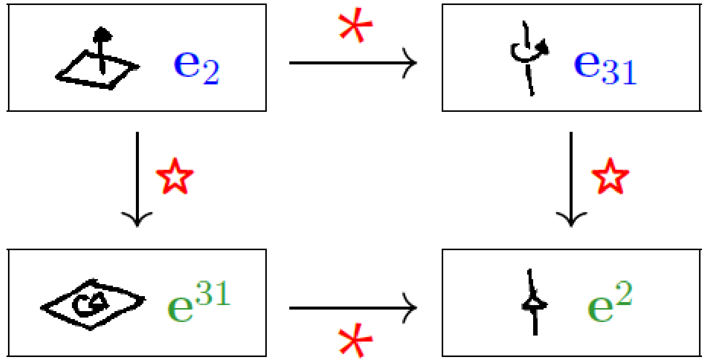

Dual Spaces are Real: Orientation Types in Geometric Algebra

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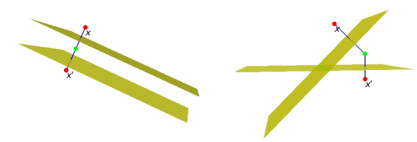
1 Duality within the Structure of Geometric Algebra

How we set up a geometric algebra $\mathbb{R}_{p,n,z}$ for an application, to obtain structure-preserving transformation operators:

- Determine what the desired symmetry operations are.
- Make those into orthogonal transformations...
- ... by multiple reflections, through Cartan-Dieudonné.
- That determines what vectors model, and hence the metric space.
- Geometric product gives the versors (orthogonal transformations).
- Higher level symmetry invariants are then the $\wedge$-blades.
- Highest element is the pseudoscalar $\mathbb{I}$, invariant (modulo scale/sign).
- Then use complementarity principles (duality) for more expressivity (OPNS/IPNS).

Example of
3D Rigid Body Motions



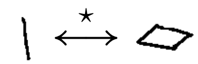
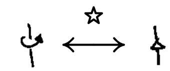
Multiple plane reflections,
so plane-based GA $\mathbb{R}_{3,0,1}$,
with 1-blade planes,
2-blade lines
and 3-blade points.

Is duality then *convenient*, rather than *essential* for modelling?
NO! There are aspects of geometrical reality not yet captured in the above.

We want all of , , ,  represented to transform algebraically and geometrically correctly.

2 The Two Dualities

- Basic wedge product $\wedge$ of the GA constructs the geometric primitives (as invariants).
- Two ways of doing *complement of wedge* (Gunn's polarity vs duality):

$(X \wedge Y)^*$	$=$	$X \cdot Y^*$	inner product, orthogonal complement	
$(X \wedge Y)^*$	$=$	$X^* \vee Y^*$	join product, orientational complement	

We have overloaded the notation for now, they may be different!

- Such constructions are truly geometric ('Clifford equivariant').
- They coexist, but 'orientational' takes probe X to dual probe X^*, so to a dual space?
- Both forms of dualization are complementation and involve the pseudoscalar $\mathbb{I}$.

3 Duals as Complementation of Blades within Same Algebra

- In numerical computations on multivectors, the dual involves complementation of indices, with a sign determined by the choice of pseudoscalar $\mathbb{I}$, since $X^* \equiv X\,\mathbb{I}$.
- For a k-blade, it gives an $(n-k)$-blade *in the same geometric algebra* $\mathcal{C}\ell(W)$.
- This complementation definition can also be mimicked in degenerate algebras (with $\mathbb{I}^2 = 0$). On ONB blades E_i, the Hodge dual $\star E_i$ is then defined by: $E_i \wedge \star E_i = \mathbb{I}$
- Both duals X^* and $\star X$ are interpreted in terms of orthogonality of subspaces.
- A^* can be used for orthogonal characterization of subspaces: $x \wedge A = 0 = x \cdot A^*$ (OPNS/IPNS).

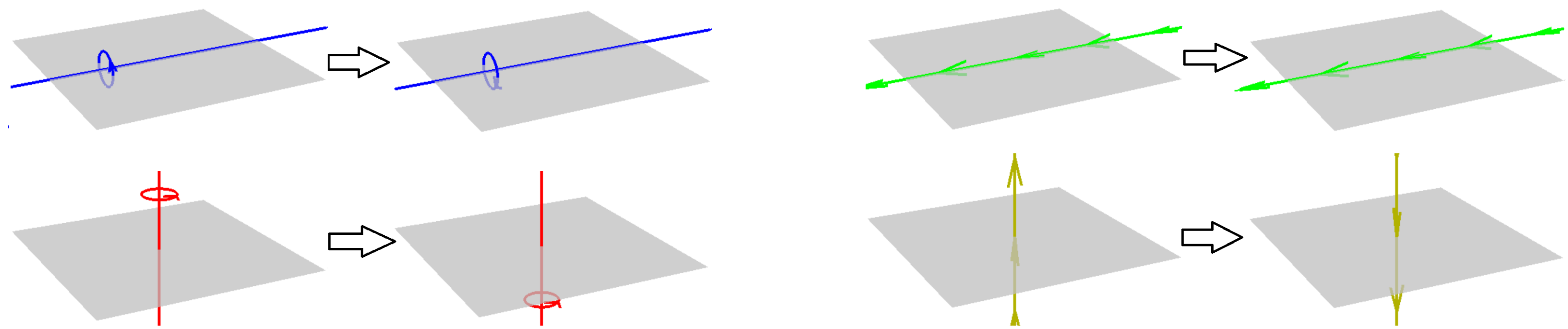
4 Dual Space: Complementary Orientations Transform Differently

- Transforming a dual by a versor may involve an extra sign:

orthogonal complement: $\underline{V}[A\,\mathbb{I}] = \underline{V}[A]\,\underline{V}[\mathbb{I}] = \underline{V}[A]\,\mathbb{I}\,\det(\underline{V})$, so $\underline{V}[A^*] = \underline{V}[A]^*\,\det(\underline{V})$.

Hodge-type dualization: $\underline{V}[E_i \wedge \star E_i] = \underline{V}[\mathbb{I}] = \mathbb{I}\,\det(\underline{V})$, so $\star\underline{V}[E_i] = \underline{V}[\star E_i]/\det(\underline{V})$.

- Such a reflection sign is also observed for differently oriented geometric elements:

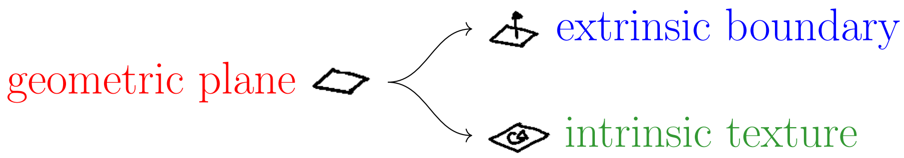


extrinsically oriented lines (axes) intrinsically oriented lines (spears)

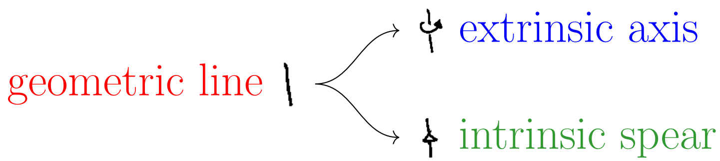
- Complementary orientation of an element is a consistent geometric interpretation of its dual.

5 Complementary Orientations

- There are two orientation types for a plane: extrinsic and intrinsic.



- Each has a sign/weight, for inside/outside (boundary spec) or handedness/chirality (texture).
- Similarly, in 3D we may want axes (lines to turn around, extrinsic) and spears (lines to move along, intrinsic).



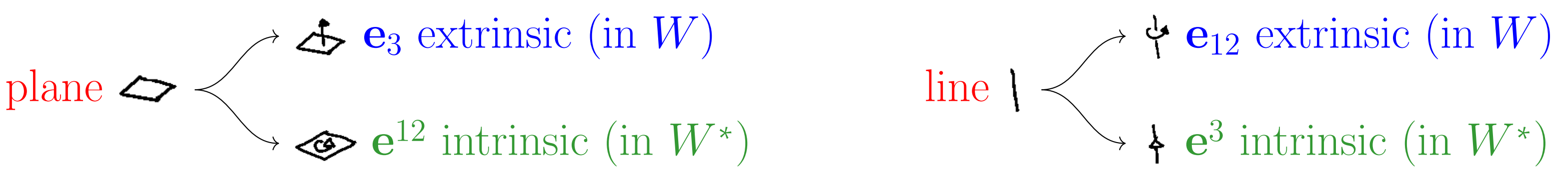
Both are needed in classical mechanics, in the GA version of 'axial' and 'polar' vectors.

- All these orientation specifications are in principle *independent*, when modelling: for planes, we may want an outside pointing border with a left-handed texture, or any of the four combinations.

6 Encoding Orientation Types Algebraically

We want all of , , , represented to transform algebraically and geometrically correctly.

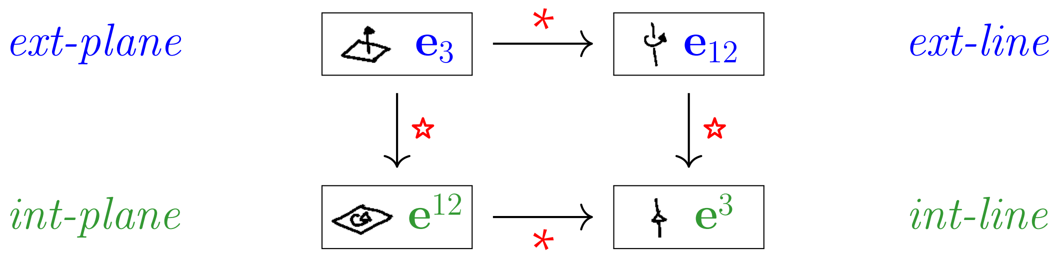
- But how to tell them apart: is algebraic element $\mathbf{e}_3$ normal vector of plane , or a spear line in direction $\mathbf{e}_3$? They transform the same, but are different as subspaces! And $\mathbf{e}_{12}$ is taken by …
- Not enough elements to encode! So need another space W^* to maintain/administrate the duals.
- In coordinate notation, let us use superscripts for those dual elements.



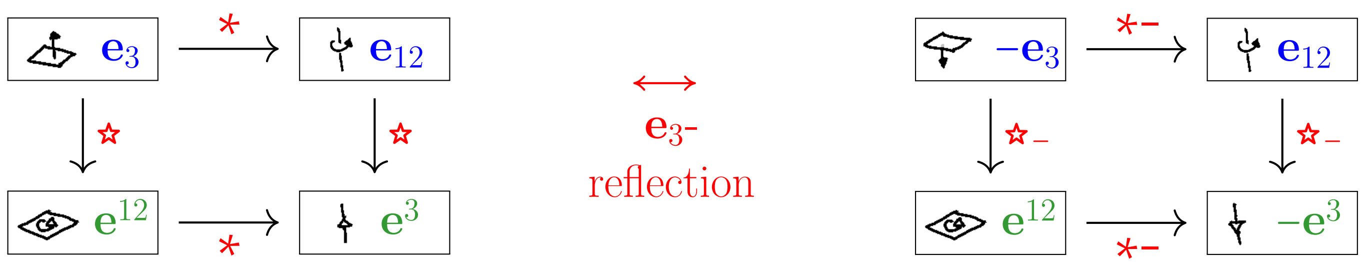
- In OGA $\mathbb{R}_3$, the geometric plane with extrinsic normal vector $\mathbf{e}_3$ has an intrinsic representation $\mathbf{e}^1 \wedge \mathbf{e}^2 = \mathbf{e}^{12}$, the wedge of two spears $\mathbf{e}^1$ and $\mathbf{e}^2$ in dual algebra $\mathbb{R}_3^*$.
- Related by duality $\star$ as $\star\mathbf{e}_3 = \mathbf{e}^{12}$, relative to the pseudoscalar $\mathbb{I} = \mathbf{e}_{123}$ or $\star 1 = \mathbf{e}^{123}$.
- Now each orientation type transforms automatically correctly under odd or even versors.
- The two types can be coupled by the pseudoscalar; but in some applications, we want as independently specifiable properties. Now studying Laguerre/Lie geometry for that purpose.

7 The Two Dualities Interpreted

Duality $*$ gives orthogonal complement within same orientation type.
Duality $\star$ gives orientational complement of same geometrical element.

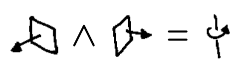
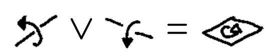
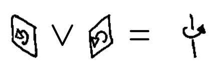


Either duality gets a sign under reflection. Therefore diagonally opposite elements have the same transformational symmetries.



8 Constructing Oriented Elements: New Space, New Algebra?

- Our usual duality relates subspace characterization to wedge and dot products (OPNS/IPNS). That gives the orthogonal complement interpretation: $(X \wedge Y)^* = X \cdot Y^*$.
- The projective duality more naturally relates to the meet and join products. This gives the orientation type interpretation: $\star(X \wedge Y) = \star X \vee \star Y$.
- For instance, an (extrinsic) axis  is the intersection (meet) of two 3D planes, and an (intrinsic) plane is the union (join) of two 3D lines. Indifferent to orientation type.

OGA $\mathbb{R}_3$	meet of planes gives ext-line	join of spears gives int-plane
space of planes (extrinsic)	 $e_1 \wedge e_2 = e_{12}$	 $e_{23} \vee e_{31} = \star e_3$
space of spears (intrinsic)	 $e^{23} \vee e^{31} = \star e^3$	 $e^1 \wedge e^2 = e^{12}$

NOTE: Classical join maps $W \times W \to W$, new join $W \times W \to W^*$. New join $X \vee Y \equiv \star X \wedge \star Y$ was redefined to involve 2 rather than 3 pseudoscalars, for better orientational reflection properties. [1]

[1] See Leo Dorst [2023] *Poincaré Duality Encodes Complementary Orientations in Geometric Algebras*, doi `10.1002/mma.9754`

9 The Metrics of $\mathcal{C}\ell(W)$ and $\mathcal{C}\ell(W^*)$

- Plane-based GA is modelled as $\mathbb{R}_{d,0,1}$, which is proper for Euclidean geometry.
- Point-based GA, as Klein dual construction, would get $\mathbb{R}^*_{1,0,d}$; but its natural use in paraxial optics (see [poster]) requires $\mathbb{R}_{d,0,1}$. There Euclidean distances are 'join strengths'.
- There thus seems no necessary relationship between the GA metric structure of a space W and its dual space W^*.
- How much freedom do we have to choose the W^*-metric given a W-metric?
  The oriented projective relationships of meet and join will work anyway.
  Subspace definition in either W or W^*, by point probing $P \wedge \star A = 0 = \star P \vee A$.
- This is the question I had hoped to answer definitively for AGACSE 2024...

10 The Metric Relationship Between the Dual Spaces in PGA

Sensible distance measures for an application need to be computable between elements of W or W^*, since they represent the same basic geometry. However, the form of the formulas may differ...

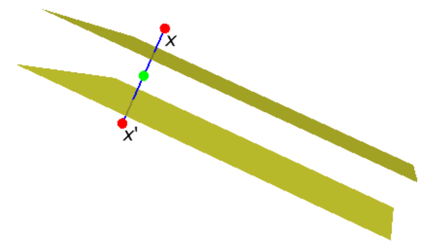
- **Plane-based PGA**: base distances on translation versor.

  Distance vector $\mathbf{d}$ of parallel planes from geometric ratio:

  $$(\mathbf{u} - \delta_2 e_0)/(\mathbf{u} - \delta_1 e_0) = 1 + (\delta_2 - \delta_1)\mathbf{u} e_0 = 1 + \mathbf{d} e_0$$

  Distance vector $\mathbf{d}$ of points from geometric ratio:

  $$(\mathbb{I}_d + \mathbf{q} e_0 \mathbb{I}_d)/(\mathbb{I}_d + \mathbf{p} e_0 \mathbb{I}_d) = 1 + (\mathbf{q} - \mathbf{p}) e_0 = 1 + \mathbf{d} e_0$$



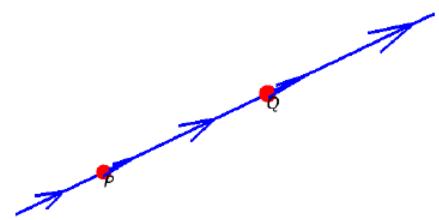
- **Point-based PGA**: base distances on join magnitude.

  Distance vector $\mathbf{d}$ of points from their join (connection):

  $$(e^0 + \mathbf{p}) \wedge (e^0 + \mathbf{q}) = \big(e^0 + \tfrac{1}{2}(\mathbf{p} + \mathbf{q})\big) \wedge (\mathbf{q} - \mathbf{p}) = (e^0 + \mathbf{c}) \wedge \mathbf{d}$$

  This is a wedge product $\wedge$, so non-metric.

  Then extract $\mathbf{d}$-coefficients from $e^0 \wedge \mathbf{d}$.



Geometric ratio in point-based PGA would give $(e_0 + \mathbf{p})(e_0 + \mathbf{q}) = (\mathbf{p} \cdot \mathbf{q}^{-1})\big(1 + e_0(\mathbf{q} - \mathbf{p})/(\mathbf{p} \cdot \mathbf{q})\big)$, perspective distance.

11 PGA as Subalgebra of CGA: Newtonian Mechanics and Optics

- The point-based distance formulas in both plane-based and point-based are strongly related; they do not even allow a scaling factor let alone a metric with a different signature?
- Plane-based $\mathbb{R}_{d,0,1}$ and Point-based $\mathbb{R}^*_{d,0,1}$ come together as subalgebras of CGA $\mathbb{R}_{d+1,1}$:
  - Plane-based $\mathbf{n} - \delta e_0$ is like $\mathbf{n} + \delta n_\infty$, so e_0 corresponds to $-n_\infty$. Subalgebra: $n_\infty \cdot X = 0$.
  - Point-based $e^0 + \mathbf{p}$ is like flat point $(n_o + \mathbf{p}) \wedge n_\infty$, so e^0 is like n_o. Subalgebra: $n_\infty \wedge X = 0$.
- Euclidean parts in $\mathbb{R}_{d+1,1} = \mathbb{R}_d \oplus \mathbb{R}_{1,1}$ can be identified for a metric relationship of W and W^*.

$$\text{sphere-based} = \{\overbrace{n_o, \underbrace{\mathbf{e}_1, \cdots, \mathbf{e}_d}^{\text{point-based}}, n_\infty}_{\text{plane-based}}\}$$

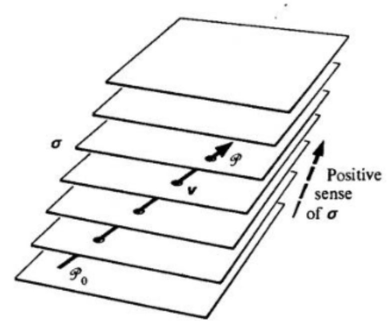
- In homogeneous coordinates, similar clue in $\begin{bmatrix} \mathsf{R} & \mathbf{t} \\ \mathbf{f}^\top & 1 \end{bmatrix}$: together projective, upper triangular is Euclidean motions, lower triangular is paraxial perspective.
- Does this generalize? Is there always a meaningful minimal superalgebra with common metric subspace? Consulting UvA colleague Patrick Forré on such issues.

12 BONUS:The Dual Space of Forms in Mathematics

- Both GA dualities differ from the k-forms, employed in standard multi-linear algebra.

- They produce a scalar when evaluated on a k-form. Extension of 1-form on vector space W with bilinear form $\langle\ ,\ \rangle$:

$$W \to W^*: \ v \to v^*: \quad v^*(w) \equiv \langle v, w \rangle, \quad \forall w \in W.$$



From *Gravitation* by Misner, Thorne, Wheeler.

- Those are a non-metric way of making duality, much used in projective geometry and (as differential forms) in calculus. No explicit pseudoscalar!

- The dual elements are viewed as linear maps, rather than as complementary subspaces.

- Their physical interpretation is in terms of quantities vs densities (heights vs contours).

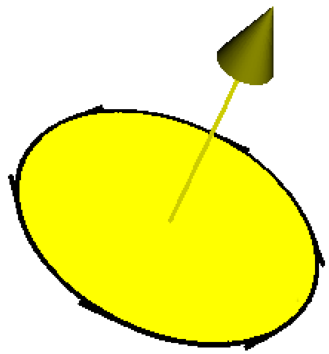
- The k-forms do form a linear space, with a dimensionality that is the same as our dual $(n-k)$-grade element (since $\binom{n}{k} = \binom{n}{n-k}$), so they can be related isomorphically, to W or W^*.

- Forré [unpublished] introduces a natural dual $A^\natural$ related to the pseudoscalar, to formally treat the mathematical differences with k-forms and study equivariance.

13 BONUS: Convention-free Duality Programming?

- GA duality involves the projection onto the pseudoscalar $\mathbb{I}$:

$$A = A\,\mathbb{I}^{-1}\,\mathbb{I} = (A \cdot \mathbb{I}^{-1}) \cdot \mathbb{I}.$$



- We keep the dual $A \cdot \mathbb{I}^{-1}$, and memorize the $\mathbb{I}$-convention used.
- More formally, we could see this as a *pseudoscalar split*:

$$\mathcal{C}\ell^{(k)}(W) \to \mathcal{C}\ell^{(n-k)}(W) \otimes \mathcal{C}\ell^{(n)}(W): \quad A \mapsto (A\,\mathbb{I}^{-1}) \otimes \mathbb{I}$$

  and concentrating on the first term of this tensor product.
- $\mathbb{I}$ division works only for non-degenate algebras, which have an invertible pseudoscalar. For degenerate, we mimic it with coordinate-based dual (Hodge dual). $E_i \wedge (\star E_i) = \mathbb{I}$.
- But complementation aspect of duality in formal proofs can be codified without getting specific.
- The tensor view may be the way to convention-free programming?

14 BONUS: Why We Use Planes Rather Than Points, as Vectors in PGA

DIMENSION-AGNOSTIC REPRESENTATION OF GEOMETRY!

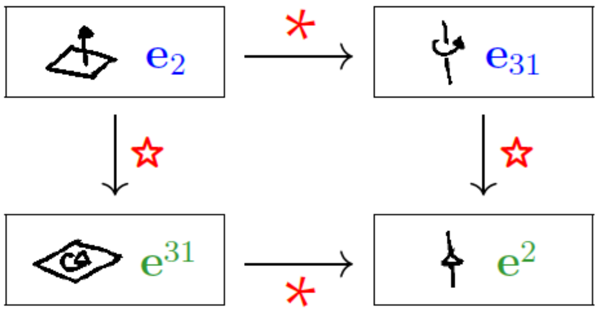
algebra	ext-plane	int-plane
OGA $\mathbb{R}_3$	$\mathbf{e}_1$	$\mathbf{e}^{23}$
PGA $\mathbb{R}_{3,0,1}$	$\mathbf{e}_1$	$e^0 \wedge \mathbf{e}^{23} = e^{023}$
CGA $\mathbb{R}_{3+1,1,0}$	$\mathbf{e}_1$	$n_o \wedge n_\infty \wedge \mathbf{e}^{23} = e^{+-23}$

For these relatively dual primitives compared to the classical approach, we should use

$$D^* = D\,\mathbb{I},$$

to correspond to the usual Hestenes dualization $A^* = A/\mathbb{I}$ $(=D)$ in geometrical semantics.

15 Summary and Future Direction



- Two compatible dualizations: orthogonal complement and orientational complement.
- Both pseudoscalar-based (unlike forms), both useful.
- They couple outer product $\wedge$ to inner product $\cdot$ and join product $\vee$, respectively.
- Consistent geometric interpretation of either and both, within one's chosen semantics.
- So far in this talk, coupled by the pseudoscalar $\mathbb{I}$.

Pure oriented geometry appears in Laguerre geometry $\mathbb{R}_{d,1,1}$ of oriented flats, and Lie sphere geometry $\mathbb{R}_{d+1,2,0}$ for oriented rounds.

- They decouple the two dualizations from their relationships by pseudoscalar $\mathbb{I}$.
- These geometries thus give more explicit control of intrinsic/extrinsic properties...
- ...and versors that act on them (Huygens wave propagation, contact transformations).
- It will make a nice retirement project, after January 2025.

ANY QUESTIONS?

16 (Appendix) Orthogonal Complement; IPNS/OPNS

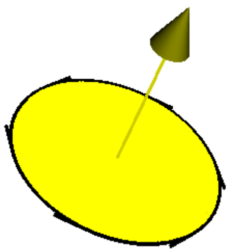
- Let us focus on blades, which represent subspaces as outer product null space:

$$[A] \equiv \mathrm{OPNS}[A] = \{\mathbf{x} \in W : \quad \mathbf{x} \wedge A = 0\}$$

- For invertible pseudoscalars, the usual view is that the dual of a blade characterizes the orthogonal complement $[A]^{\perp}$:

$$[A]^{\perp} = \mathrm{OPNS}[A^{*}]$$

- By duality of outer and inner product, we then have an alternative dual characterization of the subspace $[A]$:



$$[A] = \mathrm{IPNS}[A^{*}] = \{\mathbf{x} \in W : \quad \mathbf{x} \cdot (A^{*}) = 0\}$$

These subspace characterizations involve only the geometric locality, not the orientation (or weight).