Projective and Conformal Formulations of Electromagnetism

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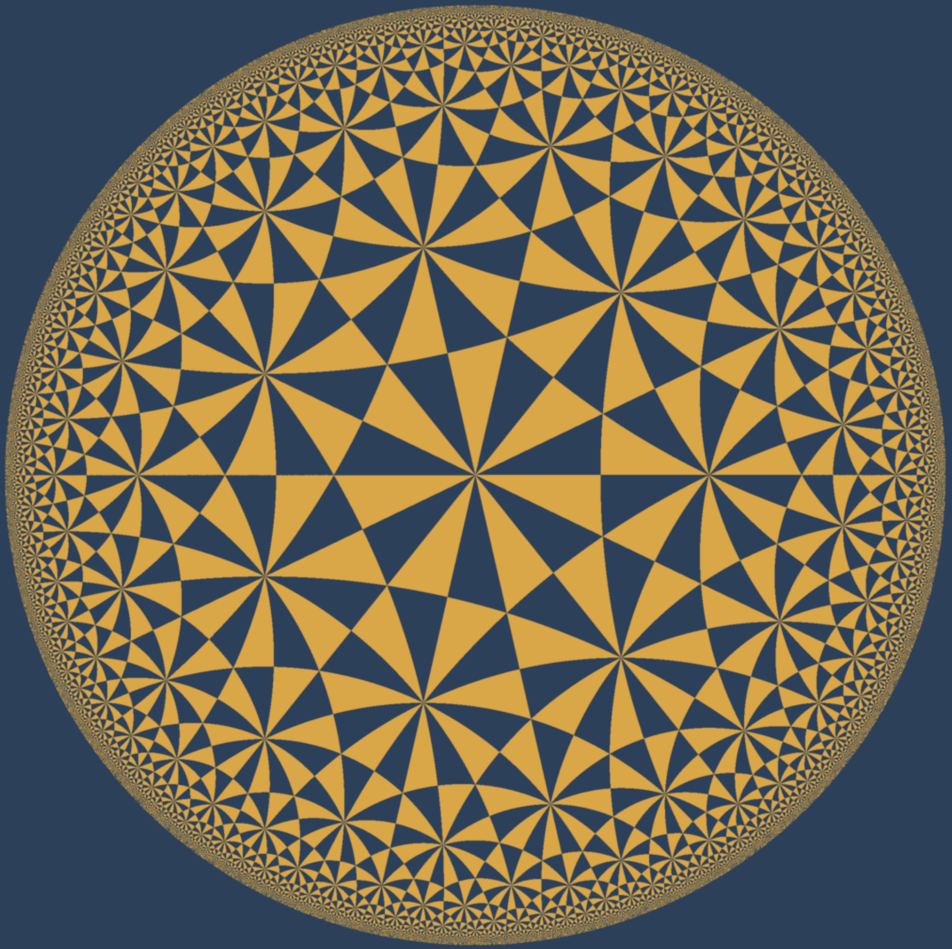


Outline

Many equations in physics exhibit some form of conformal invariance

- Electromagnetism
- Yang-Mills
- (massless) Dirac equation
- Cauchy-Riemann equations

Been suggested that conformal GA might help in exhibiting this invariance
But CGA takes place with a higher dimensional base space.

Can we find a way to make conformal GA (CSTA) useful in field theory?

Cauchy-Riemann equations

Start with the simplest example in 2D

$$\nabla\psi(x) = 0 \qquad \nabla = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} \qquad \psi = u + I_2 v$$

Get new solutions from old by translation and rotation (active viewpoint)

$$\psi'(x) = \psi(x+a)$$
$$\psi'(x) = R\psi(\tilde{R}xR)\tilde{R}$$

Note, have to back-rotate the position dependence.
Picture is different in gauge-theory gravity.

Inversion

Need one further symmetry to generate the conformal group: Inversion

$$x' = f(x) = \frac{-1}{x} = \frac{-x}{x^2}$$

Transformation

$$\underline{f}(a) = a \cdot \nabla f(x)$$

$$= \frac{-a}{x^2} + \frac{2a \cdot x\, x}{x^4}$$

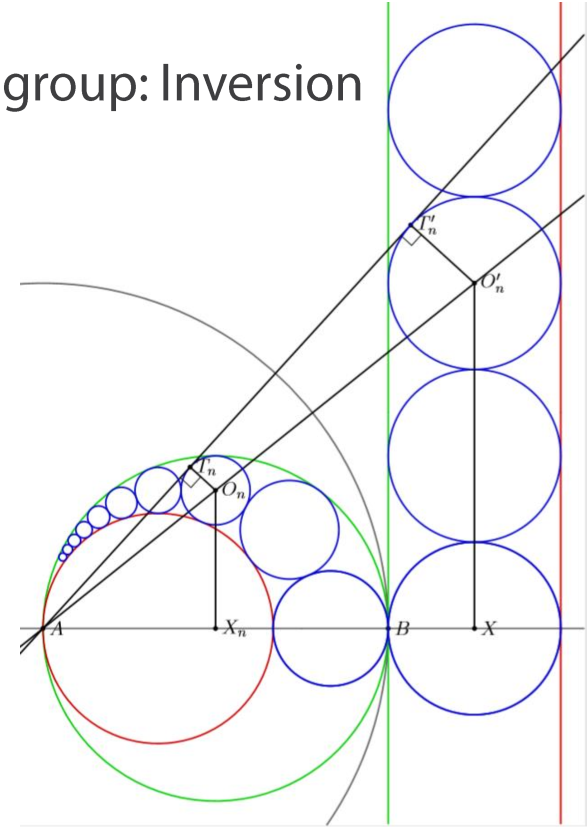
$$= \frac{xax}{x^4}$$

Directional derivatives

A position-dependent reflection

$$\nabla_{x'} = x \nabla_x \check{x}$$

The check denotes that this term is not differentiated



Inversion

Need a single-sided transformation (like a spinor transformation)

$$\psi'(x) = \frac{x}{x^2}\psi(-1/x)e_1$$ ← The real axis

$$\phi'(z) = \frac{1}{z}\phi(1/z^*)^*$$

Ignore the delta function here!

$$\begin{aligned}\nabla \frac{x}{x^2}\psi(-1/x)e_1 &= \nabla\left(\frac{x}{x^2}\right)\psi(x')e_1 + \dot{\nabla}\frac{x}{x^2}\dot{\psi}(-1/x)e_1 \\ &= \frac{x}{x^4}x\nabla_x\check{x}\psi(x')e_1 \\ &= \frac{x}{x^4}\nabla_{x'}\psi(x')e_1 = 0\end{aligned}$$

Key derivation

Special conformal transformations

Invert / translate / invert

$$x' = \frac{x + tx^2}{1 + 2t\cdot x + x^2t^2} = x\frac{1}{1+tx}$$

$$\underline{f}(a) = \frac{1}{1+xt}a\frac{1}{1+tx}$$

Position-dependent rotation and scaling

$$R = \frac{1+tx}{(1+2t\cdot x + x^2t^2)^{1/2}}$$

In d dimensions find:

$$\psi'(x) = \frac{1}{(1+2t\cdot x + x^2t^2)^{(d-1)/2}}R\psi(-1/x)$$

Conformal weight

$$\nabla\frac{1+xt}{(1+2x\cdot t + x^2t^2)^{d/2}} = 0$$

This makes it all work!

Electromagnetism

Bit different – only get conformal invariance in 4D.

$$A'(x) = \frac{1}{x^4} x A(-1/x) x$$

$$F'(x) = \frac{1}{x^6} x F(-1/x) x$$

Double-sided transformations under inversion

Conformal invariance relies on the identity (unique to bivector in spacetime)

$$\dot{\nabla} x F \dot{x} = 2 F x$$

Can include currents and have inversion invariance

$$\nabla F = J \implies \nabla F' = J'$$

$$J'(x) = \frac{1}{x^8} x J(-1/x) x$$

Different transformation law for J

Weyl (dilation) invariance

Picture once gravity is included is a bit different

In GTG picture already have local invariance under rotations and general mappings

All that is left is dilations:

$$g_{\mu\nu}dx^{\mu}dx^{\nu} \mapsto \Omega^{2}(y)g_{\mu\nu}dx^{\mu}dx^{\nu} \qquad \overline{h} \mapsto \Omega^{-1}\overline{h}$$

GTG version

Already invariant

- Dirac theory
- Yang-Mills
- Yukawa coupling (used in Higgs)
- Higgs 4^{th} order term

Not invariant

- Higgs dynamic term (needs a gauge field)
- Higgs 2^{nd} order term
- Gravity Ricci scalar term
- Conformal anomaly

Conformal geometry

Dirac was first to suggest that the Projective Null Cone (PNC) is the natural space for EM theory.

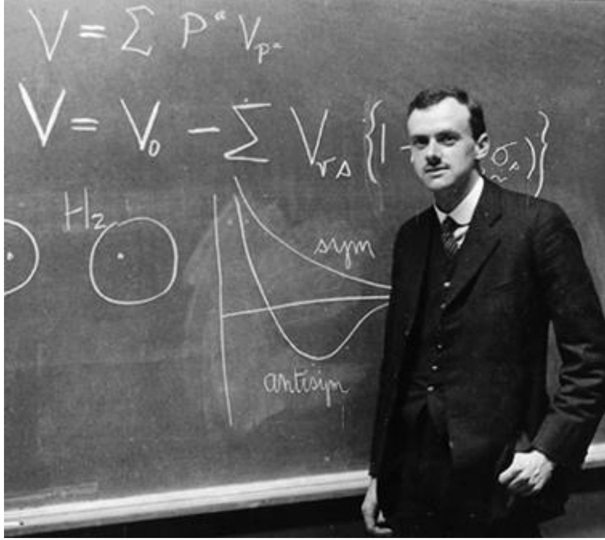
$$X^2 = 0$$

$$\psi_c(\lambda X) = \lambda^{-c}\psi_c(X)$$

c is conformal weight

Core ideas

- Define a map from PNC back to spacetime
- Find equations on the PNC that reproduce desired equations in spacetime
- Ensure the PNC equations are manifestly covariant



Conformal Invariance and Electrodynamics: Applications and General Formalism

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Notation

Working in CSTA, G(2,4)

$$X = P(X) + \tfrac{1}{2}un + \tfrac{1}{2}v\bar{n} \qquad n^2 = \bar{n}^2 = 0, \qquad n\cdot\bar{n} = 2$$

Projection back to base space

Derivative in CSTA

$$u = \bar{n}\cdot X \qquad v = n\cdot X$$

$$\nabla_X u = \bar{n} \qquad \nabla_X v = n$$

$$\nabla_X = P(\nabla_X) + \bar{n}\partial_u + n\partial_v$$

$$X\cdot\nabla_X = P(X)\cdot\nabla_X + u\partial_u + v\partial_v$$

Standard map

$$X(x) = x - \tfrac{1}{2}\bar{n} + \tfrac{1}{2}x^2 n$$

From STA to CSTA

Consequence of scale relationship

$$X\cdot\nabla_X\psi_c(X) = -c\psi_c(X)$$

This takes a bit of work to prove

Scalars

Conformal weight 0, so have

$$\phi(x) = \Phi[X(x)]$$

$$\nabla_x(\phi(x)) = \nabla_x \Phi[X(x)]$$

$$= P(\nabla_X \Phi(X)) + 2x\partial_u \phi \Big|_N$$

Restriction of a field to the PNC

$$\nabla_x(\phi(x)) = P\big(-n\cdot(X \wedge \nabla_X \Phi(X))\big)$$

Key projection operation

'Gauge' invariance

$$\Phi \mapsto \Phi + \boxed{\tfrac{1}{2}X^2\alpha}$$

This term vanishes on the PNC

$$\nabla_X \Phi_X \Big|_N \mapsto \nabla_X \Phi_X \Big|_N + \boxed{X\alpha}$$

This term potentially unknown

‘Gauge’ invariance

See from previous that vectors have are equivalent up to

$$\mathcal{A} \mapsto \mathcal{A} + X\alpha$$

Does not change STA physics

Need operators that commute with X^2:

$$X \cdot \nabla_X$$

$$X \wedge \nabla_X$$

Guess what we are going to do with these!

Also have $X \cdot \nabla_X \Phi = 0 \quad \rightarrow \quad X \cdot \mathcal{A}\big|_N = 0$

Critical to conformal invariance

Transformations

Want to use linear transformations in CSTA to achieve conformal transformations in STA:

$$\mathcal{A}(X) \mapsto \mathcal{A}'(X) = R\mathcal{A}(\tilde{R}XR)\tilde{R}$$

Lorentz transformations are obvious:

$$\begin{aligned} A'(x) &= P(-n\cdot(X\wedge R\mathcal{A}[\tilde{R}XR]\tilde{R})) \\ &= RP\Big(\mathcal{A}[X(x')] + X(x')n\cdot\mathcal{A}[X(x')]\Big)\tilde{R} \\ &= RA(\tilde{R}xR)\tilde{R} = RA(x')\tilde{R} \end{aligned}$$

R is an STA rotor here

As expected

Translations

Slightly more complicated

$$T = 1 + \tfrac{1}{2}tn$$

$$X' = \tilde{T}X(x)T = X(x+t)$$

$$\mathcal{A}(X) \mapsto \mathcal{A}'(X) = T\mathcal{A}(X')\tilde{T}$$

Basic translation in CSTA

Now $$A' = P\Big(-Tn\cdot(X'\wedge\mathcal{A}')\tilde{T}\Big)$$

$$= -P\Big(n\cdot(X'\wedge\mathcal{A}') + (t\wedge n)\cdot(n\cdot(X'\wedge\mathcal{A})) + \tfrac{1}{4}tnn\cdot(X'\wedge\mathcal{A})nt\Big)$$

$$= P\Big(-n\cdot(X'\wedge\mathcal{A}')\Big)$$

$$= A(x+t)$$ ✓

Rely on P(n)=0

Inversion

Now things get a bit tougher

Conformal weight 1

$$X' = X(-1/x) = -\frac{1}{x^2}\bar{e}X\bar{e}$$

$$\mathcal{A}'(X) = -\bar{e}\mathcal{A}(-\bar{e}X\bar{e})\bar{e} = -\frac{1}{x^2}\bar{e}\mathcal{A}(X')\bar{e}$$

$$A'(x) = P\Big(\mathcal{A}' + x\,n\cdot\mathcal{A}'\Big) = -\frac{1}{x^2}P\Big(\mathcal{A}(X') + x\,\bar{n}\cdot\mathcal{A}(X')\Big)$$

The e-bar terms cancel and transform an n to n-bar.
But result does not look anything like an inverted A field.

Inversion 2

Need to use the additional constraint

$$X' \cdot \mathcal{A}(X') = (-x^{-1} - \tfrac{1}{2}\bar{n} + 1/x^2 n) \cdot \mathcal{A}(X') = 0$$

$$\bar{n} \cdot \mathcal{A}(X') = \frac{1}{x^2}(-2x + n) \cdot \mathcal{A}(X')$$

Now have

$$A'(x) = \frac{1}{x^4} P\Big(-x^2 \mathcal{A}(X') + 2x\, x \cdot \mathcal{A}(X') - x\, n \cdot \mathcal{A}(X')\Big)$$

$$= \frac{1}{x^4} P\Big(x \mathcal{A}(X') x - x\, n \cdot \mathcal{A}(X')\Big)$$

$$= \frac{1}{x^4} x A(-1/x) x \quad \checkmark$$

Almost magical!

Bivectors

Before constructing field equations, need the map for bivectors

$$
\begin{aligned}
F = \nabla\wedge A &= P\Big(\nabla_x\wedge\mathcal{A}[X(x)] - x\wedge\nabla_x(n\cdot\mathcal{A}[X(x)]]\Big) \\
&= P\Big(\nabla_X\wedge\mathcal{A} + 2x\wedge\partial_u\mathcal{A} - x\wedge\nabla_X(n\cdot\mathcal{A}\Big) \\
&= P\Big(\mathcal{F} + X\wedge(n\cdot\ \nabla_X\mathcal{A}) - X\wedge\nabla_X(n\cdot\mathcal{A})\Big) \\
&= P\Big(\mathcal{F} + X\wedge(n\cdot\mathcal{F})\Big) \\
&= P\Big(-n\cdot(X\wedge\mathcal{F})\Big)
\end{aligned}
$$

Definition of CSTA *F*

General transformation law

'Gauge' invariance 2

Even more invariance on PNC now

$$\mathcal{A} \mapsto \mathcal{A} + \tfrac{1}{2}X^2\mathcal{B} \qquad \mathcal{F} \mapsto \mathcal{F} + X\wedge\mathcal{B}$$

Better to work with invariant quantities and operators

$$X\wedge\mathcal{F} = X\wedge\nabla\wedge\mathcal{A}$$

Contains all physical information

Can also show

$$X\cdot\mathcal{F}\Big|_N = \phi X$$

$$\nabla\cdot(X\cdot\mathcal{F}) = 2\phi$$

$$\phi \mapsto \phi - X\cdot\mathcal{B}$$

Another gauge field!

Field equations

One of the two equations is immediate:

$$X \wedge \nabla \wedge \mathcal{F} = 0$$

Covariant equation

$$\nabla_x \wedge F(x) = P\big(-n\cdot(X \wedge \nabla \wedge \mathcal{F})\big) = 0$$

Want to find a second covariant equation. Will then check it is correct.
Key identity is

$$X\nabla(XM_3) = 6XM_3 + X\dot{\nabla}X\dot{M}_3$$
$$= 6XM_3 + 2X\,X\cdot\nabla M_3 = 0$$

For any field of weight 3

Now $\mathcal{F} \mapsto \mathcal{F} + X \wedge \mathcal{B}$ ← Weight 3. We could eliminate this using above if we found a term going as $X\cdot\mathcal{B}$

Field equations

But we know $\phi \mapsto \phi - X \cdot \mathcal{B}$

Look at $X\nabla(\mathcal{F} - \phi)$

Pure bivector $\langle X\nabla\mathcal{F}\rangle = \phi$

Killed by X $= X \wedge \mathcal{J}$

$$X\nabla(\mathcal{F} - \phi) = X \wedge \mathcal{J}$$

Maxwell equations in CSTA

Note $X \cdot \mathcal{J} = 0$ $\mathcal{J} \mapsto \mathcal{J} + \alpha X$ Yet another gauge field!

To eliminate ϕ need second order equation (surprising)

$$X\nabla^2(X \wedge \mathcal{F}) = 2X\nabla(\mathcal{F} - \phi) = 2X \wedge \mathcal{J}$$

Spacetime equations

Still need to verify that our covariant equation reproduces the Maxwell equations. Have

$$\nabla_x \cdot F = P\Big((n\wedge\nabla_x)\cdot(X\wedge\mathcal{F})\Big)$$

$$n\wedge\nabla_x = n\wedge(-n\cdot(X\wedge\nabla_X))$$

Requires a short proof

Find

$$\nabla\cdot F = \tfrac{1}{2}P\langle nX\nabla\,(nX\wedge\mathcal{F})\rangle_1$$
$$= n\cdot XP\langle n\cdot(X\wedge\mathcal{J})\rangle_1 = J \quad \checkmark$$

All working as expected. Back in spacetime know we must have current conservation. Oddly this is harder to establish in CSTA.

Current conservation

Need a covariant expression of this.

Find

$$\nabla_x \cdot J(x) = -(n \cdot X)^2 (X \wedge \nabla_X) \cdot \left(\frac{\mathcal{J} \wedge n}{X \cdot n} \right)$$

$$X \nabla (X \wedge \mathcal{J}) = 0$$

Direct consequence of CSTA Maxwell equations

Together, these require

$$(X \wedge \nabla) \cdot \mathcal{J} = \mathcal{J}$$

CSTA current conservation

The current is an eigenstate of the CSTA angular momentum operator!

Now have a complete set of equations.

STA or CSTA?

Maxwell equations in CSTA

$$X\nabla(\mathcal{F}-\phi) = X\wedge\mathcal{J}$$

$$X\cdot\mathcal{F}\Big|_N = \phi X$$

$$\nabla\cdot(X\cdot\mathcal{F}) = 2\phi$$

$$(X\wedge\nabla)\cdot\mathcal{J} = \mathcal{J}$$

$$X\cdot\mathcal{J} = 0$$

Maxwell equations in STA

$$\nabla F = J$$

Making conformal invariance explicit comes at some cost!

Point charge

Can always lift objects into the 'obvious' gauge

$$\mathcal{F} = \frac{X\cdot(n\wedge F)}{(X\cdot n)^3}$$

Has correct conformal weight

For a point charge at rest at the origin $F = \frac{1}{4\pi r^3} x\wedge\gamma_0$

$$\mathcal{F} = \frac{1}{4\pi(X\cdot n\, r)^3} X\cdot(n\wedge x\wedge\gamma_0)$$

Gauge equivalent to

$$\mathcal{F} = -\frac{X\cdot L}{4\pi|X\wedge L|^3}$$

$$L = N\gamma_0$$

Worldline of the particle

Constant acceleration

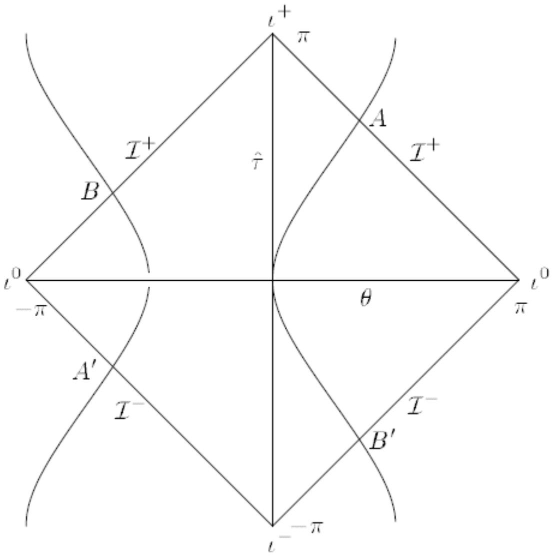
A special conformal transformation takes a line to a hyperbola.

Trajectory of a particle with constant acceleration.

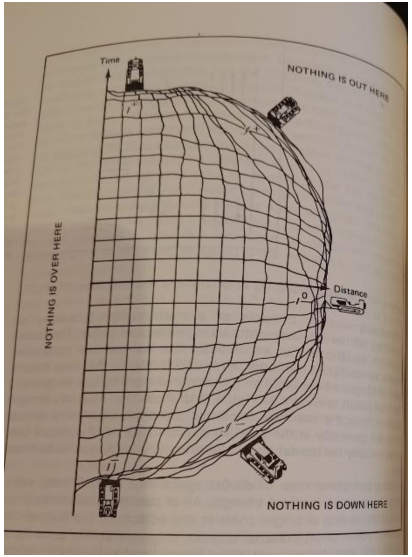
$$x + t = \tan \tfrac{1}{2}(\tau + \sigma)$$

$$t - x = \tan \tfrac{1}{2}(\tau - \sigma)$$

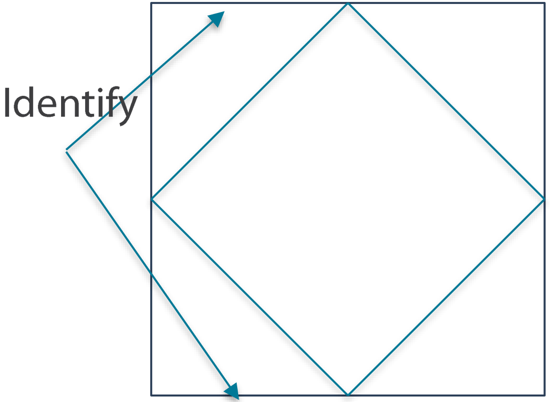
Torus (in 2D)



From Codirla & Osborn



W. J. Kaufmann



All of CSTA space

Spinors

Using ESA spinors, so $\hat{a}|\psi\rangle \mapsto a\psi\gamma_0$

In CSTA define

$$\begin{aligned}\Psi &= \Psi\tfrac{1}{2}(1+N) \\ &= (\psi_1 + e\psi_2\gamma_0)\tfrac{1}{2}(1+N)\end{aligned}$$

Pair of Dirac spinors

Projection

$$P(\Psi) = \psi_1$$
$$P(n\Psi\gamma_0) = 0$$

Ensures translation works

All spacetime transformations work. Just need inversion for full conformal symmetry

Spinor inversion

Inversions takes $\Psi(X) \mapsto \Psi'(X) = \bar{e}\Psi(-\bar{e}X\bar{e})\gamma_0$ $\qquad X' = -\frac{1}{x^2}\bar{e}X\bar{e}$

Single-sided spinor action

So $\quad P(\Psi') = \frac{1}{(x^2)^c}\psi_2(-1/x)$

To make this all work, need

$$\psi_2 = x\psi_1\gamma_0$$

$$\begin{aligned}\Psi &= (\psi_1 + ex\psi_1)\tfrac{1}{2}(1+N)\\ &= Xn\psi_1\tfrac{1}{2}(1+N)\end{aligned}$$

Get our restriction $\quad X\Psi\Big|_N = 0$

Manifestly covariant in CSTA

Spinor inversion

Now find

$$\psi'(x) = -P\left(\frac{1}{(x^2)^c}\bar{n}/2\Psi(X')\gamma_0\right)$$

$$= \frac{x}{(x^2)^{c+1}}\psi(-1/x)\gamma_0$$

Works provided c=1

Also $X\nabla(X\Psi) = 0 \quad aX\Psi = 0$

X sandwich, so only bivector remains

$$X\nabla(aX\Psi) = -4Xa\Psi + X\dot{\nabla}aX\dot{\Psi}$$

$$= -4Xa\Psi - Xa\dot{\nabla}X\dot{\Psi}$$ Now use c=1

$$= Xa(-2\Psi + X\nabla\Psi)$$ This must equal zero for all a

$$X\nabla\Psi = 2\Psi$$ Curious!

Twistors / CSTA

Can make thing more natural if we define $$\Psi_1 = \frac{Xn}{((X\cdot n)^2)}\Psi$$

This has $X\nabla\Psi_1 = 0$

If the projected psi is monogenic, have $n\nabla\Psi_1 = 0$

Together $\nabla\Psi_1 = 0$

Massless Dirac equation in CSTA

$$|\Psi\rangle = \begin{pmatrix}\psi \\ x\psi\gamma_0\end{pmatrix}$$

Looks a lot like a twistor!

$T = 1 - xn/2$ Base translator

$T\{\gamma_\mu, -\bar{n}/2\}\tilde{T}$ Fixed frame

G(1,3,1)

Summary / thanks

- You can do field theory in conformal GA, if the underlying theory is conformally invariant
- Need to stick to the null cone
- Need to take care with derivatives due to effects off the cone
- Get equations that have manifest conformal invariance
- But there are a lot of side-relations
- Spinors can be included into this framework
- Maybe a more natural setting for twistors?
- But ultimately, physics does not appear to respect conformal invariance.