CGA-Based Snake Robot Control Models

Roman Byrtus

Institute of Mathematics
Faculty of Mechanical Engineering
Brno University of Technology

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FACULTY
OF MECHANICAL
ENGINEERING

Contents

Conformal Geometric Algebra (CGA)

The Conformal Geometric Algebra (CGA) is the Clifford algebra $Cl_{N+1,1}$ along with the embedding $C : \mathbb{R}^N \ni X \mapsto M \in Cl_{N+1,1}$. The embedding of the point X in terms of the null basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N, \boldsymbol{e}_0, \boldsymbol{e}_\infty\}$ is then given by

$$X \mapsto x_1 \boldsymbol{e}_1 + \cdots + x_N \boldsymbol{e}_N + \frac{1}{2}(x_1^2 + \cdots + x_N^2)\boldsymbol{e}_\infty + \boldsymbol{e}_0. \tag{1}$$

Snake robot

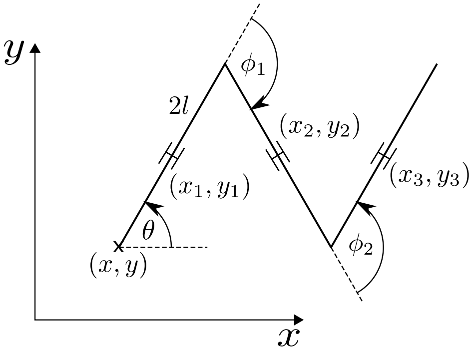


Figure: A snake robot in 2D.

Snake robot

- Robotic mechanism inspired by the locomotion of biological snakes.
- The snake robot consists of a series of links, equipped with passive wheels located in the centres, connected by actuated joints.
- The mechanism is nonholonomic, meaning there is a constraint defined on the tangent bundle TQ of the configuration space Q.

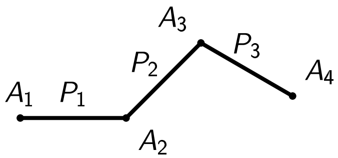


Figure: A three-link snake robot.

Kinematics

- The i-th link of the robot is represented by the point pair $P_i = A_i \wedge A_{i+1}$.
- Denote the initial configuration as P_i^0.
- Denote a transformation acting on the links as M_j in the form of $M_j = e^{-\frac{1}{2}L(q(t))}$, where $q(t)$ is a point in the configuration space at time t.
- Then the configuration of the mechanism at time t can be represented by the kinematic chain

$$P_i(t) = \prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j. \tag{2}$$

Nonholonomic constraint

- The mechanism is subject to the non-slip condition, i.e. the links' wheels are assumed not to slip sideways.
- Denoting the velocity of the i-th link's centre as v_i and the normal of the i-th link as n_i, the constraint is expressed as

$$v_i \cdot n_i = 0. \tag{3}$$

- In CGA, we express the condition as

$$\dot{p}_i \wedge P_i \wedge \boldsymbol{e}_\infty = 0, \tag{4}$$

where $\dot{p}_i$ is the velocity of the i-th link's centre $p_i = P_i \boldsymbol{e}_\infty \tilde{P}_i$.

Differential kinematics

- The nonholonomic constraint can be used to obtain forward or inverse kinematics.
- In the 2D case, results have been obtained before.
- It is possible to express $\dot{p}_i$ as

$$\dot{p}_i = \sum_{j=1}^{k} [p_i \cdot \dot{L}_j], \tag{5}$$

where $\dot{L}_j = \partial_t L_j(\boldsymbol{q}(t)) = \sum_{i=1}^{n} (\partial_{q_i} L_j)\dot{\boldsymbol{q}}_i$ is the derivative of the "axis" of the j-th transformation $M_j = e^{-\frac{1}{2}L(q(t))}$ applied to link P_i in the kinematic chain.

Differential kinematics

Denote $q(t) = [x(t), y(t), \theta(t), \phi_1(t), \phi_2(t)]$ as a point in the configuration space and $\dot{q}(t) = (\dot{x}(t), \dot{y}(t), \dot{\theta}(t), \dot{\phi}_1(t), \dot{\phi}_2(t))$ as a vector in the tangent space. Expanding the nonholonomic constraint in 2D, we would arrive at

$$
\begin{aligned}
&\left(\dot{\theta} - 2\dot{x}\sin(\theta) + 2\dot{y}\cos(\theta)\right) \boldsymbol{I} = 0, \\
&\left(\dot{\phi}_1 + 2\dot{\theta}\cos(\phi_1) + \dot{\theta} - 2\dot{x}\sin(\phi_1 + \theta) + 2\dot{y}\cos(\phi_1 + \theta)\right) \boldsymbol{I} = 0, \\
&\Big(2\dot{\phi}_1\cos(\phi_2) + \dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\theta}\cos(\phi_2) + 2\dot{\theta}\cos(\phi_1 + \phi_2) + \dot{\theta} - \\
&-2\dot{x}\sin(\phi_1 + \phi_2 + \theta) + 2\dot{y}\cos(\phi_1 + \phi_2 + \theta)\Big) \boldsymbol{I} = 0,
\end{aligned}
\tag{6}
$$

where $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_0\boldsymbol{e}_\infty$.

3D CGA Model of Planar Motion

- Moving to the 3D case, the z dimension is added in appropriate places and so we turn to 3D CGA.
- Again, it is useful to utilise $\dot{p}_i$ expressed as

$$\dot{p}_i = \sum_{j=1}^{k} [p_i \cdot \dot{L}_j], \tag{7}$$

3D CGA Model of Planar Motion

- We proceed by again expanding the nonholonomic condition $\dot{p}_i \wedge P_i \wedge \boldsymbol{e}_\infty = 0$ in order to obtain a set of differential equations with multivector coefficients.
- In order to simplify the equations obtained, we evaluate the equations in the origin ($[x, y, z] = [0, 0, 0]$) (invariance of the velocity w.r.t. the starting position in space).

Nonholonomic constraint

For the first link we obtain:

$$\begin{aligned}&\left(\dot\theta z - 2\dot x z \sin(\theta) + 2\dot y z \cos(\theta) + 2\dot z x \sin(\theta) - 2\dot z y \cos(\theta)\right) \boldsymbol{e}_1 \\ &\wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_\infty + \left(\dot\theta - 2\dot x \sin(\theta) + 2\dot y \cos(\theta)\right) \boldsymbol{e}_1 \\ &\wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty + 2\dot z \cos(\theta) \boldsymbol{e}_1 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty + \\ &2\dot z \sin(\theta) \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0. \end{aligned} \tag{8}$$

Nonholonomic constraint

For the second link we obtain:

$$\begin{aligned}&\Big(\dot{\phi}_1 z + 2\dot{\theta} z \cos\left(\phi_1\right) + \dot{\theta} z - 2\dot{x} z \sin\left(\phi_1 + \theta\right) + 2\dot{y} z \cos\left(\phi_1 + \theta\right)\\ &+ 2\dot{z} x \sin\left(\phi_1 + \theta\right) - 2\dot{z} y \cos\left(\phi_1 + \theta\right) + 2\dot{z} \sin\left(\phi_1\right)\Big)\, \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_\infty\\ &+ \left(\dot{\phi}_1 + 2\dot{\theta} \cos\left(\phi_1\right) + \dot{\theta} - 2\dot{x} \sin\left(\phi_1 + \theta\right) + 2\dot{y} \cos\left(\phi_1 + \theta\right)\right) \boldsymbol{e}_1\\ &\wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty + 2\dot{z} \cos\left(\phi_1 + \theta\right) \boldsymbol{e}_1 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0\\ &\wedge \boldsymbol{e}_\infty + 2\dot{z} \sin\left(\phi_1 + \theta\right) \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0.\end{aligned} \tag{9}$$

Nonholonomic constraint

For the third link we obtain:

$$
\begin{aligned}
&\Big(2\dot{\phi}_1 z\cos(\phi_2) + \dot{\phi}_1 z + \dot{\phi}_2 z + 2\dot{\theta} z\cos(\phi_2) + 2\dot{\theta} z\cos(\phi_1+\phi_2) + \dot{\theta} z \\
&\quad - 2\dot{x} z\sin(\phi_1+\phi_2+\theta) + 2\dot{y} z\cos(\phi_1+\phi_2+\theta) + 2\dot{z} x\sin(\phi_1+\phi_2+\theta) \\
&\quad - 2\dot{z} y\cos(\phi_1+\phi_2+\theta) + 2\dot{z}\sin(\phi_2) + 2\dot{z}\sin(\phi_1+\phi_2)\Big)\, \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \\
&\quad \wedge \boldsymbol{e}_\infty + \Big(2\dot{\phi}_1\cos(\phi_2) + \dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\theta}\cos(\phi_2) + 2\dot{\theta}\cos(\phi_1+\phi_2) \\
&\quad + \dot{\theta} - 2\dot{x}\sin(\phi_1+\phi_2+\theta) + 2\dot{y}\cos(\phi_1+\phi_2+\theta)\Big)\, \boldsymbol{e}_1 \\
&\quad \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty + 2\dot{z}\cos(\phi_1+\phi_2+\theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \\
&\quad \wedge \boldsymbol{e}_\infty + 2\dot{z}\sin(\phi_1+\phi_2+\theta)\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0.
\end{aligned}
\tag{10}
$$

Nonholonomic constraint

- We proceed by expanding the nonholonomic condition to $\dot{p}_i \wedge P_i \wedge \mathbf{e}_\infty \wedge \mathbf{e}_j = 0$, $j = 1, 2, 3$.
- $P_i \wedge \mathbf{e}_\infty \wedge \mathbf{e}_j$ defines a plane, which helps us split velocity components.

Nonholonomic constraint

Expanding $\dot{p}_i \wedge P_i \wedge \mathbf{e}_\infty \wedge \mathbf{e}_3 = 0$ we get:

$$\left(\dot{\theta} - 2\dot{x}\sin(\theta) + 2\dot{y}\cos(\theta)\right)\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_0 \wedge \mathbf{e}_\infty = 0,$$

$$\left(\dot{\phi}_1 + 2\dot{\theta}\cos(\phi_1) + \dot{\theta} - 2\dot{x}\sin(\phi_1 + \theta) + 2\dot{y}\cos(\phi_1 + \theta)\right)\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_0 \wedge \mathbf{e}_\infty = 0,$$

$$\Big(2\dot{\phi}_1\cos(\phi_2) + \dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\theta}\cos(\phi_2) + 2\dot{\theta}\cos(\phi_1 + \phi_2) + \dot{\theta}$$
$$- 2\dot{x}\sin(\phi_1 + \phi_2 + \theta) + 2\dot{y}\cos(\phi_1 + \phi_2 + \theta)\Big)\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_0 \wedge \mathbf{e}_\infty,$$

Nonholonomic constraint

Expanding $\dot{p}_i \wedge P_i \wedge \boldsymbol{e}_\infty \wedge \boldsymbol{e}_2 = 0$ we get:

$$-2\dot{z}\cos(\theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0, \tag{11a}$$
$$-2\dot{z}\cos(\phi_1 + \theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0, \tag{11b}$$
$$-2\dot{z}\cos(\phi_1 + \phi_2 + \theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0, \tag{11c}$$

Expanding $\dot{p}_i \wedge P_i \wedge \boldsymbol{e}_\infty \wedge \boldsymbol{e}_1 = 0$ we get:

$$2\dot{z}\sin(\theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0, \tag{12a}$$
$$2\dot{z}\sin(\phi_1 + \theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0, \tag{12b}$$
$$2\dot{z}\sin(\phi_1 + \phi_2 + \theta)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0. \tag{12c}$$

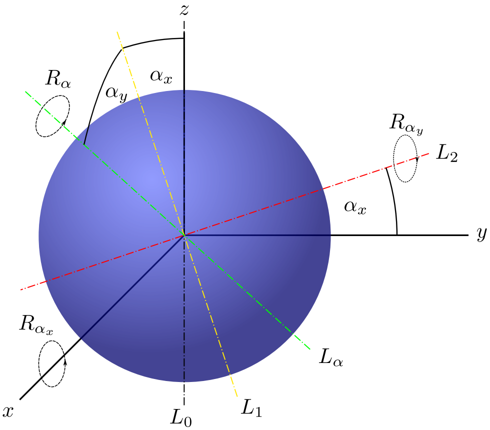
Three DOF Joint Model

- In 3D, we need to choose a way to model the joints connecting the mechanism's links.
- The links are connected by spherical joints, thus allowing pitch, yaw and roll. Denote a rotor representing the spherical joint as $R_\alpha = e^{-\frac{1}{2}\alpha L}$, where

$$L_\alpha = R_{\alpha_y} L_1 \tilde{R}_{\alpha_y} = R_{\alpha_y} R_{\alpha_x} \boldsymbol{e}_{12} \tilde{R}_{\alpha_x} \tilde{R}_{\alpha_y},$$

and $R_{\alpha_x} = e^{-\frac{1}{2}\alpha_x \boldsymbol{e}_{12}}$ and $R_{\alpha_y} = e^{-\frac{1}{2}\alpha_y L_2}$.

Sphere Joint Model



Two DOF Joint Model

- In this model, we restrict the motion realised by the joints to yaw and pitch.
- An interesting parametrisation is as follows:
- The first plane of rotation ρ_1 for the yaw motion can be represented by the three points defining the two connected links: thus, $\rho_1 = A_1 \wedge A_2 \wedge A_3 \wedge \boldsymbol{e}_\infty$.
- Let l_1 and l_2 be the lines passing through the first and second links.
- Then the axis of rotation L_{i1} for the plane ρ_1 can be expressed as

$$L_{i1} = l_1 \underline{\times} l_2,$$

where $\underline{\times}$ is the commutator product.

Two DOF Joint Model

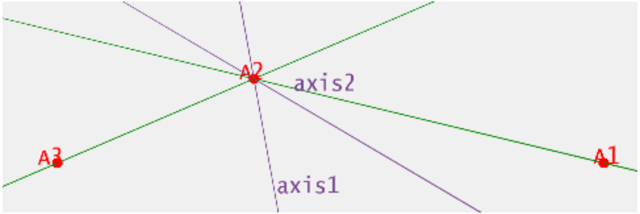


Figure: The axes of rotation $axis1, axis2$ for the link represented by points A_2, A_3.

Two DOF Joint Model

- The second plane of rotation ρ_2 for the yawing motion is the plane containing the link P_2 that is orthogonal to the first axis L_{i1}; thus its axis L_{i2} is given by

$$L_{i2} = L_{i1} \underline{\times} l_2.$$

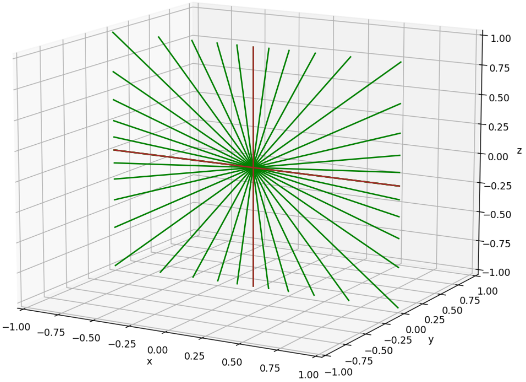
The rotation realised by the 2-DOF joint can then be expressed as

$$R_i = e^{-\frac{1}{2}\phi_i L_i},$$

with the axis L_i given by

$$L_i = \omega_i L_{i1} + (1 - |\omega_i|) L_{i2}.$$

Two DOF Joint Model

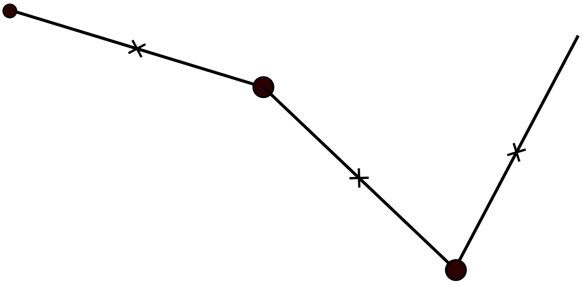


Difficulties with the approach

If we were to proceed with the full 3D CGA model, we run into a few difficulties:

- So far, all results were obtained using symbolical calculations.
- Both the 2 DOF and 3 DOF variants start to be computationally problematic.
- Difficulty in determining controllability of the mechanism.

A Purely Geometry Based Control Algorithm



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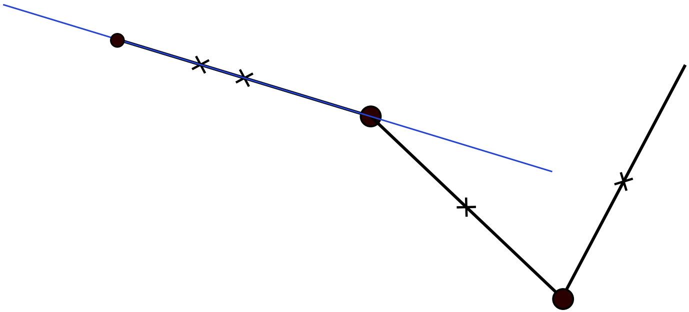

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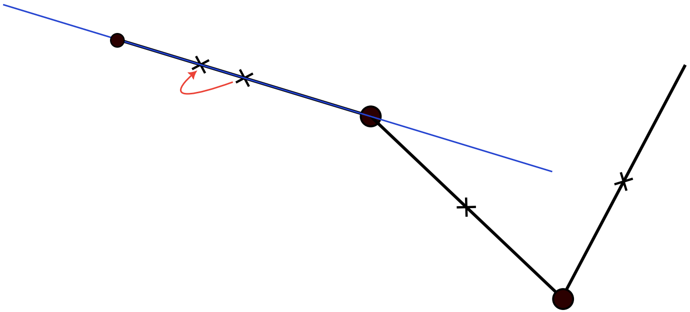

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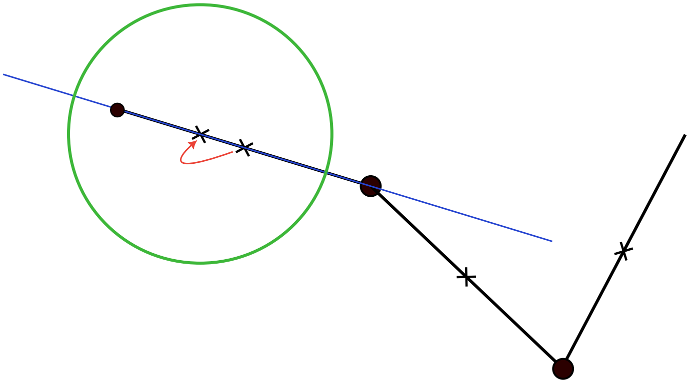

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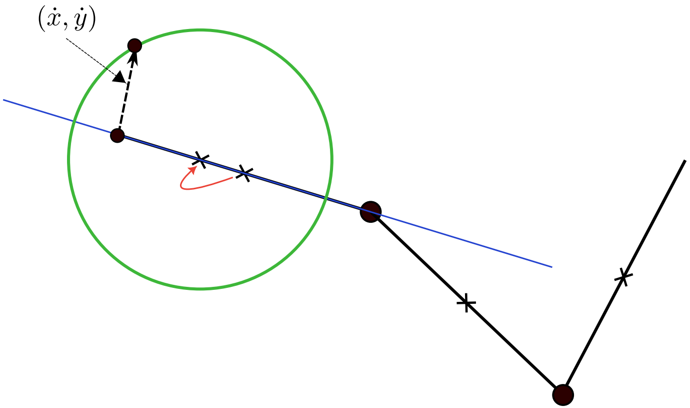

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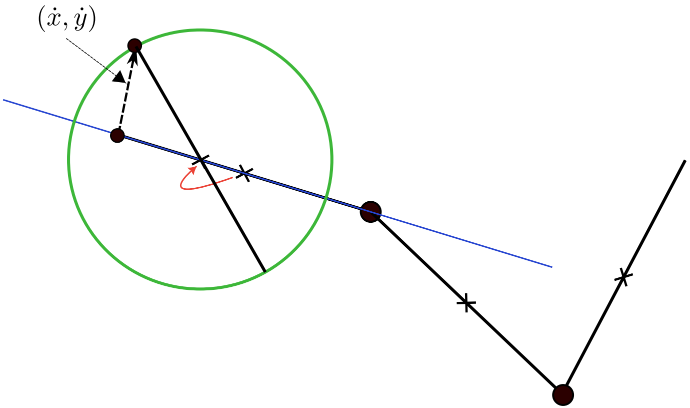

Thank you for your attention.