Paraxial Geometric Optics in 3D through Point-based Geometric Algebra

computer vision lab

Leo Dorst

Informatics Institute, University of Amsterdam

Universiteit van Amsterdam

The Vectors of HGA $\mathbb{R}_{d,0,1}$

HGA $\mathbb{R}_{d,0,1}$ is an algebra in which homogeneous points are represented as vectors.
We choose an orthonormal basis $\{e_0, \mathbf{e}_1, \cdots, \mathbf{e}_d\}$ of anti-commuting 'unit' vectors:

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \text{ and } e_0\mathbf{e}_i = -\mathbf{e}_i e_0, \text{ with } (e_0)^2 = 0 \text{ and } (\mathbf{e}_i)^2 = 1.$$

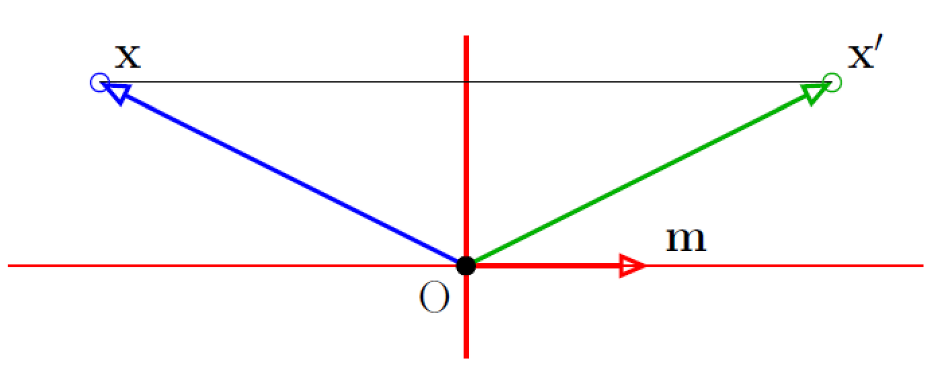
Three kinds of vectors, differing in algebraic properties and geometric semantics:

- **Directions:** A purely Euclidean vector $\mathbf{m}$ represents a 1-direction in d-space.
  Its inverse is $\mathbf{m}^{-1} \equiv \mathbf{m}/(\mathbf{m}\cdot\mathbf{m})$. Directions are 'ideal points' at infinity.
- **The origin:** The point at the origin is represented by the null vector e_0.
  Since $e_0^2 = 0$, it is *not* invertible, and therefore e_0 cannot be used as a versor.
- **Non-origin points:** A general point at $\mathbf{p}$ is of the form $P = \alpha(e_0 + \mathbf{p})$.
  Its inverse is $P^{-1} = P/(P\cdot P) = P/(\alpha^2\mathbf{p}^2)$, iff $\alpha \neq 0$ and $\mathbf{p}^2 \neq 0$.
  For versors we can use normalized points with $\alpha = 1$.

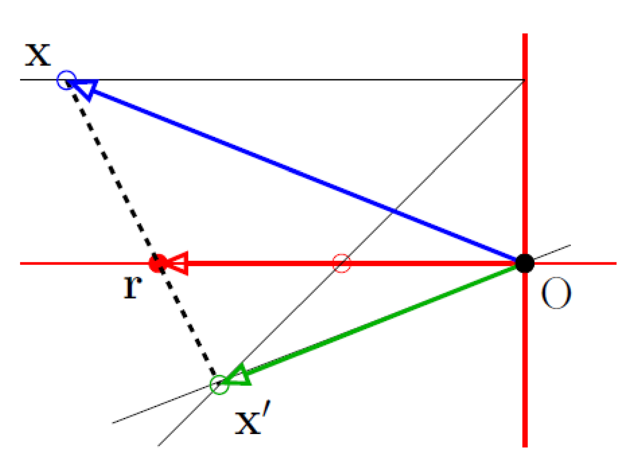
The Vector Versors of HGA

$$-\mathbf{m}\,(e_0 + \mathbf{x})\,\mathbf{m}^{-1} = e_0 - \mathbf{m}\,\mathbf{x}\,\mathbf{m}^{-1}.$$

$$-(e_0 + \mathbf{r})\,(e_0 + \mathbf{x})\,(e_0 + \mathbf{r})/\mathbf{r}^2 = (1 - 2\mathbf{r}^{-1}\cdot\mathbf{x})\,\Big(e_0 + \frac{-\mathbf{r}\mathbf{x}\mathbf{r}^{-1}}{1 - 2\mathbf{r}^{-1}\cdot\mathbf{x}}\Big)$$



plane mirror



paraxial spherical mirror

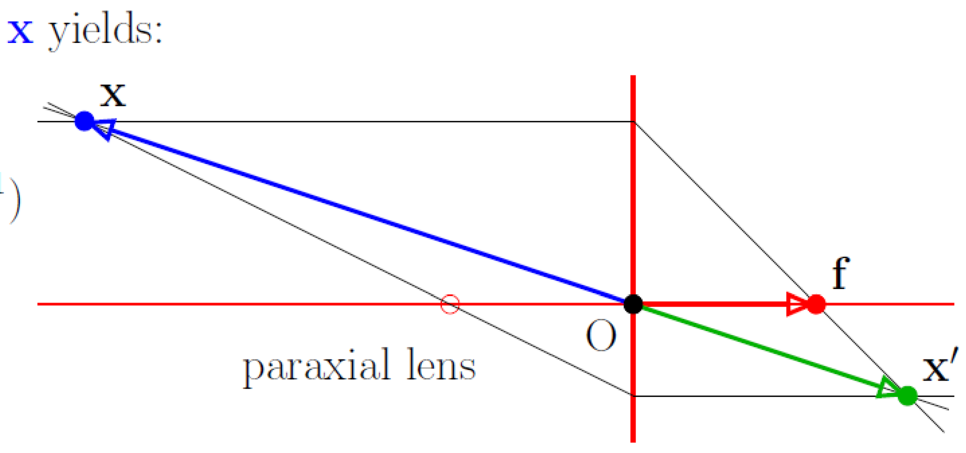
The Lensing Versor $L_\mathbf{f}$ in HGA

View a lens as the combination of a spherical mirror and a reflection, so use HGA versor $\mathbf{m}\,R$ to represent it. We should take $\mathbf{m} = \mathbf{r}$ and relate R to the focal point F by $\mathbf{f} = -\mathbf{r}/2$. That gives:

$$\mathbf{r}\,(e_0 + \mathbf{r}) = \mathbf{r}^2 - e_0\mathbf{r} = \mathbf{r}^2\,(1 - e_0\mathbf{r}^{-1}) \propto 1 - e_0\mathbf{r}^{-1} = 1 + \tfrac{1}{2}e_0\mathbf{f}^{-1} \equiv L_\mathbf{f}.$$

Now sandwiching a point $X = e_0 + \mathbf{x}$ yields:

$$\begin{aligned} &L_\mathbf{f}\,(e_0 + \mathbf{x})\,L_\mathbf{f}^{-1} = \\ &= (1 + \tfrac{1}{2}e_0\mathbf{f}^{-1})\,(e_0 + \mathbf{x})\,(1 - \tfrac{1}{2}e_0\mathbf{f}^{-1}) \\ &= e_0 + \mathbf{x} + \tfrac{1}{2}e_0\mathbf{f}^{-1}\mathbf{x} - \tfrac{1}{2}\mathbf{x}e_0\mathbf{f}^{-1} \\ &= (1 + \mathbf{f}^{-1}\cdot\mathbf{x})\,e_0 + \mathbf{x} \\ &= (1 + \mathbf{f}^{-1}\cdot\mathbf{x})\,\Big(e_0 + \frac{\mathbf{x}}{1 + \mathbf{f}^{-1}\cdot\mathbf{x}}\Big). \end{aligned}$$

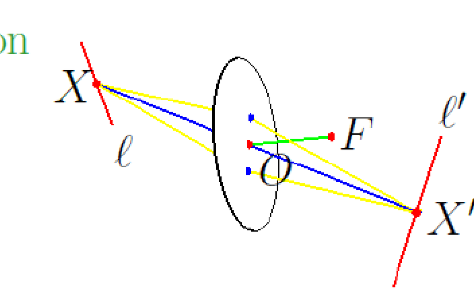


This is indeed the GA form of the familiar Euclidean lensing formula.

HGA Bonus: Equivariance of Versor Mapping

The universality of the versor form allows direct transformation of lines and planes, e.g.

$$(L_\mathbf{f}X_1/L_\mathbf{f}) \wedge (L_\mathbf{f}X_2/L_\mathbf{f}) = L_\mathbf{f}\,(X_1 \wedge X_2)/L_\mathbf{f}.$$



However, a ray tracing matrix depends on the type of flat:

point: $\begin{bmatrix}\mathbf{p}\\1\end{bmatrix} \overset{L_\mathbf{f}}{\mapsto} \begin{bmatrix}[1] & 0\\ [\mathbf{f}^{-1}]^\top & 1\end{bmatrix}\begin{bmatrix}\mathbf{p}\\1\end{bmatrix}$ line: $\begin{bmatrix}\mathbf{u}\\ \mathbf{p}\times\mathbf{u}\end{bmatrix} \overset{L_\mathbf{f}}{\mapsto} \begin{bmatrix}[1] & 0\\ [\mathbf{f}^{-1}]^\times & 1\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{p}\times\mathbf{u}\end{bmatrix}$ plane: $\begin{bmatrix}\mathbf{n}\\ -\delta\end{bmatrix} \overset{L_\mathbf{f}}{\mapsto} \begin{bmatrix}[1] & -[\mathbf{f}^{-1}]\\ \mathbf{0}^\top & 1\end{bmatrix}\begin{bmatrix}\mathbf{n}\\ -\delta\end{bmatrix}$.

Explicitly computing the matrices is better left to an HGA-to-LA compiler.

Full paper [1] at:

l.dorst@uva.nl
dorst.037@gmail.com
(after 2024)

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Optics Anywhere, via Conformal GA

HGA null vector e_0 for point at optical center is essential for lens versor.
But there are no other null points in HGA $\mathbb{R}_{d,0,1}$, so we cannot translate by versors.

CGA $\mathbb{R}_{d+1,1}$ (algebra of spheres) has *all* points as null vectors: $x = n_o + \mathbf{x} + \frac{1}{2}\mathbf{x}^2 n_\infty$.
It also has translation versors. So let us embed HGA into CGA at each location!

Changing Center by 'from c' Embedding $X|_c$

Choose any CGA point c as corresponding to the optical center e_0 of HGA.
An HGA point X at location $\mathbf{x}$ viewed from this 'origin' c is embedded as:

$$\text{'from } c\text{' map:}\quad X|_c \equiv n_o + \mathbf{x} = n_o\cdot(-n_\infty \wedge x)$$

with c and x CGA points, n_o the CGA origin. You may pronounce '$|_c$' as 'from c'.

$$\textit{structure-preserving}:\ x|_c \wedge y|_c = (x \wedge y)|_c.$$

Arbitrary HGA element X (point, line, plane, direction element) is represented as viewed 'from c' by:

$$X \ \mapsto\ X|_c = c\cdot(-n_\infty \wedge X)$$

The 'from c' mapping is 'neopotent':

$$\textit{last application counts}:\ (X|_{c_1})|_{c_2} = X|_{c_2}.$$

We can therefore always re-represent a re-represented element. In a concatenation of lenses, there is *no need to revert* from $X|_c$ to HGA X before the next step.

Jumping to a new optical center c is not a relative translation,
but an absolute teleportation.

The Lens Versor at Another Optical Center

HGA lens versor $\exp(e_0 \wedge \mathbf{f}^{-1}/2)$ at origin e_0, move to C? No HGA versor!
CGA lens versor $\exp(n_o \wedge \mathbf{f}^{-1}/2)$ at location n_o, move to c? Have CGA versor!

CGA tangent vector $n_o \wedge \mathbf{f}^{-1}$ at n_o moves to: $c \wedge (\mathbf{f}^{-1} + (\mathbf{f}^{-1}\cdot c)\,n_\infty)$.
With some algebra, we can rewrite this translated tangent multiplicatively:

$$T_c\,(n_o \wedge \mathbf{f}^{-1})\,T_c^{-1} = c/(c \wedge n_\infty \wedge \mathbf{f}) = c/(c \wedge n_\infty \wedge f),$$

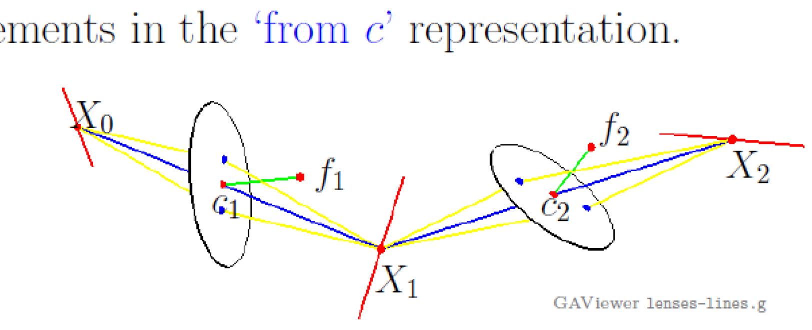
f focal point, and $\mathbf{f}$ relative vector from c to f. Indifferent to F-specification!

With that CGA tangent vector, the CGA lens versor with optical center c is:

$$\text{versor } L_{c,f} \text{ of lens at } c \text{ with focal point } f:\quad L_{c,f} = \exp(c/(c \wedge n_\infty \wedge f)/2)$$

Apply the 'at c' versor $L_{c,f}$ to elements in the 'from c' representation.

Concatenation of Elements



1. Lenses with centers at points c_i, focal points at f_i (or use $\mathbf{f}_i$). Form lens versors:

$$L_i \equiv \exp(\tfrac{1}{2}\,c_i/(c_i \wedge n_\infty \wedge f_i)) = 1 + \tfrac{1}{2}\,c_i/(c_i \wedge n_\infty \wedge f_i)$$

   or the *spherical mirror* $R_i = c_i - 2(c_i \wedge n_\infty)\cdot(c_i \wedge n_\infty \wedge f_i)$, or the *planar mirror* $M_i = (c_i \wedge n_\infty)\cdot(c_i \wedge n_\infty \wedge \mathrm{f}_i)$.

2. Embed the HGA element X into CGA by replacing its e_0 by n_o. Then iterate:

$$\begin{aligned} X_0 &= X, \\ X_i &= \underline{L_i}[X_{i-1}|_{c_i}] = L_i\,\big(c_i\cdot(-n_\infty \wedge X_{i-1})\big)\,L_i^{-1} \quad \text{for } i = 1, \cdots, n \end{aligned}$$

   or similarly for $\underline{R_i}[]$ and $\underline{M_i}[]$, remembering to include the grade involution.

3. Final result may be expressed as $X' = n_o\cdot(-n_\infty \wedge X_n)$ relative to an origin point n_o. It can be converted back to HGA by replacing n_o by e_0.

Generating Ray Transfer System Matrices

- The HGA/CGA lens versor specification lenses any flat geometric primitive.
- The 'from c' re-representation prepares it for the next optical element.
- Any flat element can be propagated through the optical system, so easy to find the total homogeneous transfer matrix for a composition of optical elements:
- Simply process appropriate i-th basis element by the iterative algorithm and denote the resulting components as i-th column of the transformation matrix.
  (E.g., matrix for imaging of arbitrary 3D line uses Plücker coordinate basis $\{e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}\}$.)
- Specification is immediately geometrical with (c_i, f_i) or $(c_i, \mathbf{f}_i)$ pairs. Due to neopotent teleportation, no intermediate 'relative Euclidean transformations'.
- This extends the 2D homogeneous matrix techniques of Corcovilos [2] to n-D.