Transformation in three/four-phase power system using GA

Johanka Brdečková

Brno University of Technology, Faculty of Mechanical Engineering

28.8.2024

Losses in three-phase power system

We are dealing with three-phase power system. The currents i_a, i_b, i_c at these phases can be represented as a vector $\boldsymbol{i} = i_a e_1 + i_b e_2 + i_c e_3$ or a point I. Similarly the voltage can be seen as a vector $\boldsymbol{v} = v_a e_1 + v_b e_2 + v_c e_3$ or a point V.

Real part of power is given by $p = \boldsymbol{v} \cdot \boldsymbol{i}$ and losses of such (4-wire) system depend on current $losses(\boldsymbol{i}) = \boldsymbol{i} \cdot \boldsymbol{i} + (i_a + i_b + i_c)^2$.

We are given some current $\boldsymbol{i}_{orig}$ (I_{orig}) and voltage $\boldsymbol{v}$. We seek I that would be optimal. It means that I preserves the original power $p = \boldsymbol{v} \cdot \boldsymbol{i}_{orig}$ and gives the minimal losses.

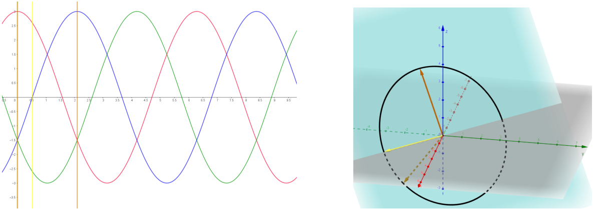

Figure: Ideal three-phase power system

Classical approach - method of Lagrange multipliers

We are looking for $\boldsymbol{i} = i_a e_1 + i_b e_2 + i_c e_3$ such that we minimize:

$$min_{\boldsymbol{i}}\{\boldsymbol{i} \cdot \boldsymbol{i} + (i_a + i_b + i_c)^2\}$$

under the constraint

$$\boldsymbol{i} \cdot \boldsymbol{v} = p,$$

where p and $\boldsymbol{v} = v_a e_1 + v_b e_2 + v_c e_3$ are given.
Lagrange function

$$L(\boldsymbol{i}, \lambda) = \boldsymbol{i} \cdot \boldsymbol{i} + (i_a + i_b + i_c)^2 - \lambda(\boldsymbol{i} \cdot \boldsymbol{v} - p)$$

Defining $\overline{\boldsymbol{n}} = 1/\sqrt{3}(e_1 + e_2 + e_3)$, the conditions for $\boldsymbol{i}$ are:

$$L'_{\boldsymbol{i}} = 2\boldsymbol{i} + 6(\overline{\boldsymbol{n}} \cdot \boldsymbol{i})\overline{\boldsymbol{n}} - \lambda \boldsymbol{v} = \boldsymbol{0}, \tag{1}$$

$$L'_{\lambda} = \boldsymbol{i} \cdot \boldsymbol{v} - p = 0. \tag{2}$$

Classical approach - method of Lagrange multipliers

$P_{\overline{\boldsymbol{n}}}$ denotes orthogonal projection of $\boldsymbol{i}$ onto $\overline{\boldsymbol{n}}$.

$$2(Id + 3P_{\overline{\boldsymbol{n}}})\boldsymbol{i} = \lambda \boldsymbol{v}$$

$$\boldsymbol{i} = 1/2\lambda(Id + 3P_{\overline{\boldsymbol{n}}})^{-1}\boldsymbol{v}$$

To avoid the inverse, we use $(Id + 3P_{\overline{\boldsymbol{n}}})^{-1} = 1/4(Id + 3P_\rho)$, where P_ρ is projection onto the plane ρ perpendicular to $\overline{\boldsymbol{n}}$.

$$\boldsymbol{i} = 1/8\lambda(Id + 3P_\rho)\boldsymbol{v}$$

To find λ we use the second condition:

$$\boldsymbol{i} \cdot \boldsymbol{v} = p$$

$$1/8\lambda(Id + 3P_\rho)\boldsymbol{v} \cdot \boldsymbol{v} = p$$

$$\lambda = \frac{8p}{((Id + 3P_\rho)\boldsymbol{v}) \cdot \boldsymbol{v}}$$

Classical approach - method of Lagrange multipliers

Finally, we get

$$\boldsymbol{i} = p\frac{(Id + 3P_\rho)\boldsymbol{v}}{((Id + 3P_\rho)\boldsymbol{v}) \cdot \boldsymbol{v}}.$$

$$P_{\overline{\boldsymbol{n}}} = \overline{\boldsymbol{n}}\overline{\boldsymbol{n}}^T = 1/3\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_\rho = Id - P_{\overline{\boldsymbol{n}}} = 1/3\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\boldsymbol{i} = p\frac{\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\boldsymbol{v}}{\boldsymbol{v}^T\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\boldsymbol{v}} = p\frac{\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}}{\begin{bmatrix} v_a & v_b & v_c \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}}$$

Adding a dimension

We are looking for $\boldsymbol{i} = i_a e_1 + i_b e_2 + i_c e_3$ such that we minimize:

$$min_{\boldsymbol{i}}\{\boldsymbol{i} \cdot \boldsymbol{i} + (i_a + i_b + i_c)^2\}$$

under the constraint

$$\boldsymbol{i} \cdot \boldsymbol{v} = p.$$

We switch to $\mathbb{R}^4$. We encode the sum of coordinates into the fourth dimension. So we will seek the current in a form

$$\boldsymbol{i}_{4D} = \boldsymbol{i} - (i_a + i_b + i_c)e_4$$

Then all points that are relevant lie in the plane ρ_{4D} with the normal $\boldsymbol{n}_{4D} = e_1 + e_2 + e_3 + e_4$. Reformulation: We minimize $\boldsymbol{i}_{4D}^2$ under the constraint

$$\boldsymbol{i}_{4D} \cdot \boldsymbol{v} = p.$$

That gives us the hyperplane τ_{4D}. We are looking for the minimum on the intersection of ρ_{4D} and τ_{4D}. To minimize the norm we use orthogonal projection of the origin.

PGA as a suitable algebra

We need hyperplanes, intersections and orthogonal projection. We seek a point

$$\boldsymbol{I}_{4D} = (\boldsymbol{i} - (i_a + i_b + i_c)e_4 + e_0)^*.$$

The solution has to lie on $\rho_{4D} = e_1 + e_2 + e_3 + e_4$. Feasible solutions lie on the plane $\tau_{4D} = \boldsymbol{v} + pe_0$. So we seek the minimum of a set

$$k_{4D} = \rho_{4D} \wedge \tau_{4D}.$$

Clearly losses can be expressed as a square of $\boldsymbol{i}_{4D}$ (the distance from origin to I_{4D}) $losses(\boldsymbol{i}) = \boldsymbol{i}_{4D} \cdot \boldsymbol{i}_{4D}$. I_{4D} we get by orthogonal projection of origin to the k_{4D}:

$$I_{4D} = (k_{4D} \cdot e_{1234}) \wedge k_{4D}.$$

After omitting the fourth coordinate:

$$\boldsymbol{i} = p\frac{(3v_a - v_b - v_c)e_1 + (3v_b - v_a - v_c)e_2 + (3v_c - v_a - v_b)e_3}{3(v_a^2 + v_b^2 + v_c^2) - 2(v_a v_b + v_b v_c + v_a v_c)} \in \mathbb{R}^3.$$

Three-phase power system in time

We consider the harmonic three-phase voltage input:

$$v_a(t) = V_{max}\cos(\omega t)$$
$$v_b(t) = V_{max}\cos\left(\omega t + \frac{2}{3}\pi\right)$$
$$v_c(t) = V_{max}\cos\left(\omega t - \frac{2}{3}\pi\right)$$

The original current is the same, just delayed by angle φ and instead of V_{max} we have I_{max}. In that case, I is simply projection of the origin onto plane τ.

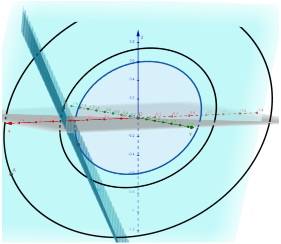

Figure: Three-phase power system

Compensation

If we use translation it is time-depending. Combination of $\mathcal{DR}$ is constant. We use $3D$ CGA.
We rotate in the plane ρ the angle of rotation is φ and then shrink circle (of $l_{orig}(t)$) to circle (of $l(t)$). The ratio of their radius is $cos(\varphi)$.

$$\mathcal{R} = \cos(\varphi/2) - \sin(\varphi/2)\frac{\boldsymbol{n}e_{123}}{\|\boldsymbol{n}\|}$$

$$\mathcal{D} = 1 + \frac{1 - |\cos(\varphi)|}{1 + |\cos(\varphi)|} e_\infty \wedge e_0$$

$$l = \mathcal{DR} l_{orig} \mathcal{R}^{-1} \mathcal{D}^{-1}$$

It can be useful to express in terms of exp():

$$\mathcal{R} = exp\left(-\frac{1}{2}\varphi\frac{e_{23} + e_{31} + e_{12}}{\sqrt{3}}\right)$$

$$\mathcal{D} = exp(-0.5ln(|cos(\varphi)|)e_\infty \wedge e_0)$$

This holds for $|\varphi| < \pi/2$.

Four-phase system

We want to transform

$$\begin{aligned} i_1(t) = & I_m cos(t\omega), & i_2(t) = I_m sin(t\omega), \\ i_3(t) = & - I_m cos(t\omega), & i_4(t) = -I_m sin(t\omega) \end{aligned} \tag{3}$$

to the signal

$$\begin{aligned} i'_1(t) = & I'_m cos(t\omega + \varphi), & i'_2(t) = I'_m sin(t\omega + \varphi), \\ i'_3(t) = & - I'_m cos(t\omega + \varphi), & i'_4(t) = -I'_m sin(t\omega + \varphi). \end{aligned} \tag{4}$$

We want to transform points $I(t) = [i_1(t), i_2(t), i_3(t), i_4(t)]$ from circle(3) to points $I'(t) = [i'_1(t), i'_2(t), i'_3(t), i'_4(t)]$ from circle (4) for any time t.

Compensation

Using the ideas above, in 4D CGA that transformation can be represented as composition of two commutative rotations

$$\mathcal{R} = \mathcal{R}_\sigma \mathcal{R}_\pi,$$

where the angle of rotation is φ:

$$\mathcal{R}_\pi = \cos(\varphi/2) - \sin(\varphi/2) e_{12},$$

$$\mathcal{R}_\sigma = \cos(\varphi/2) - \sin(\varphi/2) e_{34},$$

so

$$\begin{aligned}\mathcal{R} =& \mathcal{R}_\pi \mathcal{R}_\sigma = \mathcal{R}_\sigma \mathcal{R}_\pi = (\cos(\varphi/2) - \sin(\varphi/2) e_{12})(\cos(\varphi/2) - \sin(\varphi/2) e_{34}) \\ =& \cos^2(\varphi/2) - \cos(\varphi/2)\sin(\varphi/2)(e_{12} + e_{34}) + \sin^2(\varphi/2) e_{1234}.\end{aligned}$$

And also the dilation.

Literature

- Montoya, F. G., & Eid, A. H. (2022). Formulating the geometric foundation of Clarke, Park, and FBD transformations by means of Clifford's geometric algebra. Mathematical Methods in the Applied Sciences, 45(8), 4252-4277. https://doi.org/10.1002/mma.8038
- Francisco Casado-Machado, José L. Martínez-Ramos, Manuel Barragán-Villarejo, José María Maza-Ortega, Reduced Reference Frame Transformation Assessment in Unbalanced Three-Phase Four-Wire Systems, IEEE Access, 10.1109/ACCESS.2023.3254299, 11, (24591-24603), (2023).
- Francisco G. Montoya, Xabier Prado, Francisco M. Arrabal-Campos, Alfredo Alcayde, Jorge Mira, New mathematical model based on geometric algebra for physical power flow in theoretical two-dimensional multi-phase power circuits, Scientific Reports, 10.1038/s41598-023-28052-x, 13, 1, (2023).

- Eduardo Viciana, Francisco M. Arrabal-Campos, Alfredo Alcayde, Raul Baños, Francisco G. Montoya, All-in-one three-phase smart meter and power quality analyzer with extended IoT capabilities, Measurement, 10.1016/j.measurement.2022.112309, 206, (112309), (2023)
- Ahmad H. Eid, Francisco G. Montoya, A Systematic and Comprehensive Geometric Framework for Multiphase Power Systems Analysis and Computing in Time Domain, IEEE Access, 10.1109/ACCESS.2022.3230915, 10, (132725-132741), (2022).
- Hildenbrand, D.: Foundations of Geometric Algebra Computing. Springer Science & Business Media (2013)
- Perwass, Ch.: Geometric Algebra with Applications in Engineering (1st edn). Springer Verlag, (2009)
- Leo Dorst, Steven De Keninck: A Guided Tour to the Plane-Based Geometric Algebra PGA

Thank you for your attention.