Characteristic multivectors of Coxeter transformations give novel insights into the geometry of root systems

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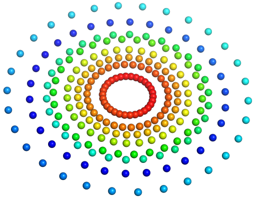
Algebraic and computational interests

- Exceptional root systems/geometries, Trinities and ADE correspondences
- Clifford algebras – characteristic MV
- Cluster algebras
- Viruses: structure, assembly, novel therapeutic approaches; computational modelling
- Data science, computational algebra, experimental mathematics



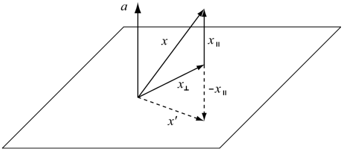
Clifford Algebra and Reflections

Vector space with an inner product

Why not work with the Clifford algebra?

Geometric product $ab \equiv a \cdot b + a \wedge b$

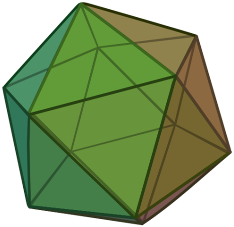
Inner product is the symmetric part $a \cdot b = \frac{1}{2}(ab + ba)$



Reflections

$$x = x_{\perp} + x_{\parallel} \rightarrow x' = x_{\perp} - x_{\parallel} = x - 2x_{\parallel} = x - 2(x \cdot n)n = -nxn$$

Groups of reflections (Coxeter groups)

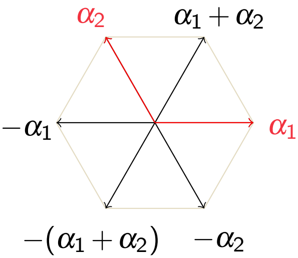


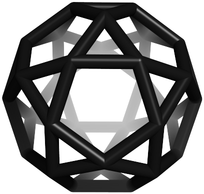
Reflection groups from generating reflections

$$\boxed{x' = -nxn} \rightarrow \boxed{x' = \pm n_k \dots n_2 n_1 x n_1 n_2 \dots n_k =: \pm \tilde{A} x A}$$

Cartan-Dieudonné theorem

Any orthogonal transformation can be written as the product of successive reflections.





Root system Φ

A set of vectors α in a vector space with an inner product such that

1. $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall \ \alpha \in \Phi$
2. $s_\alpha \Phi = \Phi \ \forall \ \alpha \in \Phi$

where the reflections and Coxeter element are $s_\alpha : v \rightarrow s_\alpha(v) = -\alpha v \alpha$ and $w = s_1 \dots s_n$

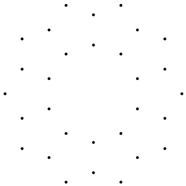
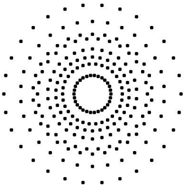
Vector space + inner product: Clifford

Cartan matrix: a rotational invariant

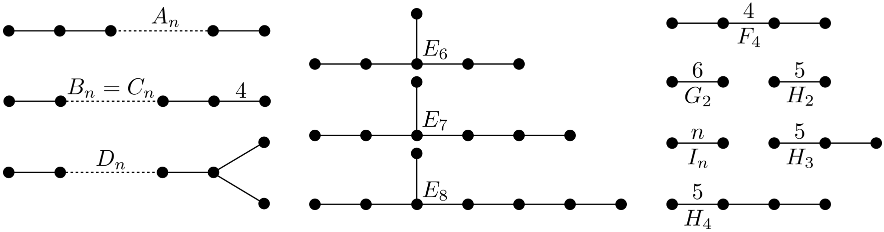
$\sim$ scalar products between simple roots.

The Coxeter Plane

- Every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane
- Coxeter elements act as rotations in these Coxeter planes

Classification of Euclidean reflection groups

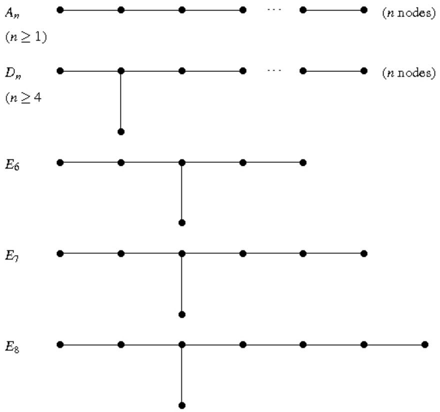


Links: none = orthogonal ($\pi/2$), unlabelled link = $\pi/3$, label $n = \pi/n$

Types

crystallographic (Weyl/Lie theory, A-G) vs non-crystallographic (I & H), simply-laced (ADE) etc

Classification of ADE diagrams – simply-laced



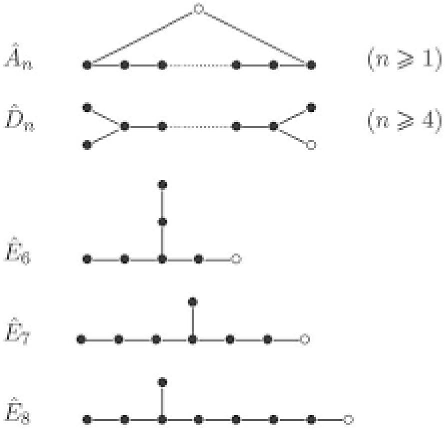
ADE pattern

Two infinite families and 3 exceptional cases.

Consider the corresponding adjacency matrices

The maximal (principal) eigenvalue of the adjacency matrix is $< 2 \Rightarrow$ ADE diagrams (Smith's theorem).

Classification of affine ADE diagrams



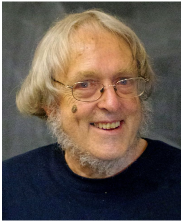
ADE pattern

Two infinite families and 3 exceptional cases.

Consider the corresponding adjacency matrices

The maximal (principal) eigenvalue of the adjacency matrix is $= 2 \Rightarrow$ affine ADE diagrams (Smith's theorem).

Forthcoming ADE book




CAMBRIDGE UNIVERSITY PRESS

ADE - patterns in mathematics

Peter Cameron, P-P Dechant, Yang-Hui He, John McKay

Simplicial Derivatives and Invariants of a linear function f

Let $\{a_k\}, k = 1, \ldots, m$ denote a frame; we denote by a^k its reciprocal frame such that $a^i \cdot a_j = \delta^i_j$. We also define $b_k = f(a_k)$.

The rth simplicial derivative is defined as

$$\partial_{(r)} f_{(r)} = \sum (a^{j_r} \wedge \cdots \wedge a^{j_1})(b_{j_1} \wedge \cdots \wedge b_{j_r})$$

with sum over $0 < j_1 < \cdots < j_r \leq m$.

Simplicial derivatives and characteristic multivectors

Originally explored by David Hestenes and Garret Sobczyk and more recently by Anthony Lasenby and Joan Lasenby et al (AGACSE Brno papers).

Characteristic polynomial

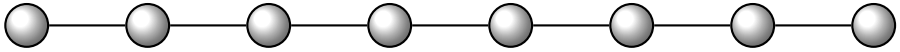
$$C_f(\lambda) = \sum_{s=0}^{m} (-\lambda)^{m-s} \partial_{(s)} * f_{(s)}$$

* denotes the scalar part of multivectors and $\partial_{(0)} * f_{(0)}$ is interpreted as 1.

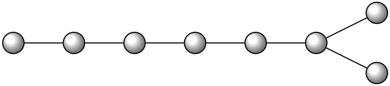
Cayley-Hamilton theorem

$$\sum_{s=0}^{m} (-1)^{m-s} \partial_{(s)} * f_{(s)} f^{m-s}(a) = 0$$

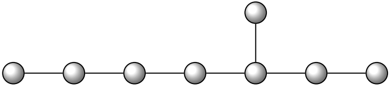
for any vector a, where $f^0(a)$ is interpreted as a.



A_8



D_8



E_8

Invariant Patterns

MV parts	0	1	2	3	4	5	6	7	8
Inv_0	X								
Inv_1	X		X						
Inv_2	X		X		X				
Inv_3	X		X		X		X		
Inv_4	X		X		X		X		X
Inv_5	X		X		X		X		
Inv_6	X		X		X				
Inv_7	X		X						
Inv_8	X								

Bivector Invariants (for an E_8 example)

$$Inv_{(2)}^{(1)} = 2a_1\wedge a_2 - 2a_2\wedge a_3 + 2a_3\wedge a_4 - 2a_4\wedge a_5 + 2a_5\wedge a_6 + 2a_5\wedge a_8 - 2a_6\wedge$$

$$Inv_{(2)}^{(2)} = -2a_1\wedge a_2 - 2a_1\wedge a_4 + 4a_2\wedge a_3 + 2a_2\wedge a_5 - 4a_3\wedge a_4 - 2a_3\wedge a_6 -$$
$$+6a_4\wedge a_5 + 2a_4\wedge a_7 - 6a_5\wedge a_6 - 4a_5\wedge a_8 + 2a_6\wedge a_7 - 2a_7\wedge a_8 =$$

$$Inv_{(2)}^{(3)} = 2a_1\wedge a_4 + 2a_1\wedge a_6 + 2a_1\wedge a_8 - 2a_2\wedge a_3 - 6a_2\wedge a_5 - 2a_2\wedge a_7 + 6a_3$$
$$-10a_4\wedge a_5 - 4a_4\wedge a_7 + 8a_5\wedge a_6 + 6a_5\wedge a_8 - 2a_6\wedge a_7 + 2a_7\wedge a_8 =$$

$$Inv_{(2)}^{(4)} = 2a_1\wedge a_2 - 2a_1\wedge a_4 - 4a_1\wedge a_6 - 2a_1\wedge a_8 + 8a_2\wedge a_5 + 4a_2\wedge a_7 - 6a_3$$
$$+12a_4\wedge a_5 + 4a_4\wedge a_7 - 8a_5\wedge a_6 - 6a_5\wedge a_8 + 2a_6\wedge a_7 - 2a_7\wedge a_8$$

Quadruvector Invariants

$$Inv^{(2)}_{(4)} = 4a_1 \wedge a_2 \wedge a_3 \wedge a_4 - 4a_1 \wedge a_2 \wedge a_4 \wedge a_5 + 4a_1 \wedge a_2 \wedge a_5 \wedge a_6 + 4a_1 \wedge a_2$$

$$+ 4a_2 \wedge a_3 \wedge a_4 \wedge a_5 - 4a_2 \wedge a_3 \wedge a_5 \wedge a_6 - 4a_2 \wedge a_3 \wedge a_5 \wedge a_8 + 4a_2 \wedge$$

$$+ 4a_3 \wedge a_4 \wedge a_5 \wedge a_8 - 4a_3 \wedge a_4 \wedge a_6 \wedge a_7 + 4a_4 \wedge a_5 \wedge a_6 \wedge a_7 - 4a_5 \wedge$$

$$Inv^{(3)}_{(4)} = -4a_1 \wedge a_2 \wedge a_3 \wedge a_4 - 4a_1 \wedge a_2 \wedge a_3 \wedge a_6 - 4a_1 \wedge a_2 \wedge a_3 \wedge a_8 + 12a_1$$

$$-12a_1 \wedge a_2 \wedge a_5 \wedge a_6 - 8a_1 \wedge a_2 \wedge a_5 \wedge a_8 + 4a_1 \wedge a_2 \wedge a_6 \wedge a_7 - 4a_1$$

$$-4a_1 \wedge a_4 \wedge a_5 \wedge a_8 + 4a_1 \wedge a_4 \wedge a_6 \wedge a_7 - 12a_2 \wedge a_3 \wedge a_4 \wedge a_5 - 4a_2$$

$$+16a_2 \wedge a_3 \wedge a_5 \wedge a_6 + 12a_2 \wedge a_3 \wedge a_5 \wedge a_8 - 8a_2 \wedge a_3 \wedge a_6 \wedge a_7 + 4a$$

$$-4a_2 \wedge a_5 \wedge a_6 \wedge a_7 - 12a_3 \wedge a_4 \wedge a_5 \wedge a_6 - 8a_3 \wedge a_4 \wedge a_5 \wedge a_8 + 8a_3$$

$$+4a_3 \wedge a_6 \wedge a_7 \wedge a_8 - 8a_4 \wedge a_5 \wedge a_6 \wedge a_7 + 4a_5 \wedge a_6 \wedge a_7 \wedge a_8 = Inv^{(5}_{(4}$$

$$Inv^{(4)}_{(4)} = 4a_1 \wedge a_2 \wedge a_3 \wedge a_4 + 8a_1 \wedge a_2 \wedge a_3 \wedge a_6 + 4a_1 \wedge a_2 \wedge a_3 \wedge a_8 - 16a_1 \wedge a$$

Sextuvector Invariants

$$Inv^{(3)}_{(6)} = 8a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6 + 8a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_8 - 8a_1 \wedge a_2$$

$$+ 8a_1 \wedge a_2 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_7 - 8a_1 \wedge a_2 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 - 8a_2 \wedge$$

$$+ 8a_2 \wedge a_3 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 - 8a_3 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 = Inv^{(5)}_{(6)}$$

$$Inv^{(4)}_{(6)} = -16a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6 - 8a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_8 + 8a_1 \wedge$$

$$-8a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_7 \wedge a_8 + 8a_1 \wedge a_2 \wedge a_3 \wedge a_6 \wedge a_7 \wedge a_8 - 16a_1 \wedge$$

$$+ 8a_1 \wedge a_2 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 + 8a_1 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 + 16a_2 \wedge$$

$$-16a_2 \wedge a_3 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 + 8a_3 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8$$

$$Inv^{(5)}_{(6)} = 8a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_6 + 8a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge a_5 \wedge a_8 - 8a_1 \wedge a_2$$

$$+ 8a_1 \wedge a_2 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_7 - 8a_1 \wedge a_2 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 - 8a_2 \wedge$$

$$+ 8a_2 \wedge a_3 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 - 8a_3 \wedge a_4 \wedge a_5 \wedge a_6 \wedge a_7 \wedge a_8 = Inv^{(3)}_{(6)}$$

The sum of all the invariants is proportional to the Coxeter element. As can also be seen from the pseudoscalar terms, that proportionality factor is -16:

$$\sum Inv_{(i)}^{(j)} = -16W$$

(this includes the scalar contributions we have seen in the context of the Cayley-Hamilton theorem and the characteristic polynomial).

$$\tilde{W} Inv_{(i)}^{(j)} W = Inv_{(i)}^{(j)}$$

Cayley-Hamilton theorem and characteristic polynomial

The characteristic equation of the Coxeter element M' can be written as

$$|M' - \lambda I| = \lambda^8 + \lambda^7 - \lambda^5 - \lambda^4 - \lambda^3 + \lambda^2 + 1 = p(\lambda)g(\lambda) = 0 \qquad (15)$$

where $p(\lambda) = \lambda^4 + \tau\lambda^3 + \tau\lambda^2 + \tau\lambda + 1 = 0$ leads to the eigenvalues of the upper block matrix and $g(\lambda) = \lambda^4 + \sigma\lambda^3 + \sigma\lambda^2 + \sigma\lambda + 1 = 0$ leads to the eigenvalues of the lower block matrix.

Cayley-Hamilton theorem and characteristic polynomial

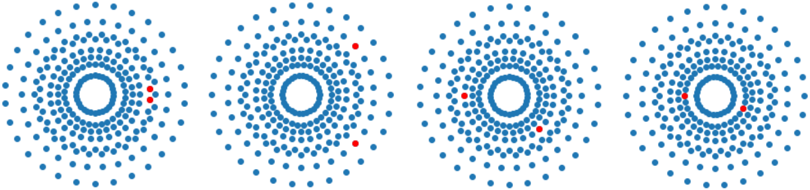
Can show that for these examples of 8D Coxeter elements and their characteristic multivectors

- Satisfy the Cayley-Hamilton theorem and give the correct characteristic polynomial (e.g. for E_8)
- Pieces are separately invariant under W (eigenMV but not eigenblades) – effectively a decomposition of W:
- $W \propto \sum \mathrm{Inv}$ cf the Lasenbys (they want to reconstruct an unknown rotation)
- In our case (W, 8D): $\mathrm{Inv}_1 = \mathrm{Inv}_7, \mathrm{Inv}_2 = \mathrm{Inv}_6, \mathrm{Inv}_3 = \mathrm{Inv}_5$

E_8 geometry in Clifford - complete factorisation

- Coxeter transformations are linear functions that have a range of invariants and invariant subspaces
- E.g. E_8 has $\boxed{1, 7, 11, 13, 17, 19, 23, 29}$ as scalar invariants (exponents - related to degrees of invariant polynomials)
- Clifford decomposition gives 4 eigen-planes

$$\boxed{W = \alpha_1 \dots \alpha_8 = \exp(\frac{\pi}{30} B_C) \exp(\frac{7\pi}{30} B_2) \exp(\frac{11\pi}{30} B_3) \exp(\frac{13\pi}{30} B_4)}$$



4 BV invariants – not necessarily blades. Relation to Coxeter planes/invariants?

$$Inv^{(1)}_{(2)} = 2a_1\wedge a_2 - 2a_2\wedge a_3 + 2a_3\wedge a_4 - 2a_4\wedge a_5 + 2a_5\wedge a_6 + 2a_5\wedge a_8 - 2a_6\wedge a$$

$$Inv^{(2)}_{(2)} = -2a_1\wedge a_2 - 2a_1\wedge a_4 + 4a_2\wedge a_3 + 2a_2\wedge a_5 - 4a_3\wedge a_4 - 2a_3\wedge a_6 -$$
$$+6a_4\wedge a_5 + 2a_4\wedge a_7 - 6a_5\wedge a_6 - 4a_5\wedge a_8 + 2a_6\wedge a_7 - 2a_7\wedge a_8 =$$

$$Inv^{(3)}_{(2)} = 2a_1\wedge a_4 + 2a_1\wedge a_6 + 2a_1\wedge a_8 - 2a_2\wedge a_3 - 6a_2\wedge a_5 - 2a_2\wedge a_7 + 6a_3$$
$$-10a_4\wedge a_5 - 4a_4\wedge a_7 + 8a_5\wedge a_6 + 6a_5\wedge a_8 - 2a_6\wedge a_7 + 2a_7\wedge a_8 =$$

$$Inv^{(4)}_{(2)} = 2a_1\wedge a_2 - 2a_1\wedge a_4 - 4a_1\wedge a_6 - 2a_1\wedge a_8 + 8a_2\wedge a_5 + 4a_2\wedge a_7 - 6a_3$$
$$+12a_4\wedge a_5 + 4a_4\wedge a_7 - 8a_5\wedge a_6 - 6a_5\wedge a_8 + 2a_6\wedge a_7 - 2a_7\wedge a_8$$

4 BV invariants: decomposition into commuting blades?

$\mathrm{Inv}_{(2)}^{(1)}, \mathrm{Inv}_{(2)}^{(2)}, \mathrm{Inv}_{(2)}^{(3)}, \mathrm{Inv}_{(2)}^{(4)}$ give 4 orthogonal bivectors

But not simple blades. Possible relation with the Coxeter planes and the decomposition in terms of commuting bivectors by Hestenes and Sobczyk / Martin Roelfs and Steven de Keninck (Graded symmetry groups: plane and simple, AACA 2023), Shirokov?

$$W_m := \frac{1}{m!}\langle B^m\rangle_{2m} = \frac{1}{m!}\underbrace{B \wedge B \wedge \ldots \wedge B}_{m}$$

$$b_i = \begin{cases} \dfrac{\lambda_i^r W_0 + \lambda_i^{r-1} W_2 + \cdots + W_k}{\lambda_i^{r-1} W_1 + \lambda_i^{r-2} W_3 + \cdots + W_{k-1}} & k \text{ even} \\[2ex] \dfrac{\lambda_i^r W_1 + \lambda_i^{r-1} W_3 + \cdots + W_k}{\lambda_i^r W_0 + \lambda_i^{r-1} W_2 + \cdots + W_{k-1}} & k \text{ odd} \end{cases},$$

Characteristic polynomials – invariants across Coxeter elements

$\mathrm{Inv}^{(1)}_{(2)}, \mathrm{Inv}^{(2)}_{(2)}, \mathrm{Inv}^{(3)}_{(2)}, \mathrm{Inv}^{(4)}_{(2)}$ give 4 orthogonal bivectors with 'characteristic polynomial' (Hestenes)

$$0 = \sum_{m=0}^{k} \langle W_m^2 \rangle_0 (-\lambda_i)^{k-m}$$

$$Inv^{(1)}_{(2)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv^{(2)}_{(2)}: \quad \lambda^4 + 8\lambda^3 + 14\lambda^2 + 7\lambda + 1$$

$$Inv^{(3)}_{(2)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv^{(4)}_{(2)}: \quad \lambda^4 + 28\lambda^3 + 134\lambda^2 + 92\lambda + 1$$

Reexpress in terms of Coxeter bivectors (non-trivial!)

$$-Inv^{(1)}_{(2)} = 1.98904 B_C + 0.415823 B_2 + 0.81347 B_3 + 1.4862 B_4$$

$$-Inv^{(2)}_{(2)} = -2.40486 B_C - 1.22929 B_2 + 0.67281 B_3 + 0.502754 B_4$$

$$-Inv^{(3)}_{(2)} = -1.4862 B_C + 1.98904 B_2 + 0.41582 B_3 - 0.813473 B_4$$

$$-Inv^{(4)}_{(2)} = 4.70463 B_C - 2.2460 B_2 + 0.90040 B_3 - 0.105104 B_4$$

- Exact solutions in terms of eigenvectors of the Cartan matrix

$$-Inv^{(1)}_{(2)} = 2\cos\frac{\pi}{30} B_C + 2\cos\frac{13\pi}{30} B_2 + 2\cos\frac{11\pi}{30} B_3 + 2\cos\frac{7\pi}{30} B_4$$

$$-Inv^{(3)}_{(2)} = -2\cos\frac{7\pi}{30} B_C + 2\cos\frac{\pi}{30} B_2 + 2\cos\frac{13\pi}{30} B_3 - 2\cos\frac{11\pi}{30} B_4$$

- The sums of squares of these coefficients add to 7, 8, 7, 28 – first term in characteristic polynomials (size); others?

Novel explicit connection between Coxeter exponents and characteristic multivectors

$\mathrm{Inv}_{(2)}^{(1)}, \mathrm{Inv}_{(2)}^{(2)}, \mathrm{Inv}_{(2)}^{(3)}, \mathrm{Inv}_{(2)}^{(4)}$ give 4 orthogonal bivectors with 'characteristic polynomial' – the first coefficient is just B^2

$$Inv_{(2)}^{(1)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 8\lambda^3 + 14\lambda^2 + 7\lambda + 1$$

$$Inv_{(2)}^{(3)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 28\lambda^3 + 134\lambda^2 + 92\lambda + 1$$

Solutions to characteristic polynomials E_8

$$Inv_{(2)}^{(1)}: \quad \lambda^4+7\lambda^3+14\lambda^2+8\lambda+1$$

$$\lambda=\frac{1}{4}\left(-7\pm\sqrt{5}\pm\sqrt{30-6\sqrt{5}}\right)$$

Ito Coxeter BV D_8 (exponents 1, 3, 5, 7, 7, 9, 11, 13)

$$-Inv^{(1)}_{(2)} = 1.9499B_1 - 1.5637B_2 - 0.8678B_3$$

$$-Inv^{(2)}_{(2)} = -2.818B_1 - 0.3862B_2 + 2.4314B_3$$

$$-Inv^{(3)}_{(2)} = -0.696B_1 + 1.082B_2 - 3.513B_3$$

$$-Inv^{(4)}_{(2)} = 3.127B_1 + 1.735B_2 + 3.900B_3$$

- Exact solutions in terms of eigenvectors of the Cartan matrix

$$-Inv^{(1)}_{(2)} = 2\cos\frac{1\pi}{14}B_1 - 2\cos\frac{3\pi}{14}B_2 - 2\cos\frac{5\pi}{14}B_3$$

$$-\frac{1}{2}Inv^{(4)}_{(2)} = 2\cos\frac{3\pi}{14}B_1 + 2\cos\frac{5\pi}{14}B_2 + 2\cos\frac{1\pi}{14}B_3$$

- The sums of squares of these coefficients add to $7, 14, 14, 28$ – first term in characteristic polynomials (size); others?

Characteristic polynomials D_8

$\mathrm{Inv}^{(1)}_{(2)}, \mathrm{Inv}^{(2)}_{(2)}, \mathrm{Inv}^{(3)}_{(2)}, \mathrm{Inv}^{(4)}_{(2)}$ give 4 orthogonal bivectors with 'characteristic polynomial' (Hestenes)

$$0 = \sum_{m=0}^{k} \langle W_m^2 \rangle_0 (-\lambda_i)^{k-m}$$

$$Inv^{(1)}_{(2)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 7\lambda$$

$$Inv^{(2)}_{(2)}: \quad \lambda^4 + 14\lambda^3 + 49\lambda^2 + 7\lambda$$

$$Inv^{(3)}_{(2)}: \quad \lambda^4 + 14\lambda^3 + 21\lambda^2 + 7\lambda$$

$$Inv^{(4)}_{(2)}: \quad \lambda^4 + 28\lambda^3 + 224\lambda^2 + 448\lambda$$

Ito Coxeter BV A_8 (exponents 1, 2, 3, 4, 5, 6, 7, 8)

$$-Inv_{(2)}^{(1)} = \sqrt{3}B_1 - 0.684B_2 + 1.9696B_3 - 1.2856B_4$$

$$-Inv_{(2)}^{(2)} = -0.6015B_2 - 2.653B_3 + 3.255B_4$$

$$-Inv_{(2)}^{(3)} = -1.1305B_2 + 0.9216B_3 - 4.987B_4$$

$$-Inv_{(2)}^{(4)} = -\sqrt{3}B_1 - 0.839B_2 + 0.364B_3 + 5.671B_4$$

- Exact solutions in terms of eigenvectors of the Cartan matrix

$$-Inv_{(2)}^{(1)} = 2\cos\frac{3\pi}{18}B_1 - 2\cos\frac{7\pi}{18}B_2 + 2\cos\frac{1\pi}{18}B_3 - 2\cos\frac{5\pi}{18}B_4$$

- The sums of squares of these coefficients add to $9, 18, 27, 36$ – first term in characteristic polynomials (size); others?

$\mathrm{Inv}_{(2)}^{(1)}, \mathrm{Inv}_{(2)}^{(2)}, \mathrm{Inv}_{(2)}^{(3)}, \mathrm{Inv}_{(2)}^{(4)}$ give 4 orthogonal bivectors with 'characteristic polynomial' (Hestenes)

$$0 = \sum_{m=0}^{k} \langle W_m^2 \rangle_0 (-\lambda_i)^{k-m}$$

$$Inv_{(2)}^{(1)}: \quad \lambda^4 + 9\lambda^3 + 27\lambda^2 + 30\lambda + 9$$

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 18\lambda^3 + 81\lambda^2 + 27\lambda$$

$$Inv_{(2)}^{(3)}: \quad \lambda^4 + 27\lambda^3 + 54\lambda^2 + 27\lambda$$

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 36\lambda^3 + 126\lambda^2 + 84\lambda + 9$$

Characteristic polynomials and invariants E_6 (1, 4, 5, 7, 8, 11)

$$Inv_{(2)}^{(1)}: \quad \lambda^3 + 5\lambda^2 + 7\lambda + 3$$

$$Inv_{(2)}^{(2)}: \quad \lambda^3 + 8\lambda^2 + 4\lambda$$

$$Inv_{(2)}^{(3)}: \quad \lambda^3 + 17\lambda^2 + 43\lambda + 3$$

$$-Inv_{(2)}^{(1)} = 2\cos\frac{2\pi}{12}\hat{B}_1 + \hat{B}_2 + \hat{B}_3$$

$$-Inv_{(2)}^{(2)} = (-1 + 2\cos\frac{2\pi}{12})\hat{B}_2 + (-1 - 2\cos\frac{2\pi}{12})\hat{B}_3$$

$$-Inv_{(2)}^{(3)} = -2\cos\frac{2\pi}{12}\hat{B}_1 + (2 - 2\cos\frac{2\pi}{12})\hat{B}_2 + (2 + 2\cos\frac{2\pi}{12})\hat{B}_3$$

Solutions to characteristic polynomials E_6

$$Inv_{(2)}^{(1)}: \quad \lambda^3 + 5\lambda^2 + 7\lambda + 3 = (\lambda+3)(\lambda+1)^2$$

$$\Rightarrow \lambda = -3, -1, -1$$

$$Inv_{(2)}^{(2)}: \quad \lambda^3 + 8\lambda^2 + 4\lambda = \lambda(\lambda + 4 + 2\sqrt{3})(\lambda + 4 - 2\sqrt{3})$$

$$\Rightarrow \lambda = -4 - 2\sqrt{3}, -4 + 2\sqrt{3}, 0$$

$$Inv_{(2)}^{(3)}: \quad \lambda^3 + 17\lambda^2 + 43\lambda + 3 = (\lambda+3)(\lambda + 7 + 4\sqrt{3})(\lambda + 7 - 4\sqrt{3})$$

$$\Rightarrow \lambda = -7 - 4\sqrt{3}, -3, -7 + 4\sqrt{3}$$

Characteristic polynomials and invariants A_6 (1, 2, 3, 4, 5, 6)

$$Inv_{(2)}^{(1)}: \quad \lambda^3 + 7\lambda^2 + 14\lambda + 7$$

$$Inv_{(2)}^{(2)}: \quad \lambda^3 + 14\lambda^2 + 21\lambda + 7$$

$$Inv_{(2)}^{(3)}: \quad \lambda^3 + 21\lambda^2 + 35\lambda + 7$$

$$-Inv_{(2)}^{(1)} = 2\cos\frac{3\pi}{14}\hat{B}_1 + 2\cos\frac{1\pi}{14}\hat{B}_2 + 2\cos\frac{5\pi}{14}\hat{B}_3$$

Characteristic polynomials and invariants D_6 (1, 3, 5, 5, 7, 9)

$$2Inv_{(2)}^{(1)} + 2Inv_{(2)}^{(2)} + Inv_{(2)}^{(3)} = 0$$

$$Inv_{(2)}^{(1)}: \quad \lambda^3 + 5\lambda^2 + 5\lambda \Rightarrow \lambda = 0, -2-\tau, -2-\sigma$$

$$Inv_{(2)}^{(2)}: \quad \lambda^3 + 10\lambda^2 + 5\lambda \Rightarrow \lambda = 0, -3-4\tau, -3-4\sigma$$

$$Inv_{(2)}^{(3)}: \quad \lambda^3 + 20\lambda^2 + 80\lambda \Rightarrow \lambda = 0, -8-4\tau, -8-4\sigma$$

Indicative of the D_6-diagram folding to H_3 – two H_3-invariant subspaces.

$$-Inv_{(2)}^{(1)} = 2\cos\frac{1\pi}{10}B_1 - 2\cos\frac{3\pi}{10}B_3$$

Characteristic multivectors of Bivector exponentials: 3D

B a general unit bivector

$$W = \cos\theta + \sin\theta B$$

a bivector exponential

$$Inv_{(2)}^{(1)} = 2\sin(2\theta)B = Inv_{(2)}^{(2)}$$

$$Inv_{(0)}^{(1)} = 3\cos^2\theta - \sin^2\theta$$

Characteristic multivectors of Bivector exponentials: 4D

$$Inv_{(2)}^{(1)} = 2\sin(2\theta)B = Inv_{(2)}^{(3)}$$

$$Inv_{(2)}^{(2)} = 4\sin(2\theta)(B + B\cdot(B\wedge B)\sin^2\theta)$$

Characteristic multivectors of Bivector exponentials: 5D

$$Inv_{(2)}^{(1)} = 2\sin(2\theta)B$$

$$Inv_{(2)}^{(2)} = 2\sin(2\theta)(-3\cos 2\theta B + 2B\cdot(B\wedge B)\sin^2\theta)$$

also in 8D...

Characteristic multivectors of Bivector exponentials: in general

Why does it seem that $Inv^{(1)}_{(2)} = 2\sin(2\theta)B$?
Easy to prove in general

$$\partial_1 f_1 = \sum e^i \tilde{W} e_i W = \sum e^i (\cos\theta - \sin\theta B) e_i (\cos\theta + \sin\theta B)$$

$$\partial_1 f_1 = n\cos^2\theta - (n-4)\sin^2\theta (B|B + B\wedge B) + \frac{1}{2}((4-n)+n)\sin(2\theta)B$$

So indeed

$$Inv^{(1)}_{(2)} = 2\sin(2\theta)B$$

in generality.

Orthogonal blades

For a transformation corresponding to orthogonal blades

$$W = \exp(\theta B_1 + \phi B_2 + \xi B_3)$$

$$= (\cos\theta + \sin\theta B_1)(\cos\phi + \sin\phi B_2)(\cos\xi + \sin\xi B_3)$$

the invariants are analogously

$$Inv_{(2)}^{(1)} = 2\sin(2\theta)B_1 + 2\sin(2\phi)B_2 + 2\sin(2\xi)B_3$$

and so on, tying in with the factorisation of the Coxeter versor and its computed invariants.

Universes as Big Data: Superstrings, Calabi-Yau Manifolds and ML

Machine-Learning Mathematical Structures

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OriginalPaper

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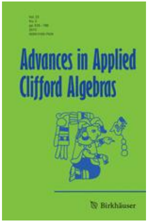
Kathryn Heal, Avinash Kulkarni, Emre Can Sertöz



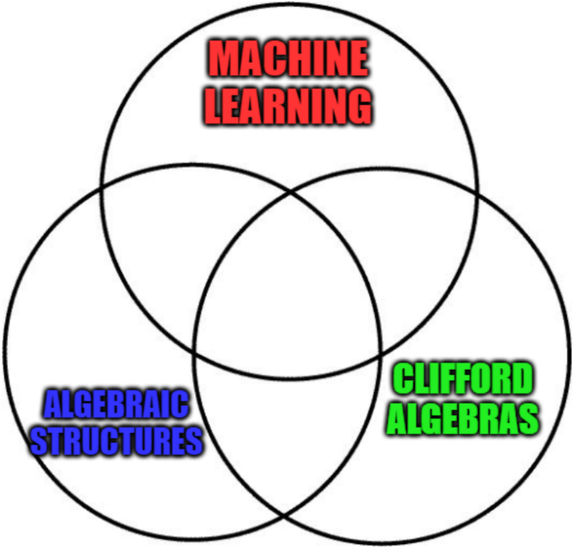
Topical Collection: Machine-learning mathematical structures

Editors: Y-H He, A Kasprzik, A Lukas, P-P Dechant, AACA

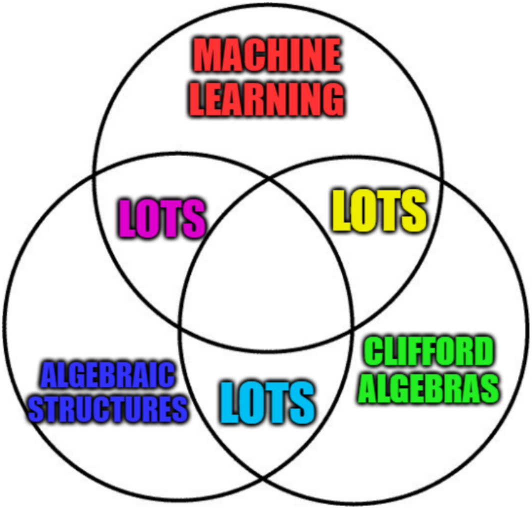
ICCA 2023 session: To machine learning and beyond – data science in mathematics, physics and engineering

Sebastià Xambó-Descamps, Isiah Zaplana Agut, YHH, PPD

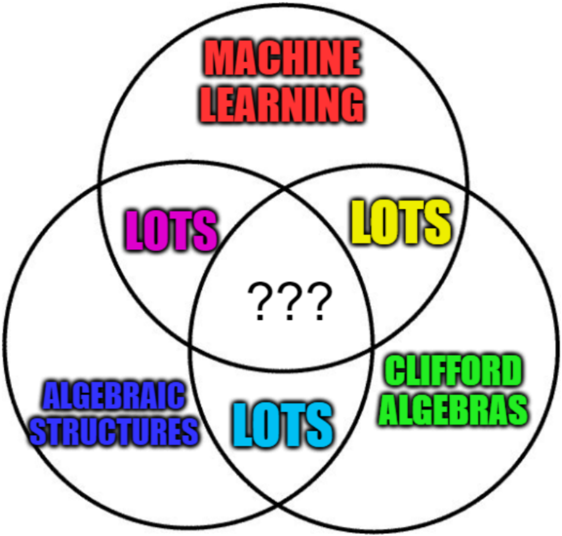
Motivation: the Topical Collection



Motivation: the Topical Collection



Motivation: the Topical Collection



A datamining pipeline

Computational Algebra

Use computational approaches (python, Sage etc) to calculate example cases. Use high-performance computing (HPC) to 'generate algebraic big data' either by

- sampling a subset of examples randomly (shotgun)
- calculating all cases exhaustively

$\Downarrow$

Dataset

$\Downarrow$

Data Science tool kit

Use standard data science tools such as NN, PCA, clustering, network analysis etc to find patterns in the data, formulate/test hypotheses etc.

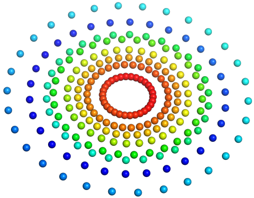
Team ML – cluster algebras and Clifford algebras

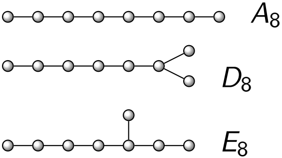
      

- Siqi Chen, Stony Brook University
- Mandy Cheung, Kavli IPMU, Japan
- Pierre-Philippe Dechant, University of Leeds
- Yang-Hui He, London Institute for Mathematical Sciences
- Elli Heyes, City/Imperial
- Edward Hirst, Queen Mary, University of London
- Jian Rong Li, University of Vienna
- Dmitrii Riabchenko, City, University of London

ML geometric invariants in Clifford algebra

- Input: a permutation of 8 vectors in 8D giving rise to a linear transformation e.g. a_1 to a_8
- Output: a set of geometric invariants of that linear transformation: nine 256D vectors
- Computational: computational algebra code computations (python, Clifford algebra package)
- Data Science: EDA, PCA, NN



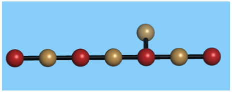


Code on GitHub

Machine Learning Clifford invariants of ADE Coxeter elements

Chen S, Dechant P-P, He Y-H, Heyes E, Hirst E, Riabchenko D, arXiv preprint arXiv:2310.00041 and Advances in Applied Clifford Algebras 34, 20 (2024)

Computational algebra, experimental mathematics, high performance computing and machine learning



$$W = \alpha_1\alpha_3\alpha_5\alpha_7\alpha_2\alpha_4\alpha_6\alpha_8 = S_\bullet S_\bullet$$

An ML problem / computational algebra and HPC

- The Coxeter elements can be computed in GA
- There are in principle $8! = 40320$ permutations $=$ 'big data'
- Calculate their invariants (galgebra package in python)

Data Science – can we machine learn the input to output mapping?

- Machine Learning and Neural Network classification
- Principal Component Analysis and Clustering
- Other computational/experimental aspects such as principal eigenvalue spectra

Machine learning

The ML problem

- Three sets of 40320 input vectors of format 'permutation' [0,1,2,3,4,5,6,7] (could use flattened root vectors instead)
- Output: $2^8 = 256$ multivector components (half redundant due to evenness) – 9 times!
- Expect great degeneracy and very good performance

Data Science results: near-perfect, low loss

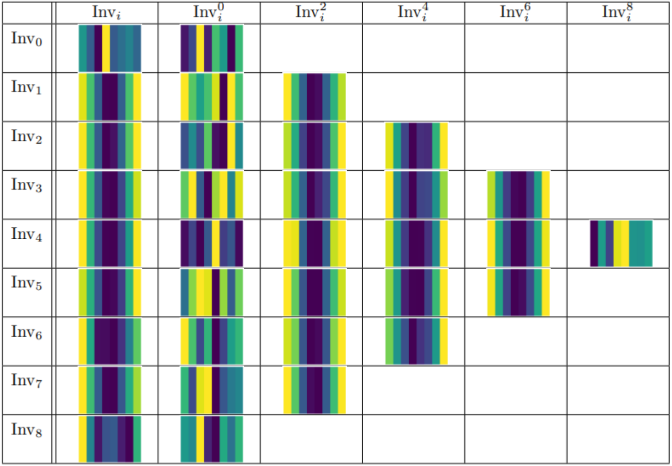
- Machine Learning: near-perfect prediction of output
- Neural Network classification: near-perfect ternary classification even for a simple 3-layer perceptron

ML prediction accuracy for invariants and subinvariants

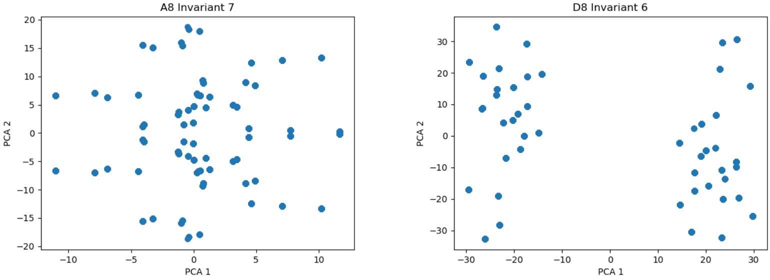
	$\mathrm{Acc}(\mathrm{Inv}_i)$	$\mathrm{Acc}(\mathrm{Inv}_i^0)$	$\mathrm{Acc}(\mathrm{Inv}_i^2)$	$\mathrm{Acc}(\mathrm{Inv}_i^4)$	$\mathrm{Acc}(\mathrm{Inv}_i^6)$	$\mathrm{Acc}(\mathrm{Inv}_i^8)$
Inv_0	1.0000	1.0000				
Inv_1	1.0000	1.0000	0.9955			
Inv_2	1.0000	1.0000	0.9912	0.9999		
Inv_3	0.9993	1.0000	0.9995	0.9999	1.0000	
Inv_4	0.9995	1.0000	0.9988	0.9891	0.9998	1.0000
Inv_5	0.9995	1.0000	0.9986	1.0000	1.0000	
Inv_6	1.0000	1.0000	1.0000	1.0000		
Inv_7	1.0000	1.0000	0.9999			
Inv_8	1.0000	1.0000				

TABLE 8. Summary of the final test accuracy (Acc) for the full invariants and each subinvariant of the 9 invariants for D_8 simple root data.

Gradient saliency for invariants and subinvariants



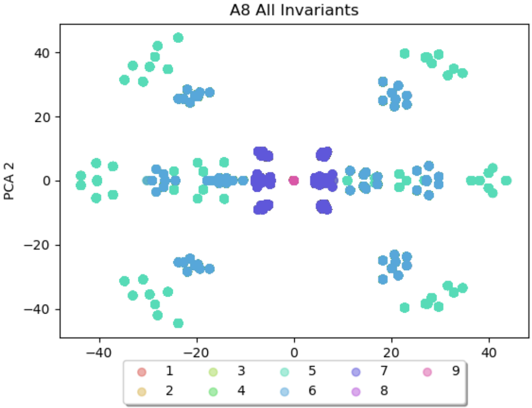
Results - PCA of Invariants



Perform PCA on the data set of invariants

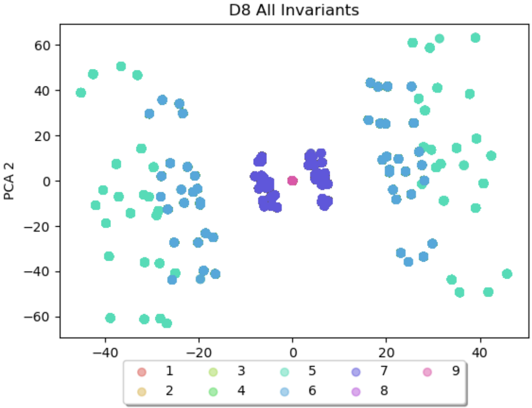
- Generally 2-fold reflection / rotation symmetry (Hodge duality?)
- Characteristic elbow drop of principal values at quite high n (but characteristic of A/D/E)

Results - PCA A_8



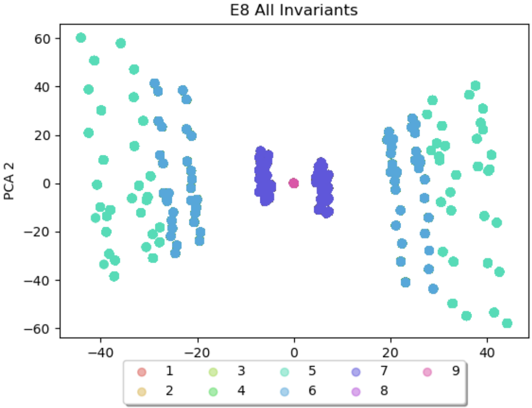
Project all invariants in the same plot.

Results - PCA D_8



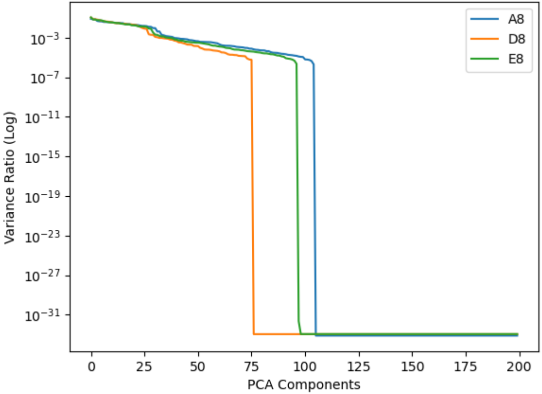
Project all invariants in the same plot.

Results - PCA E_8



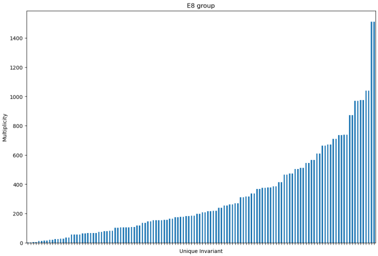
Project all invariants in the same plot.

Results - PCA Elbows



- Characteristic elbow drop of principal values at quite high n (but characteristic of A/D/E)

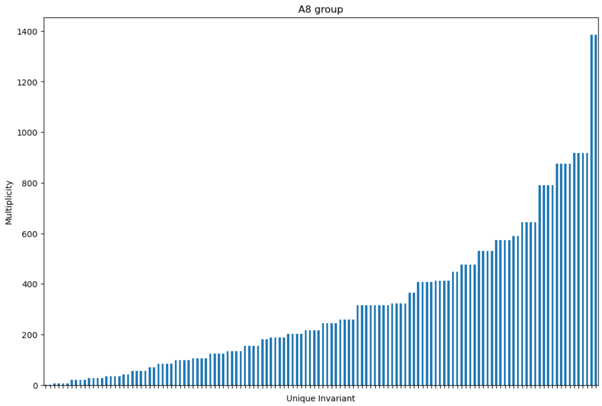
Results - frequencies E_8 (max 1511)



In fact, only 128 different sets of invariants

Corresponding to 256 inequivalent permutations, as bipartite and (anti)commutative properties mean there is an equivalence relation amongst permutations that yield the same Coxeter versor. Computations have shown analytic insights. Mostly doublets.

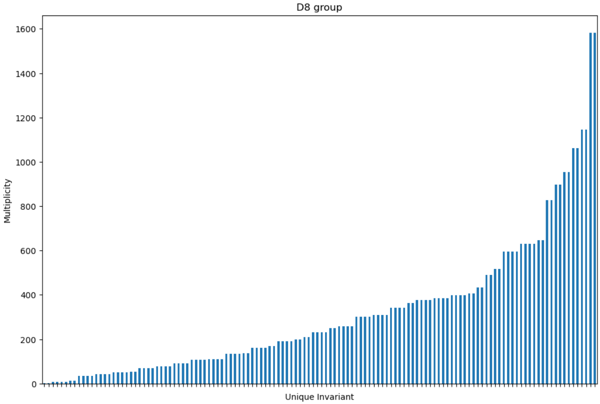
Results - frequencies A_8 (max 1385)



In fact, only 128 different sets of invariants (two unique)

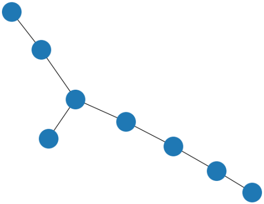
Mostly quadruplets.

Results - frequencies D_8 (max 1582)



In fact, only 128 different sets of invariants

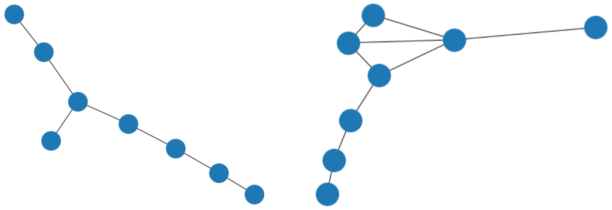
Half doublets and half quadruplets.



Lowest BV encodes the Dynkin diagram (for bipartite $W = s_{\bullet}s_{\bullet}$)

$$\mathrm{Inv}^{(1)}_{(2)} = 2a_1 \wedge a_2 - 2a_2 \wedge a_3 + 2a_3 \wedge a_4 - 2a_4 \wedge a_5 + 2a_5 \wedge a_6 + \dots$$

Results – lowest bivector

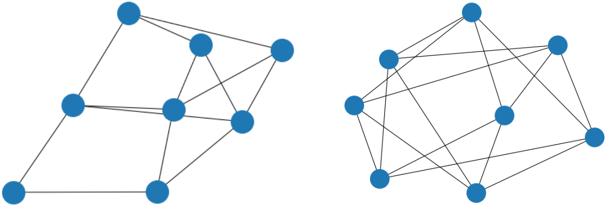


Lowest BV encodes the Dynkin diagram (for bipartite $W = s_{\bullet} s_{\bullet}$)

$$\mathrm{Inv}^{(1)}_{(2)} = 2a_1 \wedge a_2 - 2a_2 \wedge a_3 + 2a_3 \wedge a_4 - 2a_4 \wedge a_5 + 2a_5 \wedge a_6 + \dots$$

For other W permutations – get other types of diagrams

Results – other BV invariants: eigenvector centrality



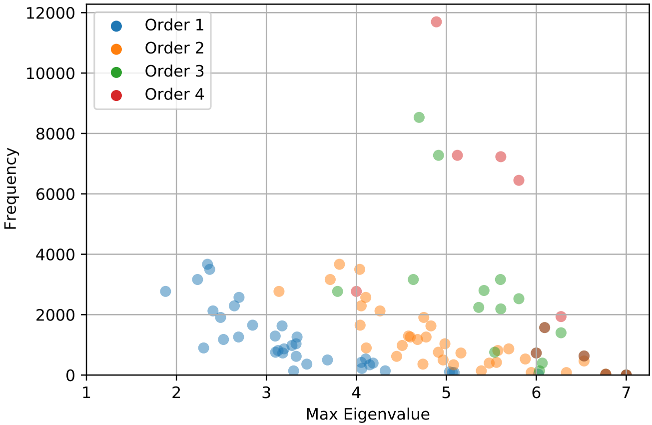
The second BV invariant is

$$\mathrm{Inv}^{(2)}_{(2)} = -2a_1 \wedge a_2 - 2a_1 \wedge a_4 + 4a_2 \wedge a_3 + 2a_2 \wedge a_5 - 4a_3 \wedge a_4 + \dots$$

Adjacency matrix

Interpret more broadly as a diagram - can study the principal eigenvalue distribution.

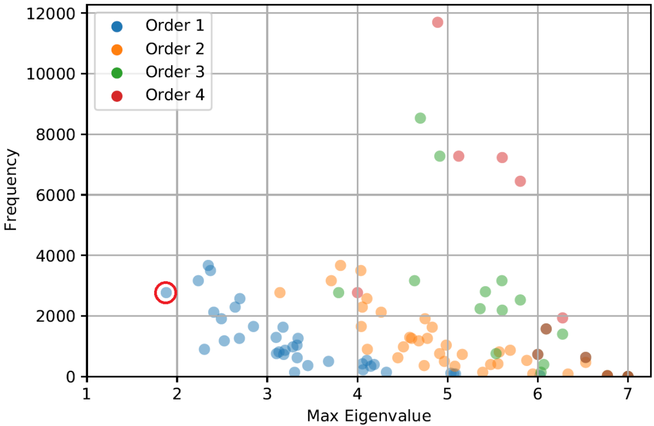
Results principal eigenvalues – A_8



For all the BV adjacency matrices, consider the principal eigenvalue

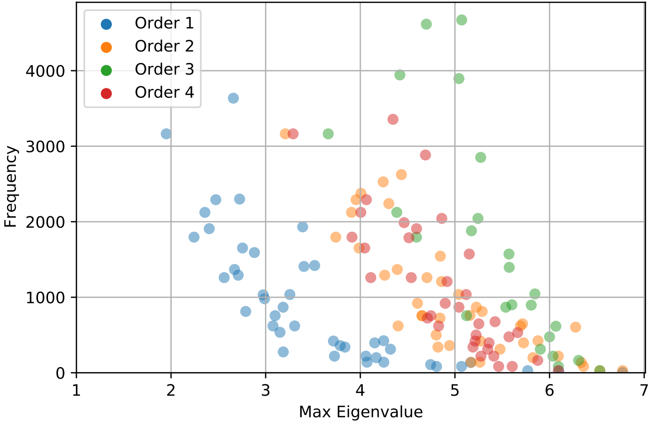
These principal eigenvalues cluster pretty well according to which invariant it came from (largely connectivity?).

Results principal eigenvalues – A_8



Smith's theorem cf earlier: the only diagrams with principal eigenvalue < 2 should be ADE – so the only one is A_8 as expected.

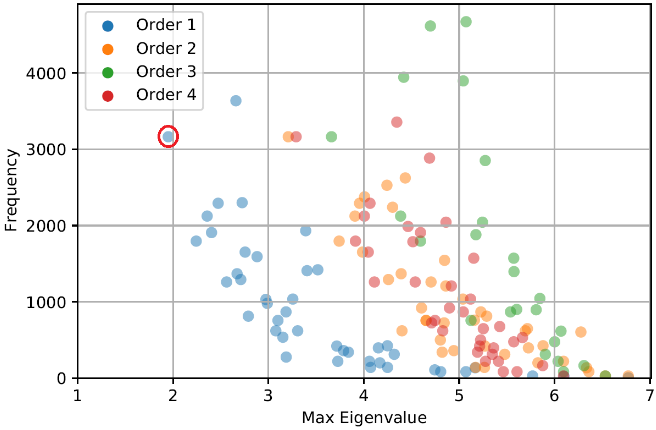
Results principal eigenvalues – D_8



For all the BV adjacency matrices, consider the principal eigenvalue

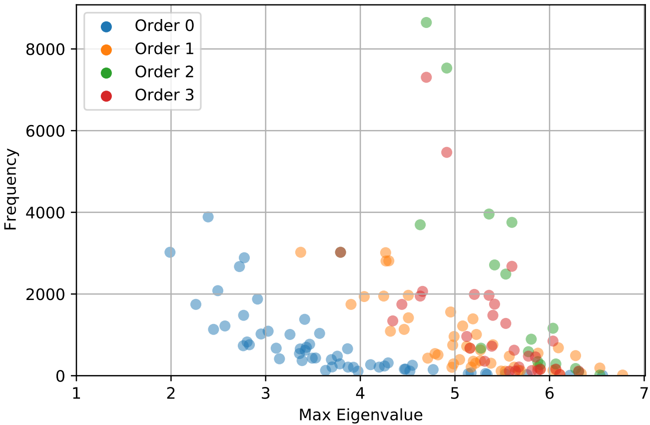
Some principal eigenvalues cluster and separate pretty well according to which invariant it came from.

Results principal eigenvalues – D_8



Smith's theorem cf earlier: the only diagrams with principal eigenvalue < 2 should be ADE – so the only one is D_8 as expected.

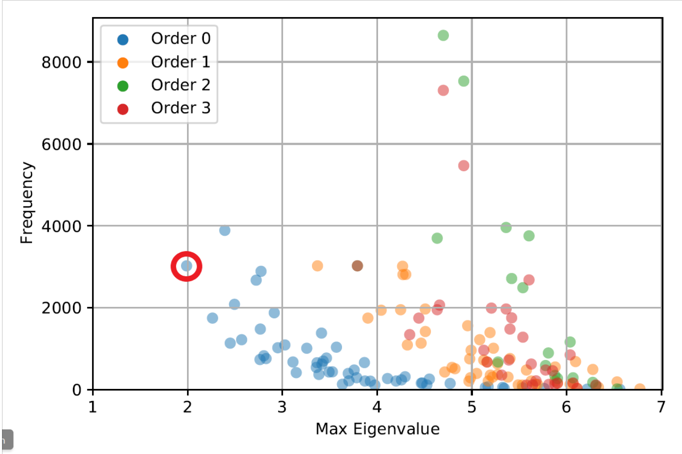
Results principal eigenvalues – E_8



For all the BV adjacency matrices, consider the principal eigenvalue

At least the lowest invariant's principal eigenvalues cluster and separate very clearly.

Results principal eigenvalues – E_8



Smith's theorem cf earlier: the only diagrams with principal eigenvalue < 2 should be ADE – so the only one is E_8 as expected.

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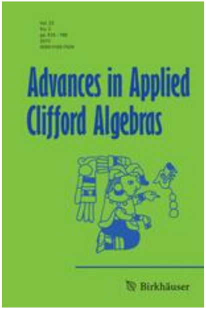


Topical Collection: Machine-learning mathematical structures

Editors: Y-H He, A Kasprzik, A Lukas, P-P Dechant, AACA

Conclusions

- Clifford algebra provides a very general way of doing (reflection) group theory (Cartan-Dieudonné)
- Clifford algebra provides a better way of understanding the geometry of Coxeter planes and invariants (degrees and exponents)
- Characteristic multivectors from simplicial derivatives of Coxeter elements – geometric interpretation
- Some new results on invariants of bivector exponentials in general and the Coxeter plane geometry in particular
- Computational algebra, data science and experimental mathematics can be used to guide intuition, extend our reach, and help formulate hypotheses

Thank you!

Some papers with further details

- Dechant P-P, From the Trinity (A_3, B_3, H_3) to an ADE correspondence, PRSA 474 (2220), 20180034
- Dechant P-P. Clifford Spinors and Root System Induction: H_4 and the Grand Antiprism. AACA. 2021 Jul;31(3):57.
- Chen S, Dechant P-P, He Y-H, Heyes E, Hirst E, Riabchenko D, Machine Learning Clifford invariants of ADE Coxeter elements, AACA 2024
- P-P Dechant, Y-H He, Machine-learning a virus assembly fitness landscape, PLoS ONE 16(5): 2021
- Dechant P-P, He YH, Heyes E, Hirst E. Cluster Algebras: Network Science and Machine Learning, J. Comp. Alg 2023
- Cheung MW, Dechant P-P, He YH, Heyes E, Hirst E, Li JR. Clustering Cluster Algebras with Clusters. ATMP 2024

Clifford algebra: no need for complexification

- Turns out in Clifford algebra we can factorise W into orthogonal (commuting/anticommuting) components $\boxed{W = \alpha_1 \dots \alpha_n = W_1 \dots W_n}$ with $\boxed{W_i = \exp(\pi m_i B_i / h)}$
- Here, B_i is a bivector describing a plane with $\boxed{B_i^2 = -1}$
- For v orthogonal to the plane described by I_i we have $\boxed{v \to \tilde{W}_i v W_i = \tilde{W}_i W_i v = v}$ so cancels out
- For v in the plane we have $\boxed{v \to wv \tilde{W}_i v W_i = \tilde{W}_i^2 v = \exp(2\pi m_i B_i / h) v}$
- Thus if we decompose W into orthogonal eigenspaces, in the eigenvector equation all orthogonal bits cancel out and one gets the complex eigenvalue from the respective eigenspace

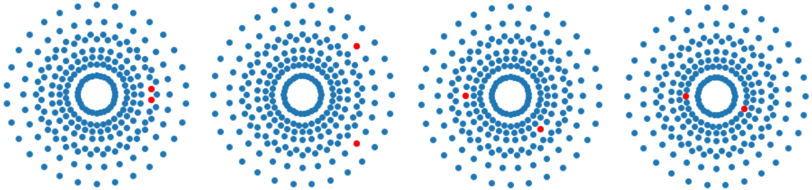
8D case: E_8

- E.g. H_4 has exponents $1, 11, 19, 29$, E_8 has

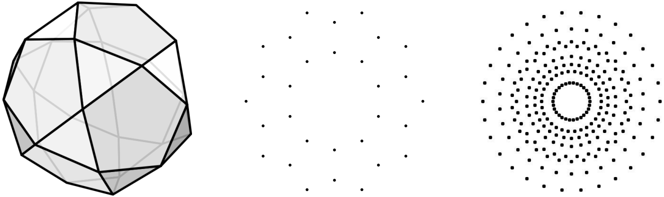
$$\boxed{1, 7, 11, 13, 17, 19, 23, 29}$$

- Coxeter versor decomposes into orthogonal components

$$\boxed{W = \alpha_1 \dots \alpha_8 = \exp(\frac{\pi}{30} B_C) \exp(\frac{7\pi}{30} B_2) \exp(\frac{11\pi}{30} B_3) \exp(\frac{13\pi}{30} B_4)}$$



- Every (for our purposes) Coxeter group has a Coxeter plane.
- A way to visualise Coxeter groups in any dimension by projecting their root system onto the Coxeter plane



Coxeter Elements, Degrees and Exponents

Standard exposition

"In order to bring the eigenvalues of the Coxeter element w into the picture, we have to complexify the situation".

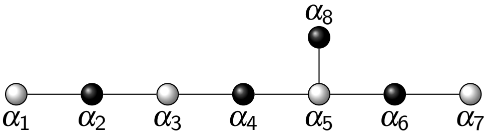
- The Coxeter element has complex eigenvalues of the form $\boxed{\exp(2\pi mi/h)}$ where m are called exponents
- Standard theory complexifies the real Coxeter group setting in order to find complex eigenvalues, then takes real sections again.
- In particular, 1 and $h-1$ are always exponents
- Turns out that actually exponents and degrees are intimately related ($\boxed{m=d-1}$). The construction is slightly roundabout but uniform, and uses the Coxeter plane.

Sums of powers of coeffs – obvious from cyclotomic stuff?

power	2	4	6	8	10
$E_8^{(1)}$	7	21	73	269	1022
$E_8^{(2)}$	8	36	197	1124	6478
$E_8^{(3)}$	7	21	73	269	1022
$E_8^{(4)}$	28	516	10972	240644	5315228
$D_8^{(1)}$	7	21	70	245	882
$D_8^{(2)}$	14	98	707	5194	38759
$D_8^{(3)}$	14	154	1883	23226	286699
$D_8^{(4)}$	28	336	4480	62720	903168
$A_8^{(1)}$	9	27	90	315	1134
$A_8^{(2)}$	18	162	1539	15066	150903
$A_8^{(3)}$	27	621	15390	382725	9519282
$A_8^{(4)}$	36	1044	33300	1070244	34420356

The Coxeter Plane

- In particular, can show every (for our purposes) Coxeter group has a Coxeter plane
- Existence relies on the fact that all groups in question have tree-like Dynkin diagrams, and thus admit an alternate colouring
- Essentially just gives two sets of mutually commuting generators



- Essentially just gives two sets of orthogonal = mutually commuting generators but anticommuting root vectors α_w and α_b (duals ω or α^b and α^w)
- Cartan matrices are positive definite, and thus have a Perron-Frobenius (all positive) eigenvector λ_i (principal eigenvalue).
- Take linear combinations of components of this eigenvector as coefficients of two vectors from the orthogonal sets $v_w = \sum \lambda_w \alpha^w$ and $v_b = \sum \lambda_b \alpha^b$

$$v_w = \lambda_1\alpha^1 + \lambda_7\alpha^7 + \lambda_3\alpha^3 + \lambda_5\alpha^5, v_b = \lambda_2\alpha^2 + \lambda_6\alpha^6 + \lambda_4\alpha^4 + \lambda_8\alpha^8$$

- Their outer product/Coxeter plane bivector $B_C = v_b \wedge v_w$ describes an invariant plane where w acts by rotation by $2\pi/h$.

$$8W_{(2)} = -a_1\wedge a_2 + a_1\wedge a_4 - a_1\wedge a_8 - a_3\wedge a_4 + a_3\wedge a_8 - a_5\wedge a_8 + a_6\wedge a_7 + a_7$$

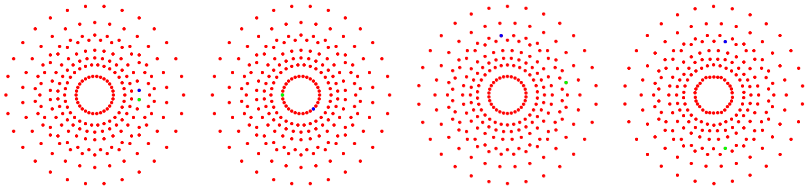
$$\begin{aligned}4W_{(4)} = &- a_1\wedge a_2\wedge a_3\wedge a_4 + a_1\wedge a_2\wedge a_3\wedge a_8 - a_1\wedge a_2\wedge a_5\wedge a_8 + a_1\wedge a_2\wedge\\ &+ a_1\wedge a_4\wedge a_5\wedge a_8 - a_1\wedge a_4\wedge a_6\wedge a_7 - a_1\wedge a_4\wedge a_7\wedge a_8 + a_1\wedge a_6\wedge\\ &+ a_3\wedge a_4\wedge a_6\wedge a_7 + a_3\wedge a_4\wedge a_7\wedge a_8 - a_3\wedge a_6\wedge a_7\wedge a_8 + a_5\wedge a_6\wedge\end{aligned}$$

$$\begin{aligned}2W_{(6)} = &- a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5\wedge a_8 + a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_6\wedge a_7 + a_1\wedge a_2\wedge\\ &- a_1\wedge a_2\wedge a_3\wedge a_6\wedge a_7\wedge a_8 + a_1\wedge a_2\wedge a_5\wedge a_6\wedge a_7\wedge a_8 - a_1\wedge a_4\wedge\\ &+ a_3\wedge a_4\wedge a_5\wedge a_6\wedge a_7\wedge a_8\end{aligned}$$

$$W_{(8)} = a_1\wedge a_2\wedge a_3\wedge a_4\wedge a_5\wedge a_6\wedge a_7\wedge a_8$$

- Turns out instead of just taking Perron-Frobenius eigenvector, can just take the other eigenvectors of the Cartan matrix too
- These give 4 orthogonal planes

$$W = \alpha_1 \dots \alpha_8 = \exp(\frac{\pi}{30}B_C)\exp(\frac{7\pi}{30}B_2)\exp(\frac{11\pi}{30}B_3)\exp(\frac{13\pi}{30}B_4)$$



Related to the Lasenbys' Brno talks about eigenbivectors of matrices from complex eigenvectors.

Solutions to characteristic polynomials E_8

$$Inv^{(2)}_{(2)}: \quad \lambda^4+8\lambda^3+14\lambda^2+7\lambda+1$$

$$\lambda=\frac{1}{2}\left(-4\pm\sqrt{5}\pm\sqrt{15-6\sqrt{5}}\right)$$

Solutions to characteristic polynomials E_8

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 28\lambda^3 + 134\lambda^2 + 92\lambda + 1$$

$$\lambda = -7 \pm 2\sqrt{5} \pm 2\sqrt{15 - 6\sqrt{5}}$$

Solutions to characteristic polynomials D_8

$$Inv^{(1)}_{(2)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 7\lambda$$

$$\Rightarrow \lambda^3 + 7\lambda^2 + 14\lambda + 7$$

$$\lambda = 0, -3.8019, -2.4450, -0.75302$$

Solutions to characteristic polynomials D_8

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 14\lambda^3 + 49\lambda^2 + 7\lambda$$

$$\lambda = 0, -7.9390, -5.9119, -0.14914$$

Solutions to characteristic polynomials D_8

$$Inv^{(3)}_{(2)}: \quad \lambda^4 + 14\lambda^3 + 21\lambda^2 + 7\lambda$$

$$\lambda = 0, -12.345, -1.17092, -0.48427$$

Solutions to characteristic polynomials D_8

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 28\lambda^3 + 224\lambda^2 + 448\lambda$$

$$\lambda = 0, -15.208, -9.7802, -3.0121$$

Solutions to characteristic polynomials A_8

$$Inv_{(2)}^{(1)}: \quad \lambda^4 + 9\lambda^3 + 27\lambda^2 + 30\lambda + 9$$

A mess in terms of cubic roots of one

$$\lambda = -3, -3.8794, -1.6527, -0.46791$$

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 18\lambda^3 + 81\lambda^2 + 27\lambda$$

$$\lambda = 0, -10.596, -7.0419, -0.36184$$

Solutions to characteristic polynomials A_8

$$Inv_{(2)}^{(3)}: \quad \lambda^4 + 27\lambda^3 + 54\lambda^2 + 27\lambda$$

$$\lambda = 0, -24.873, -1.2781, -0.84936$$

(some square roots of 37!)

Solutions to characteristic polynomials A_8

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 36\lambda^3 + 126\lambda^2 + 84\lambda + 9$$

$$\lambda = -3, -32.163, -0.70409, -0.13247$$