x^\star \Gamma^n \mathcal{F} = (\mathcal{F} \otimes_A \cdots \otimes_A \mathcal{F})^{\mathfrak{S}_n}.$$

With Lemma 7.8 (applied to $V = \mathcal{F}$, $K = A$ and $\alpha = \tau_s$) we see that the trace of τ_s on $x^\star \Gamma^n \mathcal{F}$ is the coefficient of T^n in

$$L(X, \mathcal{F}, \tau) = \det_A(1 - \tau_s T \mid \mathcal{F})^{-1}$$

which finishes the proof. □

3. Proof of the trace formula for L-functions

Again, A is a finite field extension of $\mathbf{F}_q$. We are now going to prove Theorem 7.3. We repeat that we need to show the identity

$$L(X, \mathcal{F}, T) = \prod_{n \geq 0} L(Y, \mathrm{R}^n f_! \mathcal{F}, T)^{(-1)^n}$$

for a separated morphism $f\colon X \to Y$ of schemes of finite type over $\mathbf{F}_q$, and for an A-crystal $\mathcal{F}$ on X. Equivalently, we have to show that the triangle

$$\begin{array}{ccc} \mathrm{K}_0(X, A) & \xrightarrow{\mathrm{R}f_!} & \mathrm{K}_0(Y, A) \\ & {\scriptstyle L(X,-,T)} \searrow \quad \swarrow {\scriptstyle L(Y,-,T)} & \\ & 1 + TA[[T]] & \end{array} \tag{34}$$

commutes.

We start with the crucial case of a projective curve.

Proposition 7.10. *If X is a projective curve over $Y = \operatorname{Spec} \mathbf{F}_q$ then (34) commutes.*

Proof. By Theorem 7.7 we have

$$L(X, \mathcal{F}, T) = \sum_{n \geq 0} \sum_{x \in (\operatorname{Sym}^n X)(\mathbf{F}_q)} (\operatorname{tr}_{A, \Gamma_A^n \mathcal{F}} x) T^n.$$

Applying the trace formula of Theorem 5.4 to the $\Gamma_A^n \mathcal{F}$ this becomes

$$L(X, \mathcal{F}, T) = \sum_{n \geq 0} \sum_{i \geq 0} (-1)^i \operatorname{tr}_A \left(\tau_s \mid \mathrm{H}^i(\operatorname{Sym}^n X, \Gamma_A^n \mathcal{F}) \right) T^n.$$

In Theorem 6.8 we have computed the cohomology of $\Gamma^n_A\mathcal{F}$ on $\operatorname{Sym}^n X$, and applying this we find that $L(X,\mathcal{F},T)$ equals

$$\sum_{n\geq 0}\sum_{i\geq 0}(-1)^i \operatorname{tr}_A\left(\tau_s \mid \Gamma^{n-i}\mathrm{H}^0(X,\mathcal{F})\right)\cdot \operatorname{tr}_A\left(\tau_s \mid \wedge^i \mathrm{H}^1(X,\mathcal{F})\right)T^n.$$

Rearranging terms, we can rewrite this as

$$\Big(\sum_{n\geq 0}\operatorname{tr}_A(\tau_s \mid \Gamma^n\mathrm{H}^0(X,\mathcal{F}))T^n\Big)\Big(\sum_{n\geq 0}(-1)^n\operatorname{tr}_A(\tau_s \mid \wedge^n\mathrm{H}^1(X,\mathcal{F}))T^n\Big),$$

and by Lemma 7.8 as

$$\det{}_A\left(1-\tau_s T \mid \mathrm{H}^0(X,\mathcal{F})\right)^{-1}\cdot \det{}_A\left(1-\tau_s T \mid \mathrm{H}^1(X,\mathcal{F})\right),$$

which is precisely $L(Y,\mathrm{R}f_!\mathcal{F},T)$. □

The proof of the general case proceeds by a series of lemmas that ultimately reduce the theorem to Proposition 7.10.

We first show that Theorem 7.3 may be verified fiber by fiber.

Lemma 7.11. *Let $f\colon X\to Y$ be a separated morphism of schemes of finite type. If (34) commutes for the map $i_y^\star f\colon X_y\to \operatorname{Spec} k(y)$ for every closed point $y\in Y$, then it commutes for the map $f\colon X\to Y$.*

Proof. Let $\mathcal{F}$ be an A-crystal on X. By the definition of L-function as a product over the closed points we have

$$L(X,\mathcal{F},T)=\prod_{y\in|Y|}L(X_y,i_y^\star\mathcal{F},T).$$

and

$$L(Y,\mathrm{R}^n f_!\mathcal{F},T)=\prod_{y\in Y}\det{}_A\left(1-\tau_s T \mid i_y^\star\mathrm{R}^n f_!\mathcal{F}\right).$$

Theorem 2.35 (proper base change) gives isomorphisms

$$i_y^\star\mathrm{R}^n f_!\mathcal{F}=\mathrm{R}^n(i_y^\star f)_\star i_y^\star\mathcal{F}$$

and multiplying the identity of the trace formula for $i_y^\star\mathcal{F}$ on $X_y\to\operatorname{Spec} k(y)$ over all y gives the trace formula for $\mathcal{F}$ on $f\colon X\to Y$. □

Lemma 7.12. *Let $f\colon X\to Y$ and $g\colon Y\to Z$ be separated morphisms between schemes of finite type over $\mathbf{F}_q$. If (34) commutes for f and g, then it also commutes for $gf\colon X\to Z$. Conversely, if it commutes for gf and for g then it also commutes for f.*

PROOF. By the A-coefficients version of Proposition 3.3 we have $\mathrm{R}g_!\mathrm{R}f_! = \mathrm{R}(gf)_!$ as maps $\mathrm{K}_0(X,A) \to \mathrm{K}_0(Z,A)$. Now consider the diagram

$$\begin{array}{ccccc} \mathrm{K}_0(X,A) & \xrightarrow{\mathrm{R}f_!} & \mathrm{K}_0(Y,A) & \xrightarrow{\mathrm{R}g_!} & \mathrm{K}_0(Z,A) \\ & \searrow & \downarrow & \swarrow & \\ & & 1+TA[[T]] & & \end{array}$$

If the two inner triangles commute, then also the outer diagram commutes, and if the right and outer triangle commutes, then so does the left one. □

LEMMA 7.13. *Let $f\colon X \to Y$ be a separated morphism between schemes of finite type over $\mathbf{F}_q$. Let $i\colon Z \rightarrowtail X$ be a closed immersion, and let $j\colon U \hookrightarrow X$ be the open complement. If (34) commutes for $fi\colon Z \to Y$ and $fj\colon U \to Y$ then it commutes for $f\colon X \to Y$.*

PROOF. Let $\mathcal{F}$ be an A-crystal on X. We have

$$L(X,\mathcal{F},T) = L(Z, i^\star\mathcal{F}, T)L(U, j^\star\mathcal{F}, T)$$

in $1+TA[[T]]$ and by the short exact sequence

$$0 \to j_!j^\star\mathcal{F} \to \mathcal{F} \to i_\star i^\star\mathcal{F} \to 0$$

we have

$$[j_!j^\star\mathcal{F}] + [i_\star i^\star\mathcal{F}] = [\mathcal{F}]$$

in $K_0(X,A)$. Applying $\mathrm{R}f_!$ we find

$$\mathrm{R}f_![\mathcal{F}] = \mathrm{R}(fj)_![j^\star\mathcal{F}] + \mathrm{R}(fi)_![i^\star\mathcal{F}]$$

in $K_0(Y,A)$ and we see that the statement for $\mathcal{F}$ along $f\colon X \to Y$ follows from that for $j^\star\mathcal{F}$ and $i^\star\mathcal{F}$ along $fj\colon U \to Y$ and $fi\colon Z \to Y$ respectively. □

LEMMA 7.14. *Assume the fibers of $f\colon X \to Y$ have dimension ≤ 1. Then (34) commutes.*

PROOF. By Lemma 7.11 we may assume $Y = \operatorname{Spec} k$ for a finite extension k of $\mathbf{F}_q$. Considering the composition $X \to \operatorname{Spec} k \to \operatorname{Spec} \mathbf{F}_q$ and the second statement of Lemma 7.12, we see that we may reduce to the case $Y = \operatorname{Spec} \mathbf{F}_q$. By Lemma 7.13 we may assume that X is irreducible. If $\dim X = 0$ then (34) commutes for trivial reasons. If

$\dim X = 1$ then choose an open immersion $j\colon X \hookrightarrow \overline{X}$ into a projective curve. Applying Proposition 7.10 to $j_!\mathcal{F}$ on $\overline{X}$ shows that (34) commutes for $f\colon X \to \operatorname{Spec} \mathbf{F}_q$. $\square$

We now combine these ingredients to finish the proof.

PROOF OF THEOREM 7.3. Let $d(f)$ be the maximum of the dimensions of the irreducible components of the fibers $f^{-1}(y)$ with $y \in Y$. We prove the statement by induction on $d(f)$. For $d(f) = 0$ the theorem follows immediately from Lemma 7.11.

Assume the theorem has been shown for all g with $d(g) < d$ and let $f\colon X \to Y$ be a map with $d(f) = d$. By Lemma 7.11 we may assume $Y = \operatorname{Spec} k$, and by 7.13 we can assume X irreducible. Choose an affine open dense $U \subset X$. Since the dimensions of the irreducible components of the complement Z are less than d, we only need to show the theorem for the map $U \to \operatorname{Spec} k$. Choose a closed immersion $U \rightarrowtail \mathbf{A}^n_k$. Then the map $U \to \operatorname{Spec} k$ factors as

$$U \rightarrowtail \mathbf{A}^n_k \twoheadrightarrow \mathbf{A}^{n-1}_k \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{Spec} k,$$

and since each arrow has fibers of dimension ≤ 1 the theorem follows by 7.14. $\square$

Notes

Theorems 7.3 and 7.5 are shown in [**11**] (in slightly larger generality), using a different method. They reduce to the case of an affine smooth X, and use Serre duality to deduce the result from Anderson's 'elementary approach' [**3**]. The approach taken here, using symmetric powers to reduce to the trace formula, is based on Deligne, see [**5**, Exp. XVII] and [**16**, fonctions L].

For more background on L-functions associated to τ-sheaves, see [**25, 47, 11**].

Exercises

EXERCISE 7.1. Let X be a proper scheme over $\operatorname{Spec} \mathbf{F}_2$. Assume that $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbf{F}_2$ and $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Show that $\#X(\mathbf{F}_4)$ is congruent to 1 modulo 4.

EXERCISE 7.2 ($\star$). Let G be a linear algebraic group G over $\mathbf{F}_q$ and let V be a finite-dimensional representation of $G(\mathbf{F}_q)$. Show that there is a crystal $\mathcal{F}$ on G so that for all $g \in G(\mathbf{F}_q)$ the pair $g^\star(\mathcal{F}, \tau)$

is isomorphic with the pair (V, g) in the category of $\mathbf{F}_q$-vector spaces equipped with an endomorphism.

EXERCISE 7.3 ($\star$). Show that the homomorphism

$$\mathrm{K}_0(X, A) \to \prod_{x \in |X|} 1 + TA[[T]]$$

defined by

$$\mathcal{F} \mapsto \big(L(\operatorname{Spec} k(x), i_x^\star \mathcal{F}, T)\big)_x$$

is injective. (Hints: use Noetherian induction to reduce to locally free $\mathcal{F}$ with τ an isomorphism, then use the Brauer-Nesbitt theorem and the Chebotarev density theorem [**44**, Thm 7]).

CHAPTER 8

Special values of L-functions

We start this chapter with the motivating example of the Goss zeta function.

1. Example: the Goss zeta function of a scheme over $\mathbf{F}_q[t]$

In this section A denotes the polynomial ring $\mathbf{F}_q[t]$, and F its fraction field $\mathbf{F}_q(t)$. As in the previous chapters we denote $\operatorname{Spec} A$ by C.

If M is a finite* A-module then there are monic $f_i \in A$ such that

$$M \cong \bigoplus_i A/(f_i).$$

We define the *characteristic element* of the module M to be the monic element

$$[M] := \prod_i f_i$$

of A. It is independent of the choice of decomposition of M into cyclic A-modules. In fact:

Lemma 8.1. $[M] = \det_A\big(\, t \otimes 1 - 1 \otimes t \;\mid A \otimes_{\mathbf{F}_q} M \,\big)$. □

In other words, if $P(X) \in \mathbf{F}_q[X]$ is the characteristic polynomial of the endomorphism t of the $\mathbf{F}_q$-vector space M, then $[M] = P(t)$ in A.

Now let X be a scheme of finite type over C. Then for every closed point $x \in X$ the residue field $k(x)$ is both a finite field and an A-algebra, so that it is in particular a finite A-module. The *Goss zeta function* of $X \to C$ at an integer n is the power series

$$\zeta_G(X/C, n, T) := \prod_{x \in |X|} \Big(1 - T^{d_x} [k(x)]^{-n}\Big)^{-1} \in 1 + TF[[T]].$$

*We use *finite* in the set-theoretic sense, so M is torsion and finitely generated.

For example, if $X = C$, then

$$\zeta_G(C/C, n, T) = \prod_g \left(1 - g^{-n} T^{\deg g}\right)^{-1}$$

where g ranges over the monic irreducible polynomials in A. By unique factorization we also have

$$\zeta_G(C/C, n, T) = \sum_f f^{-n} T^{\deg f}$$

where f ranges over the monic polynomials in $\mathbf{F}_q[t]$.

Remark 8.2. The Goss zeta function of a scheme over $\operatorname{Spec} A$ is analogous to the Hasse-Weil zeta function of a scheme over $\operatorname{Spec} \mathbf{Z}$. In both cases these are formed as a product over the closed points. In the Hasse-Weil zeta function, the factor at a point x depends on the cardinality of $k(x)$, which is the positive generator of the fitting ideal of the $\mathbf{Z}$-module $k(x)$. In the Goss zeta function, this is replaced by the characteristic element of the A-module $k(x)$.

We will now express these Goss zeta functions as L-functions of τ-sheaves with coefficients in A. Consider the τ-sheaf $\mathcal{C}$ in $\mathbf{Coh}_\tau(C, A)$ given by $\mathcal{C} = (\mathcal{O}_{C\times C}, \delta)$ with

$$\delta = (t \otimes 1 - 1 \otimes t) \colon (\mathrm{id} \otimes \sigma)^\star \mathcal{O}_{C\times C} = \mathcal{O}_{C\times C} \to \mathcal{O}_{C\times C}.$$

We denote by $\mathcal{C}^{\otimes n}$ the n-fold tensor product $\mathcal{C} \otimes_A \cdots \otimes_A \mathcal{C}$. We have $\mathcal{C}^{\otimes n} = (\mathcal{O}_{C\times C}, \delta^n)$.

Proposition 8.3. *If $\pi \colon X \to C$ is a map of finite type and n a positive integer then*

$$L(X, \pi^\star \mathcal{C}^{\otimes n}, T) = \zeta_G(X/C, -n, T)$$

in $1 + TA[[T]]$.

Proof. By the definition of L and ζ_G as a product over closed points of X, it suffices to show this for $X = \operatorname{Spec} k$ with k a finite extension of $\mathbf{F}_q$. The map $\pi \colon X \to C$ then corresponds to a map $A \to k$. Let θ be the image of t in k. The characteristic polynomial of t acting (as multiplication by θ) on the $\mathbf{F}_q$-vector space k is

$$(X - \theta)(X - \theta^q) \cdots (X - \theta^{q^{d-1}}) \in \mathbf{F}_q[X],$$

so we have

$$[k] = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}}) \in A \subset A \otimes k.$$

On the other hand, the action of τ_s^d on the rank one $A \otimes k$-module $\pi^\star \mathcal{C}^{\otimes n}$ is multiplication by

$$(t \otimes 1 - 1 \otimes \theta^{q^{d-1}})^n \cdots (t \otimes 1 - 1 \otimes \theta^q)^n (t \otimes 1 - 1 \otimes \theta)^n$$

so that

$$L(\operatorname{Spec} k, \pi^\star \mathcal{C}^n, T) = (1 - T^d [k]^n)^{-1},$$

as we had to show. □

Let $\overline{C} \cong \mathbf{P}^1$ be the natural compactification of $C = \operatorname{Spec} \mathbf{F}_q[t]$. Let ∞ be the added point at infinity. Let n be a positive integer. Let d be an integer with $d > n/(q-1)$. Consider the object $\mathcal{F}_{n,d}$ of $\mathbf{Coh}_\tau(\overline{C}, A)$ given by

$$\mathcal{F}_{n,d} = \left(\operatorname{pr}_{\overline{C}}^\star \mathcal{O}_{\overline{C}}(-d\infty), \delta^n \right).$$

Note that δ^n does indeed define a map $(\mathrm{id} \times \sigma)^\star \mathcal{F}_{n,d} \to \mathcal{F}_{n,d}$ because $d \geq n/(q-1)$, and that the strict inequality $d > n/(q-1)$ guarantees that $\infty^\star \mathcal{F}_{n,d}$ is nilpotent. In fact, the A-crystal $\mathcal{F}_{n,d}$ does not depend on the choice of d and could be regarded as the extension by zero of the A-crystal $\mathcal{C}^{\otimes n}$ along $C \hookrightarrow \overline{C}$.

The previous proposition now implies immediately:

COROLLARY 8.4. *Let $\pi \colon X \to C$ be a proper map. Let $\bar\pi \colon \overline{X} \to \overline{C}$ be a proper map extending π. Let n be a positive integer and let $d > n/(q-1)$. Then $L(X, \pi^\star \mathcal{C}^{\otimes n}, T) = L(\overline{X}, \bar\pi^\star \mathcal{F}_{n,d}, T)$.* □

Applying the trace formula of Theorem 7.5 we obtain a cohomological expression for the Goss zeta function at negative integers.

PROPOSITION 8.5. *Let $\pi \colon X \to C$ be a proper map. Let $\bar\pi \colon \overline{X} \to \overline{C}$ be a proper map extending π. Let n be a positive integer and let $d > n/(q-1)$. Then*

$$\zeta_G(X/C, -n, T) = \prod_{i \geq 0} \det\nolimits_A \left(1 - \tau_s T \mid \mathrm{H}^i(C \times \overline{X}, (id \times \bar\pi)^\star \mathcal{F}_{n,d}) \right)^{(-1)^{i+1}}.$$

in $1 + TA[[T]]$. □

The cohomological expression also implies rationality.

COROLLARY 8.6. *Let $X \to C$ be a proper map and let n be a positive integer. Then $\zeta_G(X/C, -n, T) \in F(T)$.* □

In the coming sections we will describe the order of vanishing as well as the leading coefficient of $\zeta_G(X/C, -n, T)$ at $T = 1$, under one additional hypothesis.

2. Extensions of τ-sheaves

Let X be a scheme over $\mathbf{F}_q$ and C the spectrum of an $\mathbf{F}_q$-algebra A. For a quasi-coherent $\mathcal{O}_{C\times X}$-module $\mathcal{F}$ we define the quasi-coherent τ-sheaf $\Theta(\mathcal{F})$ on $C \times X$ as follows:

$$\Theta(\mathcal{F}) := \bigoplus_{n\geq 0} (\mathrm{id}\times\sigma^n)^\star \mathcal{F}$$

and

$$\tau(s_0, s_1, s_2, \ldots) := (0, s_0, s_1, \ldots).$$

LEMMA 8.7. *The functor Θ is a left adjoint to the forgetful functor $\mathbf{QCoh}_\tau(X, A) \to \mathbf{QCoh}(C \times X)$ that maps $(\mathcal{G}, \tau)$ to $\mathcal{G}$.*

PROOF. We claim that the map

$$\mathrm{Hom}_{\mathbf{QCoh}_\tau(X,A)}\big(\Theta(\mathcal{F}),\, (\mathcal{G},\tau)\big) \;\to\; \mathrm{Hom}_{\mathbf{QCoh}(C\times X)}\big(\mathcal{F},\, \mathcal{G}\big)$$

given by

$$\varphi \mapsto \big[\, s \mapsto \varphi(s_0)\,\big] \tag{35}$$

is an isomorphism. We show this by constructing an inverse isomorphism. Given a morphism $\psi\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_{C\times X}$-modules we denote by ψ_n the $\mathcal{O}_{C\times X}$-linear maps

$$\psi_n := (\mathrm{id}\times\sigma^n)^\star\psi\colon (\mathrm{id}\times\sigma^n)^\star\mathcal{F} \to (\sigma^n)^\star\mathcal{G}.$$

Note that $\psi_0 = \psi$. Now the map

$$\mathrm{Hom}_{\mathbf{QCoh}(C\times X)}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathbf{QCoh}_\tau(X,A)}(\Theta(\mathcal{F}), (\mathcal{G},\tau))$$

given by

$$\psi \mapsto \Big[\, (s_n)_n \mapsto \sum_{n\geq 0} \tau^n \psi_n(s_n)\,\Big]$$

is a two-sided inverse to (35), which proves the adjunction. □

COROLLARY 8.8. *Let X be a scheme over $\mathbf{F}_q$ and let A be an $\mathbf{F}_q$-algebra. Then the category $\mathbf{QCoh}_\tau(X, A)$ has enough injectives.*

PROOF. By Grothendieck's theorem [**26**, 1.10.1] [**45**, Tag 05AB] it suffices to verify that $\mathbf{QCoh}_\tau(X, A)$ satisfies:

(1) filtered direct limits exist and are exact;
(2) there is a $\mathcal{U}$ in $\mathbf{QCoh}_\tau(X, A)$ so that for every nonzero map $\varphi\colon \mathcal{F} \to \mathcal{G}$ there is a map $\mathcal{U} \to \mathcal{F}$ so that the composition $\mathcal{U} \to \mathcal{G}$ is nonzero.

An abelian category satisfying (1) and (2) is called a *Grothendieck category*, and a $\mathcal{U}$ as in (2) is called a *generator*. By the hypothesis $C \times X$ is noetherian, and therefore $\mathbf{QCoh}(C \times X)$ is a Grothendieck category [**45**, Tag 077P]. But then $\mathbf{QCoh}_\tau(X, A)$ inherits property (1) directly from $\mathbf{QCoh}(C \times X)$, while $\Theta(\mathcal{U})$ for a generator $\mathcal{U}$ of $\mathbf{QCoh}(C \times X)$ will be a generator of $\mathbf{QCoh}_\tau(X, A)$. □

Let X be a scheme over $\mathbf{F}_q$, and let $\mathcal{F}$ be a quasi-coherent τ-sheaf with A-coefficients on X. Let $\mathbf{1}_{X,A}$ be the unit τ-sheaf

$$\mathbf{1}_{X,A} = (\mathcal{O}_{C\times X}, \mathrm{id})\,.$$

Since $\mathbf{QCoh}_\tau(X, A)$ has enough injectives, we may consider the derived functors

$$\mathrm{Ext}^i(\mathbf{1}, -)\colon\ \mathbf{QCoh}_\tau(X, A) \to \mathbf{Mod}\, A$$

of the functor $\mathrm{Hom}(\mathbf{1}, -)$. To say more about these extension groups, we will need additional hypotheses on X and A.

Proposition 8.9. *If X is smooth over $\mathbf{F}_q$ then the functor $\mathcal{F} \mapsto \sigma^\star\mathcal{F}$ on quasi-coherent $\mathcal{O}_X$-modules is exact.*

Proof. Let R be a regular local ring containing $\mathbf{F}_q$, of finite type over $\mathbf{F}_q$, with maximal ideal $\mathfrak{m}$ and residue field k. Note that k is a finite extension of $\mathbf{F}_q$, and hence perfect. By the regularity we can find a collection of generators $a_1, \ldots, a_d$ of $\mathfrak{m}$ that are k-linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. Let M be the R-module whose underlying additive group is R, and on which R acts via the map $R \to R$, $s \mapsto s^q$. Note that M is generated by the q^d elements

$$\big(a_1^{e_1} \cdots a_d^{e_d}\big)_{0 \le e_i < q}$$

as an R-module. We claim that M is in fact free of rank q^d over R. Indeed, it suffices to show that the elements $a_1^{e_1} \cdots a_d^{e_d}$ are k-linearly independent in $M/\mathfrak{m}M = R/\mathfrak{m}^q R$. This follows at once by the regularity of R, which implies that the natural k-linear maps

$$\mathrm{Sym}_k^n\,(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

are isomorphisms.

Now let X be a smooth scheme over $\mathbf{F}_q$. Then by the above the $\mathcal{O}_X$-module $\mathcal{M} := \mathcal{O}_X$ with action via the q-th power map is locally free, and hence flat. Since the functor $\sigma^\star$ coincides with the functor $\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}$, we conclude that it is exact. □

REMARK 8.10. More generally, Kunz [**37**] has shown that for a noetherian scheme over $\mathbf{F}_q$ the functor $\sigma^\star$ is exact if and only if X is regular.

COROLLARY 8.11. *Let X be a smooth scheme over $\mathbf{F}_q$, let A be an $\mathbf{F}_q$-algebra. Then the functor $\Theta\colon \mathbf{QCoh}(C \times X) \to \mathbf{QCoh}_\tau(X, A)$ is exact.* □

COROLLARY 8.12. *Let X be a smooth scheme over $\mathbf{F}_q$, let A be an $\mathbf{F}_q$-algebra. If $(\mathcal{F}, \tau)$ is injective in $\mathbf{QCoh}_\tau(X, A)$ then $\mathcal{F}$ is injective in the category of quasi-coherent $\mathcal{O}_{C\times X}$-modules.*

PROOF. By the assumptions Θ is exact, so that the forgetful functor has an exact left adjoint. This implies that the forgetful functor maps injectives to injectives, see [**45**, Tag 015Y]. □

THEOREM 8.13. *Assume that X is a smooth scheme over $\mathbf{F}_q$. Let $(\mathcal{F}, \tau)$ be an object of $\mathbf{QCoh}_\tau(X, A)$. Then there is a long exact sequence of A-modules*

$$\begin{aligned} 0 \quad \longrightarrow \quad & \mathrm{Ext}^0(\mathbf{1}, (\mathcal{F}, \tau)) \longrightarrow \mathrm{H}^0(C \times X, \mathcal{F}) \xrightarrow{1-\tau_s} \mathrm{H}^0(C \times X, \mathcal{F}) \\ \longrightarrow \quad & \mathrm{Ext}^1(\mathbf{1}, (\mathcal{F}, \tau)) \longrightarrow \mathrm{H}^1(C \times X, \mathcal{F}) \xrightarrow{1-\tau_s} \mathrm{H}^1(C \times X, \mathcal{F}) \\ \longrightarrow \quad & \cdots \end{aligned}$$

functorial in $(\mathcal{F}, \tau)$.

PROOF. Consider the quasi-coherent τ-sheaf $\Theta(\mathcal{O}_X)$. By the adjunction of Lemma 8.7 we have

$$\mathrm{Hom}(\Theta(\mathcal{O}_X), (\mathcal{F}, \tau)) = \Gamma(C \times X,\, \mathcal{F})$$

and by Corollary 8.12 we even have

$$\mathrm{Ext}^i(\Theta(\mathcal{O}_X), (\mathcal{F}, \tau)) = \mathrm{H}^i(C \times X,\, \mathcal{F}),$$

where Ext^i denotes the extension group in the category $\mathbf{QCoh}_\tau(X, A)$.

The map

$$\delta\colon \Theta(\mathcal{O}_X) \to \Theta(\mathcal{O}_X),\ (f_0, f_1, \ldots) \mapsto (f_0, f_1 - f_0, f_2 - f_1, \ldots)$$

is a morphism of τ-sheaves on X and the sequence

$$0 \to \Theta(\mathcal{O}_X) \xrightarrow{\delta} \Theta(\mathcal{O}_X) \to \mathbf{1} \to 0, \tag{36}$$

with the map $\Theta(\mathcal{O}_X) \to \mathbf{1}$ given by $(f_0, f_1, \ldots) \mapsto f_0 + f_1 + \cdots$, is an exact sequence of quasi-coherent τ-sheaves.

Applying $\mathrm{Hom}(-, (\mathcal{F}, \tau))$ to the exact sequence (36) yields the long exact sequence of the theorem. □

COROLLARY 8.14. *Let X be proper smooth over $\mathbf{F}_q$. Let A be of finite type over $\mathbf{F}_q$. If $\mathcal{F}$ is a coherent τ-sheaf with coefficients in A then the A-modules $\mathrm{Ext}^i(\mathbf{1}, \mathcal{F})$ are finitely generated.* □

3. Determinants and Fitting ideals

Let A be a Dedekind domain. If M is a finitely generated torsion A-module then there exists ideals $I_i \subset A$ so that $M \cong \oplus_i A/I_i$. The *Fitting ideal* of M is the ideal $[M] = \prod_i I_i \subset A$. It is independent of the choice of decomposition.

Alternatively, we may write $[M] = \prod \mathfrak{p}^{e(\mathfrak{p})}$ where $e(\mathfrak{p})$ is the length of the $A_\mathfrak{p}$-module $A_\mathfrak{p} \otimes_A M$.

Note that there is a slight conflict of notation. What was denoted by $[M]$ in section §1 (for $A = \mathbf{F}_q[t]$) is the unique monic generator of the ideal that we are now denoting by the same symbol $[M]$. Of course, for a general Dedekind domain, the ideal $[M]$ need not be principal.

LEMMA 8.15. *Let $0 \to M_1 \to \cdots \to M_d \to 0$ be an exact sequence of finitely generated torsion A-modules. Then*

$$[M_1][M_3][M_5]\cdots = [M_2][M_4][M_6]\cdots$$

as ideals in A. □

LEMMA 8.16. *Let n be a non-negative integer. Let R be a discrete valuation ring with fraction field L. Let H be a free R-module of rank n. Let*

$$0 \to H \xrightarrow{u} H \to T \to 0$$

be a short exact sequence of R-modules with T a torsion module. Then the length of T is the valuation of $\det_F(\,\mathrm{id} \otimes U \mid F \otimes_R H\,)$. □

LEMMA 8.17. *Let H be a locally free finitely generated A-module. Let u be an endomorphism of H such that $\mathrm{id} \otimes u$ is an automorphism of $F \otimes_A H$. Then $[\mathrm{coker}\, u]$ is a principal ideal, generated by $\det_A(\mathrm{id} \otimes u \mid F \otimes_A H)$.*

PROOF. It suffices to verify that for all primes $\mathfrak{p}$ the localization $[\mathrm{coker}\, u]_{(\mathfrak{p})}$ is generated by $\det_A(\mathrm{id} \otimes u \mid F \otimes_A H)$, which is the statement of the preceding lemma. □

LEMMA 8.18. *Let H be a finitely generated A-module. Let u be an endomorphism of H. Assume that the eigenvalue 1 of u acting on $F \otimes_A H$ is semi-simple. Then we have*

$$\det_F\big(1 - uT \mid F \otimes_A H\big) = \lambda(1-T)^r + \textit{higher order terms}$$

where

$$r = \mathrm{rk}_A \ker(1-u) = \mathrm{rk}_A \operatorname{coker}(1-u)$$

and

$$\lambda A = [\operatorname{coker}(1-u)_{\mathrm{tors}}]/[\ker(1-u)_{\mathrm{tors}}]$$

as A-submodules of F.

The semi-simplicity assumption means that we have a decomposition

$$F \otimes_A H = V \oplus W$$

of $F[u]$-modules such that $u = 1$ on V and $1 - u$ is an automorphism of W.

PROOF OF LEMMA 8.18. Let $V \subset H$ be the submodule of u-invariants. Then we have an induced exact sequence

$$0 \to K/V \to H/V \xrightarrow{1-u} H/V \to Q/V \to 0.$$

with K/V and Q/V torsion, and

$$\det_A(1 - uT \mid H) = (1-uT)^r \det_A(1 - uT \mid H/V).$$

We can therefore replace H by H/V and reduce to the case where $r = 0$. We then have that $1-u$ is an automorphism of $F \otimes_A V$ and that K and Q are torsion modules. We need to show that

$$\det_F\big(1 - u \mid F \otimes_A H\big) = \lambda$$

with $\lambda A = [Q]/[K]$.

Let H_{tors} be the torsion submodule of H. Let K' and Q' be the kernel and cokernel of $1-u$ on H_{tors}, so that we have an exact sequence

$$0 \to K' \to H_{\mathrm{tors}} \xrightarrow{1-u} H_{\mathrm{tors}} \to Q' \to 0$$

of torsion A-modules. We have $K' = K$, and by Lemma 8.15 we also have $[K'] = [Q']$, so that $[Q'] = [K]$. The cokernel Q' sits inside Q and we have an exact sequence

$$0 \to H/H_{\mathrm{tors}} \xrightarrow{1-u} H/H_{\mathrm{tors}} \to Q/Q' \to 0.$$

By Lemma 8.17 we conclude

$$\det_F\big(\,1-u \mid F\otimes_A H\,\big)A = [Q]/[Q'] = [Q]/[K]$$

as A-submodules of F. □

4. Special values of L-functions of τ-sheaves

Let $C = \operatorname{Spec} A$ be an integral curve over $\mathbf{F}_q$, with function field F. In particular A is a Dedekind domain. Let X be a proper smooth scheme over $\operatorname{Spec}\mathbf{F}_q$ and $\mathcal{F}$ be a coherent τ-sheaf on X with coefficients in A. As in the previous chapter, we assume

(H2) *there is an $\mathcal{O}_X$-module $\mathcal{F}_0$ so that $\mathcal{F} \cong \operatorname{pr}_X^\star \mathcal{F}_0$ as $\mathcal{O}_{C\times X}$-modules.*

A triple $(A, X, \mathcal{F})$ of particular interest is the triple $(\mathbf{F}_q[t], \overline{X}, \bar\pi^\star\mathcal{F}_{n,d})$ of Proposition 8.5. In this section we will express the order of vanishing and leading coefficient of $L(X, \mathcal{F}, T)$ at $T = 1$ in terms of the extension groups $\operatorname{Ext}^i(\mathbf{1}, \mathcal{F})$.

Under our assumptions the $\mathrm{H}^i(C\times X, \mathcal{F})$ are free A-modules of finite rank on which τ_s acts linearly. We make the following additional hypothesis

(SS) *the eigenvalue 1 of τ_s acting on $F\otimes_A \mathrm{H}^\bullet(C\times X, \mathcal{F})$ is semi-simple.*

The $\operatorname{Ext}^i(\mathbf{1}, \mathcal{F})$ are finitely generated modules over the Dedekind domain A, so we can talk about their rank $\operatorname{rk}_A \operatorname{Ext}^i(\mathbf{1}, \mathcal{F})$ and about the Fitting ideal of their torsion submodule $\big[\operatorname{Ext}^i(\mathbf{1}, \mathcal{F})_{\mathrm{tors}}\big]$.

THEOREM 8.19. *Let $C = \operatorname{Spec} A$ be an integral curve over $\mathbf{F}_q$, with function field F. Let X be a proper smooth scheme over $\operatorname{Spec}\mathbf{F}_q$ and $\mathcal{F}$ be a coherent τ-sheaf on X with coefficients in A. Assume* (H2) *and* (SS). *Then we have*

$$L(X, \mathcal{F}, T) = \lambda\cdot(T-1)^r + \textit{higher order terms}$$

where

$$r = \sum_{i\geq 0}(-1)^i i \operatorname{rk}_A \operatorname{Ext}^i(\mathbf{1}, \mathcal{F}),$$

and $\lambda \in F^\times$ satisfies

$$\lambda A = \prod_{i\geq 0}\big[\operatorname{Ext}^i(\mathbf{1}, \mathcal{F})_{\mathrm{tors}}\big]^{(-1)^i}$$

as A-modules in F.

Proof of Theorem 8.19. We split the long exact sequence of Theorem 8.13 in exact sequences

$$0 \to K^i \to \mathrm{H}^i(C \times X, \mathcal{F}) \xrightarrow{1-u} \mathrm{H}^i(C \times X, \mathcal{F}) \to Q^i \to 0$$

and

$$0 \to Q^{i-1} \to \mathrm{Ext}^i(\mathbf{1}, \mathcal{F}) \to K^i \to 0.$$

indexed by i. By Theorem 7.5 we have

$$L(X, \mathcal{F}, T) = \prod_i \det_F \big(1 - \tau_s T \mid F \otimes_A \mathrm{H}^i(C \times X, \mathcal{F}) \big)^{(-1)^{i+1}} \tag{37}$$

in $F(T)$ and by Lemma 8.18 we have

$$\det_F \big(1 - \tau_s T \mid F \otimes_A \mathrm{H}^i(C \times X, \mathcal{F}) \big) = \lambda_i (1-T)^{r_i} + \text{higher order terms}$$

with $r_i = \mathrm{rk}\, K^i = \mathrm{rk}\, Q^i$ and $\lambda_i A = [Q^i_{\mathrm{tors}}]/[K^i_{\mathrm{tors}}]$. Multiplying out the factors in (37) we find

$$\begin{aligned} \lambda A = \prod_i \lambda_i^{(-1)^{i+1}} A &= \prod_i \big[Q^i_{\mathrm{tors}}\big]^{(-1)^{i+1}} \big[K^i_{\mathrm{tors}}\big]^{(-1)^i} \\ &= \prod_i \big(\big[Q^{i-1}_{\mathrm{tors}}\big] \big[K^i_{\mathrm{tors}}\big] \big)^{(-1)^i} \\ &= \prod_i \big[\mathrm{Ext}^i(\mathbf{1}, \mathcal{F})_{\mathrm{tors}}\big]^{(-1)^i} \end{aligned}$$

for the leading coefficient and

$$\begin{aligned} r = \sum_i (-1)^{i+1} r_i &= \sum_i (-1)^{i+1} \big((i+1)\, \mathrm{rk}_A\, K^i - i\, \mathrm{rk}_A\, Q^i \big) \\ &= \sum_i (-1)^i i\, \mathrm{rk}_A\, K^i + i\, \mathrm{rk}_A\, Q^i \\ &= \sum_i (-1)^i i\, \mathrm{rk}_A\, \mathrm{Ext}^i(\mathbf{1}, \mathcal{F}) \end{aligned}$$

for the order of vanishing. □

5. An example failing the semi-simplicity hypothesis

We end with an example of a Goss zeta value where the hypothesis (SS) is not satisfied, and where the conclusion of Theorem 8.19 does not hold. It is based on the observation that the $\mathbf{F}_2$-linear map $\mathbf{F}_4 \to \mathbf{F}_4$, $x \mapsto x^2$ has two eigenvalues 1, but is not semi-simple.

Let $A = \mathbf{F}_2[t]$, and $C = \operatorname{Spec} A$ and denote the quadratic cover $\operatorname{Spec} \mathbf{F}_4[t]$ by X. We will consider the Goss zeta function $\zeta_G(X/C, -1, T)$ and its behavior at $T = 1$.

Consider the sheaf $\pi^\star \mathcal{F}_{1,2}$ on $\overline{X} = \mathbf{P}^1_{\mathbf{F}_4}$. By Corollary 8.4 we have

$$\zeta_G(X/C, -1, T) = L(\mathbf{P}^1_{\mathbf{F}_4}, \pi^\star \mathcal{F}_{1,2}, T).$$

By construction we have an isomorphism

$$\pi_\star \pi^\star \mathcal{F}_{1,2} = \mathbf{F}_4 \otimes_{\mathbf{F}_2} \mathcal{F}_{1,2}.$$

An easy computation shows that $\mathrm{H}^0(\mathbf{P}^1_{\mathbf{F}_2[t]}, \mathcal{F}_{1,2}) = 0$ and that $\mathrm{H}^1(\mathbf{P}^1_{\mathbf{F}_2[t]}, \mathcal{F}_{1,2})$ is free of rank 1 over $\mathbf{F}_2[t]$, on which τ_s acts as the identity. It follows that the action of τ_s on the rank 2 module $\mathrm{H}^1(\mathbf{P}^1_{\mathbf{F}_4[t]}, \pi^\star \mathcal{F}_{1,2}) = \mathbf{F}_4 \otimes_{\mathbf{F}_2} \mathrm{H}^1(\mathbf{P}^1_{\mathbf{F}_2[t]}, \mathcal{F}_{1,2})$ is given by a matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and hence $\pi^\star \mathcal{F}_{1,2}$ does not satisfy (SS). It follows from this computation that

$$\zeta_G(X/C, -1, T) = 1 + T^2$$

in $1 + TA[[T]]$, so that the order of vanishing at $T = 1$ is 2. On the other hand, using the long exact sequence of Theorem 8.13 we see that $\operatorname{Ext}^i(\mathbf{1}_{\overline{X},A}, \pi^\star \mathcal{F}_{1,2})$ has rank 0, 1, 1 for $i = 0, 1, 2$ respectively, so that

$$\sum_i (-1)^i i \operatorname{rk}_A \operatorname{Ext}^i(\mathbf{1}_{\overline{X},A}, \pi^\star \mathcal{F}_{1,2}) = 1$$

and the conclusion of Theorem 8.19 fails to hold.

I do not know what to conjecture or expect for the order of vanishing and leading coefficient of the L-function of a τ-sheaf that does not satisfy (SS).

Notes

The main result of this chapter, Theorem 8.19, is essentially due to V. Lafforgue [**38**, §2]. Compared to [**38**], we have provided more detailed arguments, and have generalized the result to higher-dimensional X.

It is a positive-characteristic analogue of results of Kato [**33**] and Milne and Ramachandran [**42**]. In fact, by analogy with Lichtenbaum's conjectures [**40**], one could compare the groups $\operatorname{Ext}^1(\mathbf{1}_{\overline{X}}, \pi^\star \mathcal{F}_{n,d})$ and

$\mathrm{Ext}^2(\mathbf{1}_{\overline{X}}, \pi^\star\mathcal{F}_{n,d})$ for a finite $X \to C$ with the Quillen K-groups K_{2n-1} and K_{2n-2} of a number field, respectively.

The hypothesis that X be smooth and our hypothesis (H2) simplify the arguments and statement, but are not essential. The semi-simplicity hypothesis however, is crucial. It is trivially satisfied if the order of vanishing of the L-fuction is at most one, which covers $\zeta_G(C/C, -n, T)$ for $C = \mathrm{Spec}\,\mathbf{F}_q[t]$ and many other basic cases, see [**48**, §3].

At *positive* integers, the Goss zeta function is no longer a rational function. Yet, if v is a place of the function field F, then seen as a power series over the completion F_v, it has an analytic continuation, and its value at $T = 1$ is well-defined. These values are typically transcendental [**53**]. They are related to periods of (generalizations of) Drinfeld modules and shtukas, see [**4, 38, 46, 21**]. The most powerful results in this direction are again applications of the Woods Hole trace formula.

Exercises

EXERCISE 8.1. Let X be a scheme of finite type over $\mathbf{F}_q$. Let $\mathcal{F}$ be a coherent τ-sheaf on X, let n be an integer. Show that there is a natural isomorhpism

$$\mathrm{Ext}^n_{\mathbf{Coh}_\tau X}(\mathbf{1}, \mathcal{F}) = \mathrm{Ext}^n_{\mathbf{Crys}\, X}(\mathbf{1}, \mathcal{F})$$

of Yoneda extension groups.

EXERCISE 8.2 ("Beilinson's basic lemma"). Let X be of finite type over $\mathbf{F}_q$. Let $\mathcal{F}$ be a crystal on X. Show that there is a non-empty affine subscheme $j\colon U \to X$ such that $\mathrm{Ext}^n(\mathbf{1}, j_!j^\star\mathcal{F}) = 0$ for all $n \neq 1$.

EXERCISE 8.3. Let $C = \mathrm{Spec}\,\mathbf{F}_q[t]$. Let n be a positive integer. Show that $\zeta_G(C/C, -n, T)$ vanishes at $T = 1$ if and only if n is divisible by $q - 1$. (Compare with the Riemann zeta function at negative integers.)

APPENDIX A

The trace formula for a transversal endomorphism

In this appendix we give a proof of the Woods Hole trace formula for a transversal endomorphism f of a proper smooth scheme X over a field, see Theorem A.4. The only published proof of this theorem is by Illusie in SGA5 [**27**, Exp. III, 6.12], using Grothendieck-Serre duality. However, the proof is rather convoluted. We also use Grothendieck-Serre duality, but deduce the formula in a more direct way.

It should be noted that special cases of the trace formula admit simpler proofs. This is the case if the endomorphism is the Frobenius endomorphism (see these notes, or [**22**], or [**3**]), or if it is an automorphism of finite order co-prime to the characteristic (see [**9**] or [**18**]). In these cases the Grothendieck group of pairs $(\mathcal{F}, \varphi)$ of a coherent $\mathcal{O}_X$-module with a map $f^\star\mathcal{F} \to \mathcal{F}$ becomes a manageable object, which allows a reduction to trivial instances of the formula.

1. Extensions

1.1. Extension groups and cup product. Let X be a noetherian scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Then we have higher extension groups $\mathrm{Ext}^p_X(\mathcal{F}, \mathcal{G})$, which can be defined either as the derived functors of either $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)$ or $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{G})$. Equivalently, they can be defined as Yoneda extension groups in either the category of $\mathcal{O}_X$-modules or the category of quasi-coherent $\mathcal{O}_X$-modules. The groups $\mathrm{Ext}^p_X(\mathcal{F}, \mathcal{G})$ are modules over the ring $\Gamma(X, \mathcal{O}_X)$.

If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are quasi-coherent $\mathcal{O}_X$-modules then we have cup product pairings

$$\mathrm{Ext}^p_X(\mathcal{F}, \mathcal{G}) \times \mathrm{Ext}^q_X(\mathcal{G}, \mathcal{H}) \longrightarrow \mathrm{Ext}^{p+q}_X(\mathcal{F}, \mathcal{H})$$

which are bilinear over $\Gamma(X, \mathcal{O}_X)$.

1.2. Base-change of extensions. Let X be a noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be coherent $\mathcal{O}_X$-modules. Assume that $\mathcal{F}$ is acyclic for $-\otimes\mathcal{H}$. For example, this is the case if $\mathcal{H}$ is locally free. Then we have for every p a natural map

$$\operatorname{Ext}^p_X(\mathcal{F},\,\mathcal{G}) \longrightarrow \operatorname{Ext}^p_X(\mathcal{F}\otimes\mathcal{H},\,\mathcal{G}\otimes\mathcal{H}). \tag{38}$$

Similarly, let $f\colon X\to Y$ be a morphism of noetherian schemes, let $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_Y$-modules and assume that $\mathcal{F}$ is acyclic for $f^\star$. Then for every p there is a natural map

$$\operatorname{Ext}^p_Y(\mathcal{F},\,\mathcal{G}) \longrightarrow \operatorname{Ext}^p_X(f^\star\mathcal{F},\,f^\star\mathcal{G}). \tag{39}$$

In the special case $\mathcal{F}=\mathcal{O}_X$ we obtain a map

$$\mathrm{H}^p(Y,\,\mathcal{G}) \longrightarrow \mathrm{H}^p(X,\,f^\star\mathcal{G})$$

which is simply the pull-back map on cohomology.

Pull-back of extensions is compatible with product of extensions: if $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are coherent $\mathcal{O}_Y$-modules and if $\mathcal{F}$ and $\mathcal{G}$ are $f^\star$-acyclic, then for every p and q the diagram

$$\begin{array}{ccc}
\operatorname{Ext}^p_Y(\mathcal{F},\mathcal{G})\otimes\operatorname{Ext}^q_Y(\mathcal{G},\mathcal{H}) & \longrightarrow & \operatorname{Ext}^{p+q}_Y(\mathcal{F},\mathcal{H}) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^p_X(f^\star\mathcal{F},f^\star\mathcal{G})\otimes\operatorname{Ext}^q_X(f^\star\mathcal{G},f^\star\mathcal{H}) & \longrightarrow & \operatorname{Ext}^{p+q}_X(f^\star\mathcal{F},f^\star\mathcal{H})
\end{array}$$

commutes.

1.3. Künneth formula. Let k be a field and let X and Y be noetherian schemes over $\operatorname{Spec} k$. Consider the product $X\times Y$ over k. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent $\mathcal{O}_X$-modules and $\mathcal{G}_1$ and $\mathcal{G}_2$ coherent $\mathcal{O}_Y$-modules. Then for each i the $\mathcal{O}_{X\times Y}$-module

$$\mathcal{F}_i\boxtimes\mathcal{G}_i := \pi_X^\star\mathcal{F}_i\otimes\pi_Y^\star\mathcal{G}_i$$

is coherent. Combining the natural maps of (38) and (39) we obtain maps

$$\operatorname{Ext}^p_X(\mathcal{F}_1,\mathcal{F}_2) \longrightarrow \operatorname{Ext}^p_{X\times Y}(\mathcal{F}_1\boxtimes\mathcal{G}_1,\mathcal{F}_2\boxtimes\mathcal{G}_1)$$

and

$$\operatorname{Ext}^q_Y(\mathcal{G}_1,\mathcal{G}_2) \longrightarrow \operatorname{Ext}^q_{X\times Y}(\mathcal{F}_2\boxtimes\mathcal{G}_1,\mathcal{F}_2\boxtimes\mathcal{G}_2).$$

The images can be cupped together into a map

$$\operatorname{Ext}^p_X(\mathcal{F}_1,\mathcal{F}_2)\otimes_k\operatorname{Ext}^q_Y(\mathcal{G}_1,\mathcal{G}_2) \longrightarrow \operatorname{Ext}^{p+q}_{X\times Y}(\mathcal{F}_1\boxtimes\mathcal{G}_1,\,\mathcal{F}_2\boxtimes\mathcal{G}_2).$$

The *Künneth formula* is a theorem stating that for every n the map

$$\bigoplus_{p+q=n} \mathrm{Ext}^p_X(\mathcal{F}_1, \mathcal{F}_2) \otimes_k \mathrm{Ext}^q_Y(\mathcal{G}_1, \mathcal{G}_2) \xrightarrow{\sim} \mathrm{Ext}^n_{X\times Y}(\mathcal{F}_1 \boxtimes \mathcal{G}_1,\, \mathcal{F}_2 \boxtimes \mathcal{G}_2) \tag{40}$$

is an isomorphism.

2. Grothendieck-Serre duality

Grothendieck-Serre duality is a generalization of Serre duality to arbitrary proper maps of noetherian schemes. Serre duality covers the case of a proper smooth morphism to the spectrum of a field. The statement is most natural (and certainly most economical) in the language of derived categories. We give the statement below, in section 2.1. In the subsequent sections we explicitly spell out a few consequences of Grothendieck-Serre duality that will be used in the proof of the Woods Hole trace formula. These consequences are given in the language of extension groups and do not refer to the derived category.

We refer to [**30**] for more details.

2.1. Duality in the derived category. For a scheme X we denote by $\mathrm{D}^b(X)$ the bounded derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. Let X and Y be noetherian schemes. Let $f\colon X \to Y$ be a proper morphism. Then we have a total derived push-forward functor

$$\mathrm{R}f_\star \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y).$$

It maps complexes with coherent cohomology to complexes with coherent cohomology. *Grothendieck-Serre duality* states that there is a functor

$$f^! \colon \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$$

and for every $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$ and $\mathcal{G}^\bullet \in \mathrm{D}^b(Y)$ an isomorphism

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathrm{R}f_\star\mathcal{F}^\bullet,\, \mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}^b(Y)}(\mathcal{F}^\bullet,\, f^!\mathcal{G}^\bullet) \tag{41}$$

functorial in $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. If $f\colon X \to Y$ and $g\colon Y \to Z$ are proper morphisms of noetherian schemes then we have natural isomorphisms $\mathrm{R}g_\star \mathrm{R}f_\star \cong \mathrm{R}(gf)_\star$ and $(gf)^! \cong f^! g^!$, compatible with (41).

For some classes of f the functor $f^!$ can be explicitly computed. In particular, if $f\colon X \to Y$ is proper smooth of relative dimension n then the functor $f^!$ is given by

$$f^!\mathcal{G}^\bullet = (\Omega^n_{X/Y} \otimes \mathrm{L}f^\star\mathcal{G})[n]$$

and if $i\colon X \to Y$ is a regular closed immersion of codimension d then

$$i^!\mathcal{G}^\bullet = (\mathcal{N}^d_{X/Y} \otimes \mathrm{L}i^\star\mathcal{G})[-d]$$

where $\mathcal{N}^d_{X/Y} := \wedge^d\mathcal{N}_{X/Y}$ is the determinant of the normal bundle, see 2.5 below for more details.

2.2. Duality for proper smooth variety. Let X be proper and smooth of dimension n over $\operatorname{Spec} k$. Then Serre duality provides a canonical map

$$\mathrm{tr}_X\colon \mathrm{H}^n(X, \Omega^n_{X/k}) \to k.$$

For every coherent $\mathcal{O}_X$-module $\mathcal{F}$ and for every p the resulting map

$$\mathrm{H}^p(X, \mathcal{F}) \otimes \mathrm{Ext}^{d-p}_X(\mathcal{F}, \Omega^n_{X/k}) \longrightarrow \mathrm{H}^n(X, \Omega^n_{X/k}) \xrightarrow{\mathrm{tr}_X} k$$

is a perfect pairing. This follows from Grothendieck-Serre duality, together with the explicit computation of $f^!\mathcal{O}_Y$ for the map $f\colon X \to Y = \operatorname{Spec} k$.

2.3. Duality for proper smooth map. Let X and Y be noetherian schemes and let $f\colon X \to Y$ be proper and smooth of relative dimension n. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is *acyclic for* $f_\star$, and let $\mathcal{G}$ be a locally free $\mathcal{O}_Y$-module. Then Grothendieck-Serre duality provides for every p a canonical isomorphism

$$\mathrm{Ext}^{n+p}_X(\mathcal{F},\, f^\star\mathcal{G} \otimes \Omega^n_{X/Y}) \xrightarrow{\sim} \mathrm{Ext}^p_Y(f_\star\mathcal{F},\, \mathcal{G}). \tag{42}$$

If $Y = \operatorname{Spec} k$ then these groups vanish unless $p = 0$. For $p = 0$ and $\mathcal{G} = \mathcal{O}_Y$ we find an isomorphism

$$\mathrm{Ext}^n_X(\mathcal{F},\, \Omega^n_{X/k}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{H}^0(X, \mathcal{F}),\, k)$$

which coincides with the isomorphism of Serre duality for X/k.

2.4. Künneth and duality for a coordinate projection. Let k be a field. Let X and Y be proper smooth of relative dimension n and m respectively. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module and $\mathcal{G}$ a coherent $\mathcal{O}_X$-module. We have

$$\pi_X^\star\mathcal{G} \otimes \Omega^m_{X\times Y/X} = \mathcal{G} \boxtimes \Omega^m_Y.$$

Duality for π_X gives, as a special case, an isomorphism

$$\mathrm{Ext}^m_{X\times Y}(\mathcal{O}_X \boxtimes \mathcal{F},\, \mathcal{G} \boxtimes \Omega^m_Y) \xrightarrow{\sim} \oplus_p \mathrm{Ext}^p_X(\mathcal{O}_X \otimes_k \mathrm{H}^p(Y,\mathcal{F}),\, \mathcal{G}).$$

This isomorphism is compatible with duality for Y in a sense that we now make precise. Let p be an integer and consider the square
(43)

$$\begin{array}{ccc} \mathrm{Ext}^m_{X\times Y}(\mathcal{O}_X \boxtimes \mathcal{F},\, \mathcal{G} \boxtimes \Omega^m_Y) & \longleftarrow & \mathrm{Ext}^p_X(\mathcal{O}_X, \mathcal{G}) \otimes \mathrm{Ext}^{n-p}_Y(\mathcal{F}, \Omega^m_Y) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^p_X(\mathcal{O}_X \otimes_k \mathrm{H}^p(Y,\mathcal{F}),\, \mathcal{G}) & \longleftarrow & \mathrm{Hom}_k(\mathrm{H}^p(Y,\mathcal{F}),\, \mathrm{H}^p(X,\mathcal{G})). \end{array}$$

Here the top map is the Künneth map, the left map is duality for π_X and the right map is duality for Y. Then this square *commutes* for p even and *anticommutes* for p odd.

2.5. Duality for regular closed immersion. Let X be a noetherian scheme. A closed immersion $i\colon Z \to X$ is said to be *regular of codimension* d if the ideal sheaf $\mathcal{I}$ is locally generated by a regular sequence of length d. For example, if X and Z are smooth of relative dimension n resp. $n-d$ over a field k then any closed immersion $i\colon Z \to X$ is regular of codimension d.

Let $i\colon Z \to X$ be a regular closed immersion of codimension d. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Then the conormal sheaf $\mathcal{C}_{Z/X} := i^\star\mathcal{I}/\mathcal{I}^2$ and the normal sheaf $\mathcal{N}_{Z/X} := \mathcal{C}^\vee_{Z/X}$ are locally free $\mathcal{O}_Z$-modules of rank d. We denote their determinants by $\mathcal{C}^d_{Z/X}$ and $\mathcal{N}^d_{Z/X}$. These are mutually inverse invertible $\mathcal{O}_Z$-modules.

We have $i^!\mathcal{G} = \mathrm{L}i^\star\mathcal{G} \otimes \mathcal{N}^d_{Z/X}[-d]$, so Grothendieck-Serre duality provides for every coherent $\mathcal{O}_Z$-module $\mathcal{F}$ and every coherent $\mathcal{O}_X$-module $\mathcal{G}$ which is acyclic for $i^\star$ a canonical isomorphism

$$\alpha_{Z/X}\colon\ \mathrm{Ext}^p_Z(\mathcal{F}, i^\star\mathcal{G} \otimes \mathcal{N}^d_{Z/X}) \xrightarrow{\sim} \mathrm{Ext}^{p+d}_X(i_\star\mathcal{F}, \mathcal{G}), \tag{44}$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

2.6. Compatibility. Now assume $f\colon X \to \operatorname{Spec} k$ and $g\colon Z \to \operatorname{Spec} k$ be proper smooth of relative dimension n and $n-d$ respectively.

Let $i\colon Z \to X$ be a closed immersion.

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle f} \\ & & \operatorname{Spec} k. \end{array}$$

We have $g^! = i^! f^!$, and a related compatibility between Serre duality on X and Z, and Grothendieck-Serre duality for i. An explicit instance of this compatibility is as follows. The short exact sequence

$$0 \to \mathcal{C}_{Z/X} \to i^\star \Omega^1_{X/k} \to \Omega^1_{Z/k} \to 0,$$

induces an isomorphism

$$i^\star \Omega^n_{X/k} \otimes \mathcal{N}^d_{Z/X} \cong \Omega^{n-d}_{Z/k},$$

and the square

$$\begin{array}{ccc} \operatorname{Ext}^{n-d}_Z(\mathcal{O}_Z, i^\star \Omega^n_{X/k} \otimes \mathcal{N}^d_{Z/X}) & \xrightarrow{\alpha_{Z/X}} & \operatorname{Ext}^n_X(i_\star \mathcal{O}_Z, \Omega^n_{X/k}) \\ \downarrow{\scriptstyle \operatorname{tr}_Z} & & \downarrow \\ k & \xleftarrow[\operatorname{tr}_X]{} & \operatorname{Ext}^n_X(\mathcal{O}_X, \Omega^n_{X/k}) \end{array} \tag{45}$$

commutes.

2.7. Transversal base change. Let $f\colon X' \to X$ be a morphism of noetherian schemes over $\operatorname{Spec} k$. Let $i\colon Z \to X$ be a regular closed immersion of codimension d. Let $i'\colon Z' \to X'$ be the base change of i along f, so that we have a cartesian square

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

Then $i'\colon Z' \to X'$ is a closed immersion, but it need not be regular of codimension d.

Assume now that $f\colon X' \to X$ is a regular closed immersion which is *transversal* to Z. Then $i'\colon Z' \to X'$ is a regular closed immersion of codimension d and the canonical map

$$f'^\star \mathcal{C}_{Z/X} \to \mathcal{C}_{Z'/X'}$$

is an isomorphism of locally free $\mathcal{O}_{Z'}$-modules of rank d.

Let $\mathcal{F}$ be a locally free $\mathcal{O}_Z$-module of finite rank, and let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is acyclic for $f'^\star$ and $i_\star\mathcal{F}$ is acyclic for $f^\star$. We obtain a square

$$\begin{array}{ccc} \operatorname{Ext}^p_Z(\mathcal{F}, i_\star\mathcal{G}\otimes\mathcal{N}^d_{Z/X}) & \xrightarrow{\alpha} & \operatorname{Ext}^{p+d}_X(i_\star\mathcal{F},\mathcal{G}) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^p_{Z'}(f'^\star\mathcal{F}, i'^\star f^\star\mathcal{G}\otimes\mathcal{N}^d_{Z'/X'}) & \xrightarrow{\alpha} & \operatorname{Ext}^{p+d}_{X'}(i'_\star f'^\star\mathcal{F}, f^\star\mathcal{G}). \end{array} \tag{46}$$

where the vertical maps are the base change maps as in (39). This square commutes.

3. A local computation

Let X be separated and smooth of relative dimension d over a field k. Let $f\colon X\to X$ be a morphism over k.

Let $\Gamma = (\mathrm{id}, f)\colon X\to X\times X$ be the graph of f and $\Delta\colon X\to X\times X$ be the diagonal. These are closed immersions. Assume Γ and Δ intersect transversally in $X\times X$. Let C be their intersection, so that we have a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X\times X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle\Delta} \\ C & \xrightarrow{i} & X \end{array}$$

In particular $if = i$ and $\Gamma i = \Delta i$.

We have $\Omega^1_{X\times X} = \Omega^1_X\boxtimes\mathcal{O}_X\oplus\mathcal{O}_X\boxtimes\Omega^1_X$. Taking exterior powers gives

$$\Omega^d_{X\times X} = \oplus_p\,\Omega^p_X\boxtimes\Omega^{d-p}_X.$$

Consider the composition of natural maps

$$\mathcal{C}^d_\Gamma \xrightarrow{\mathrm{d}} \Gamma^\star\Omega^d_{X\times X} \twoheadrightarrow \Gamma^\star\left(\mathcal{O}_X\boxtimes\Omega^d_X\right) = f^\star\Omega^d_X.$$

Pulling back along i we obtain a map

$$\pi_0\colon i^\star\mathcal{C}^d_\Gamma \to i^\star f^\star\Omega^d_X = i^\star\Omega^d_X$$

of invertible $\mathcal{O}_C$-modules. (The π_0 is to indicate the map originates in the projection $\Omega^d_{X\times X} \twoheadrightarrow \mathcal{O}_X \boxtimes \Omega^d_X$ on the component for $p = 0$.) On the other hand, transversal base change gives a map

$$\Delta^\star \colon i^\star \mathcal{C}^d_\Gamma \to \mathcal{C}^d_i$$

Finally, the map f induces a map $\mathrm{d}f \colon i^\star\Omega^1_X \to i^\star\Omega^1_X$ and hence a map $\mathrm{d}f \colon i^\star\Omega^d_X \to i^\star\Omega^d_X$.

Proposition A.1. *The following diagram of invertible $\mathcal{O}_C$-modules*

$$\begin{array}{ccc} i^\star\mathcal{C}^d_\Gamma & \xrightarrow{\Delta^\star} & \mathcal{C}^d_i \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \mathrm{d}} \\ i^\star\Omega^d_X & \xrightarrow{\det(1-\mathrm{d}f)} & i^\star\Omega^d_X \end{array}$$

commutes.

Note that the four arrows are isomorphisms.

Proof. This is a local computation. Let $x \in C$ be a point. Let $t_1, \dots, t_d$ be a system of local parameters of X near x. Put $x_i := t_i \otimes 1$ and $y_i := 1 \otimes t_i$. Then

$$(x_1, \dots, x_d, y_1, \dots, y_d)$$

is a system of local parameters of $X \times X$ near (x, x). Near this point the closed subscheme Γ is cut out by the regular sequence $(y_1 - f(x_1), \dots, y_d - f(x_d))$, so that $i^\star\mathcal{C}^d_\Gamma$ is generated (near x) by

$$s := (y_1 - f(x_1)) \wedge \cdots \wedge (y_d - f(x_d)).$$

The image of s under π_0 is $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d$. The image of s under $\Delta^\star$ is

$$(t_1 - f(t_1)) \wedge \cdots \wedge (t_d - f(t_d))$$

which is mapped to the section $\det(1 - \mathrm{d}f) \cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d$ of $i^\star\Omega^d_X$. □

4. Duality for the graph of a morphism

Let X and Y be proper smooth of dimension n (resp. m) over k. Let $f \colon X \to Y$ be a morphism over k. Consider the graph of f, that is the map

$$\Gamma = (\mathrm{id}, f) \colon X \to X \times Y.$$

Note that Γ is a regular closed immersion of codimension m and that $\mathcal{C}_\Gamma = f^\star\Omega_Y$.

Let $\mathcal{F}$ be a an $\mathcal{O}_Y$-module and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume $\mathcal{F}$ and $\mathcal{G}$ are locally free of finite rank.

Since $\pi_X\Gamma = \mathrm{id}_X$ we have

$$\Gamma^\star(\mathcal{G} \boxtimes \Omega_Y^m) \otimes \mathcal{N}_\Gamma^m = \mathcal{G}$$

so duality for Γ provides an isomorphism

$$\mathrm{Hom}_X(f^\star\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Ext}^m_{X\times Y}(\Gamma_\star f^\star\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^m). \tag{47}$$

Composing with the canonical map $\mathcal{O}_X \boxtimes \mathcal{F} \to \Gamma_\star f^\star\mathcal{F}$ we obtain a map

$$\theta\colon\ \mathrm{Hom}_X(f^\star\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Ext}^m_{X\times Y}(\mathcal{O}_X\boxtimes\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^m).$$

The Künneth formula and Serre duality for Y give a decomposition

$$\mathrm{Ext}^m_{X\times Y}(\mathcal{O}_X\boxtimes\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^m) \xrightarrow{\sim} \oplus_p \mathrm{Hom}_k(\mathrm{H}^p(Y,\mathcal{F}),\, \mathrm{H}^p(X,\mathcal{G})).$$

PROPOSITION A.2. *Let $\varphi\colon f^\star\mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let p be a positive integer. Then the image of $\theta(\varphi)$ in the component*

$$\mathrm{Hom}_k(\mathrm{H}^p(Y,\mathcal{F}),\, \mathrm{H}^p(X,\mathcal{G}))$$

of $\mathrm{Ext}^m_{X\times Y}(\mathcal{O}_X\boxtimes\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^m)$ *is $(-1)^p$ times the map*

$$\mathrm{H}^p(Y,\mathcal{F}) \xrightarrow{f^\star} \mathrm{H}^p(X, f^\star\mathcal{F}) \xrightarrow{\varphi} \mathrm{H}^p(X,\mathcal{G})$$

coming from functoriality of cohomology.

PROOF. Let $\pi_X : X\times Y \to X$ be the projection. Note that $\pi_X\Gamma = \mathrm{id}_X$. In particular, $\Gamma_\star f^\star\mathcal{F}$ is $\pi_{X,\star}$-acyclic, and we have $\pi_{X,\star}\Gamma_\star f^\star\mathcal{F} = f^\star\mathcal{F}$. Therefore duality for the projection $X\times Y \to X$ gives an isomorphism

$$\mathrm{Ext}^m_{X\times Y}(\Gamma_\star f^\star\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^m) \xrightarrow{\sim} \mathrm{Hom}_X(f^\star\mathcal{F},\, \mathcal{G}).$$

This is the inverse of the isomorphism (47). It therefore suffices to show that the square

$$\begin{array}{ccc}
\mathrm{Ext}^m_{X\times Y}(\Gamma_\star f^\star\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^n) & \longrightarrow & \mathrm{Ext}^m_{X\times Y}(\mathcal{O}_X\boxtimes\mathcal{F},\, \mathcal{G}\boxtimes\Omega_Y^n) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(f^\star\mathcal{F},\, \mathcal{G}) & \longrightarrow & \mathrm{Hom}(\mathrm{H}^p(Y,\mathcal{F}), \mathrm{H}^p(X,\mathcal{G}))
\end{array}$$

commutes for even p and anticommutes for odd p. Here the top and bottom maps are the canonical ones. The left map is duality for π_X and the right map is Künneth combined with duality for Y/k. This then follows from functoriality of duality for π_X (applied to the map

$\mathcal{O}_X \boxtimes \mathcal{F} \to \Gamma_\star f^\star \mathcal{F})$, and the commutativity resp. anticommutativity of the square (43). □

COROLLARY A.3. *Assume moreover that* $X = Y$ *and* $\mathcal{F} = \mathcal{G}$. *Let* $\varphi\colon f^\star\mathcal{F} \to \mathcal{F}$ *be a morphism of* $\mathcal{O}_X$*-modules. Then* $\theta(\varphi)$ *is mapped under the composition*

$$\operatorname{Ext}^n_{X\times X}(\mathcal{O}_X \boxtimes \mathcal{F},\, \mathcal{F} \boxtimes \Omega^n_X) \xrightarrow{\Delta^\star} \operatorname{Ext}^n_X(\mathcal{F}, \mathcal{F} \otimes \Omega^n_X) \xrightarrow{\operatorname{tr}_X} k$$

to the alternating sum $\sum_p (-1)^p \operatorname{tr}_k(\varphi,\, \mathrm{H}^p(X,\mathcal{F}))$. □

5. Woods Hole trace formula

THEOREM A.4. *Let* X *be proper smooth over* $\operatorname{Spec} k$ *and let* f *be an endomorphism of* X *over* k. *Let* $\mathcal{F}$ *be a locally free* $\mathcal{O}_X$*-module of finite rank. Let* $\varphi\colon f^\star\mathcal{F} \to \mathcal{F}$ *be a map of* $\mathcal{O}_X$*-modules. Assume that the graph of* f *intersects the diagonal in* $X \times X$ *transversally. Then*

$$\sum_p (-1)^p \operatorname{tr}_k(\varphi,\, \mathrm{H}^p(X,\mathcal{F})) = \sum_{f(x)=x} \frac{\operatorname{tr}_k(\varphi,\, x^\star\mathcal{F})}{\det_k(1 - \mathrm{d}f,\, x^\star\Omega^1_{X/k})}$$

in k.

PROOF. As before, we denote by $\Gamma = (\mathrm{id}, f)\colon X \to X \times X$ the graph of f and by $\Delta\colon X \to X \times X$ the diagonal. We have a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & X \times X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \Delta} \\ C & \xrightarrow{i} & X \end{array}$$

Consider the locally free $\mathcal{O}_{X\times X}$-module $\mathcal{H} := \mathcal{F} \boxtimes \mathcal{F}^\vee$.

Step 1. The natural map $\mathrm{d}\colon \mathcal{C}_\Gamma \to \Gamma^\star\Omega^1_{X\times X}$ induces a map

$$\mathcal{C}^d_\Gamma \to \wedge^d \Gamma^\star \Omega^1_{X\times X} = \Gamma^\star\Omega^d_{X\times X} \tag{48}$$

and hence a section of $\Gamma^\star\Omega^d_{X\times X} \otimes \mathcal{N}^d_\Gamma$. Using Grothendieck-Serre duality (44) for Γ this induces a map

$$\mathrm{H}^0(X, \Gamma^\star\mathcal{H}) \longrightarrow \operatorname{Ext}^d_{X\times X}(\Gamma_\star\mathcal{O}_X,\, \Omega^d_{X\times X} \otimes \mathcal{H})$$

and composing with the canonical map $\mathcal{O}_{X\times X} \to \Gamma_\star\mathcal{O}_X$ we obtain a map

$$\gamma_1\colon \mathrm{H}^0(X, \Gamma^\star\mathcal{H}) \longrightarrow \mathrm{H}^d(X \times X,\, \Omega^d_{X\times X} \otimes \mathcal{H}).$$

By transversal base change we have a natural isomorphism $\mathcal{C}_i^d \cong \Delta^\star\mathcal{C}_\Gamma^d$, and hence pulling back the map in (48) along Δ we obtain a map

$$\mathcal{C}_i^d \longrightarrow i^\star\Gamma^\star\Omega^d_{X\times X} = i^\star\Delta^\star\Omega^d_{X\times X}.$$

This gives a section of $i_\star\mathcal{N}_i^d \otimes \Delta^\star\Omega^d_{X\times X}$ and hence, just as above, Grothendieck-Serre duality for i yields a map

$$\gamma_2\colon \mathrm{H}^0(C, i^\star\Gamma^\star\mathcal{H}) \to \mathrm{H}^0(X, \Delta^\star\Omega^d_{X\times X}\otimes\Delta^\star\mathcal{H}).$$

By transversal base change (46) the square

$$\begin{array}{ccc} \mathrm{H}^0(X,\Gamma^\star\mathcal{H}) & \xrightarrow{\Delta^\star} & \mathrm{H}^0(C, i^\star\Delta^\star\mathcal{H}) \\ \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} \\ \mathrm{H}^d(X\times X, \Omega^d_{X\times X}\otimes\mathcal{H}) & \xrightarrow{\Delta^\star} & \mathrm{H}^d(X, \Delta^\star\Omega^d_{X\times X}\otimes\Delta^\star\mathcal{H}) \end{array} \tag{49}$$

commutes.

Step 2. We have $\Delta^\star\mathcal{H} = \mathcal{F}\otimes\mathcal{F}^\vee$, so there is a canonical trace map

$$\mathrm{tr}_\mathcal{F}\colon \Delta^\star\mathcal{H} \to \mathcal{O}_X$$

which locally sends a section of $\mathcal{H}$ to the trace of the corresponding endomorphism of $\mathcal{F}$. We have a natural commutative square

$$\begin{array}{ccc} \mathrm{H}^0(C, i^\star\Delta^\star\mathcal{H}) & \xrightarrow{\mathrm{tr}_\mathcal{F}} & \mathrm{H}^0(C,\mathcal{O}_C) \\ \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} \\ \mathrm{H}^d(X, \Delta^\star\Omega^d_{X\times X}\otimes\Delta^\star\mathcal{H}) & \xrightarrow{\mathrm{tr}_\mathcal{F}} & \mathrm{H}^d(X, \Delta^\star\Omega^d_{X\times X}) \end{array} \tag{50}$$

where γ_3 is the analogue of the map γ_2 (for $\mathcal{F} = \mathcal{O}_X$).

Step 3. The natural map $\mathrm{d}\colon \mathcal{C}_i \to i^\star\Omega^1_X$ induces a section of $i^\star\Omega^d_X\otimes\mathcal{N}_i^d$ and therefore, with Grothendieck-Serre duality for i, a map

$$\gamma_4\colon \mathrm{H}^0(C,\mathcal{O}_C) \longrightarrow \mathrm{H}^d(X,\Omega^d_X).$$

Now by the local computation in Proposition A.1 also the diagram

$$\begin{array}{ccc} \mathrm{H}^0(C,\mathcal{O}_C) & \xrightarrow{\det(1-\mathrm{d}f)^{-1}} & \mathrm{H}^0(C,\mathcal{O}_C) \\ \downarrow{\scriptstyle\gamma_3} & & \downarrow{\scriptstyle\gamma_4} \\ \mathrm{H}^d(X,\Delta^\star\Omega^d_{X\times X}) & \xrightarrow{\pi_0} & \mathrm{H}^d(X,\Omega^d_X). \end{array} \tag{51}$$

commutes.

Step 4. Joining the commutative squares (49), (50) and (51) and using the compatibility of (45) we find a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^0(X, \Gamma^\star \mathcal{H}) & \longrightarrow & \mathrm{H}^0(C, \mathcal{O}_C) & & \\
\downarrow \gamma_1 & & \downarrow \gamma_4 & \overset{\mathrm{tr}_C}{\searrow} & k \\
 & & & \underset{\mathrm{tr}_X}{\nearrow} & \\
\mathrm{H}^d(X \times X, \Omega^d_{X \times X} \otimes \mathcal{H}) & \longrightarrow & \mathrm{H}^d(X, \Omega^d_X) & &
\end{array} \tag{52}
$$

We have $\Gamma^\star \mathcal{H} = \mathcal{F} \otimes f^\star \mathcal{F}^\vee$, so that φ induces a section $\varphi \in \mathrm{H}^0(X, \Gamma^\star \mathcal{H})$. We will compute its image $T(\varphi)$ in k under the commutative diagram (52) in two ways, obtaining the left-hand and right-hand sides of the trace formula.

Tracing the diagrams along the top maps and through tr_C we find

$$T(\varphi) = \sum_{x \in C} \frac{\mathrm{tr}_k(\varphi,\, x^\star \mathcal{F})}{\det_k(1 - \mathrm{d}f,\, x^\star \Omega^1_{X/k})},$$

which is the right-hand-side of the Woods Hole trace formula.

We now compute the image of φ via the bottom path. The bottom map factors over

$$\mathrm{H}^d(X \times X,\, (\mathcal{O}_X \boxtimes \Omega^d_X) \otimes \mathcal{H})$$

which using Künneth and Serre duality for X can be identified with

$$\oplus_p \operatorname{Hom}_k(\mathrm{H}^p(X, \mathcal{F}), \mathrm{H}^p(X, \mathcal{F})).$$

By Proposition A.2 and Corollary A.3 we find

$$T(\varphi) = \sum_p (-1)^p \,\mathrm{tr}_k(\varphi,\, \mathrm{H}^p(X, \mathcal{F}))$$

and equating the two obtained expressions for $T(\varphi)$ finishes the proof of the Woods Hole trace formula. □

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