The Zeros of Heron's Formula in Orthocentric Tetrahedra

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Abstract

Metric vector spaces with a Lorentzian signature $\mp 1, \pm 1, \ldots, \pm 1$ are not only the foundation of special relativity, but also contain covariant models for hyperbolic, inversive and de Sitter geometries, which themselves contain models of Euclidean geometry. This in turn opens up many possible conceptual windows into modeling in engineering and the natural sciences, which are ideally suited to analysis via geometric algebra and upon which the second author's life's work has been intensely focused.

Recently the first author succeeded in generalizing one of the most basic formulae in Euclidean distance geometry, namely Heron's formula, to tetrahedra (as well as higher-dimensional simplices), but then made the disconcerting discovery that its zeros do not generally correspond to quadrangles in the Euclidean plane. Instead, they correspond to collinear quadruples of vertices separated by infinite distances but with generically well-defined distance ratios. This presents the problem of finding a suitable geometric model in which these zeros can be explicitly represented in some reasonably natural way.

Abstract (cont.)

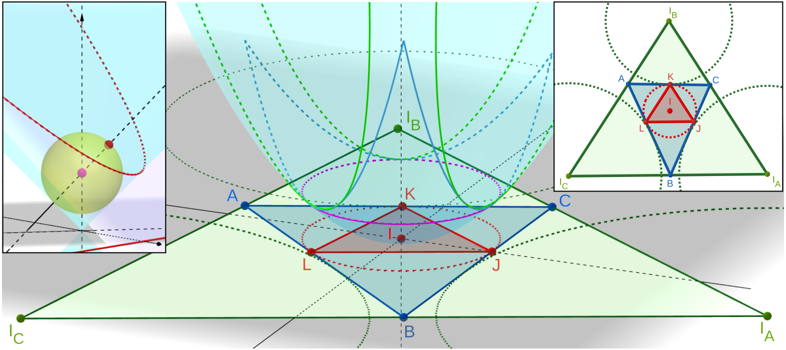
It also raises the question of how it is that triangles and tetrahedra can be so drastically different. Various authors, notably Mowaffaq Hajja, Horst Martini & Allan L. Edmonds have studied this question deeply, and concluded that the key property distinguishing tetrahedra from triangles is the fact that the altitudes of their vertices over the co-dimension 1 faces are not generally concurrent. Tetrahedra for which they are concurrent are known as **orthocentric** tetrhedra, and their distance geometry is astonishingly simple compared to general tetrahedra.

In particular, the principal algebraic property distinguishing triangles from tetrahedra is that the former are naturally embedded in an **orthocentric system** wherein their vertices and excenters are exchanged by inversion in a "polar circle" with imaginary radius. This elegant involution, which Udo Hertrich-Jeromin, Alastair King & Jun O'Hara rediscovered and renamed "conformal duality," does not however generalize to higher dimensions.

The talk concludes by showing, apparently for the first time, how such orthocentric systems generalize to three-dimensional Euclidean space (and beyond?).

$\mathcal{G}_{3,1}$: A Vector Space Model with Many Different Geometric Interpretations

- Its even subalgebra is $\mathrm{O}^+(3,1) \approx \mathrm{O}^+(1,3) \approx \mathrm{SL}(2,\mathbb{C})$;
- So it represents the Lorentz group of space-time (like $\mathcal{G}_{1,3}$)
- And $\mathbb{R}^{3,1}$ contains models of several well-known geometries:
  - Lines inside the null cone: hyperbolic 3-space;
  - Lines on the null cone: the inversive (conformal) plane;
  - A parabolic section of the null cone: the Euclidean plane.



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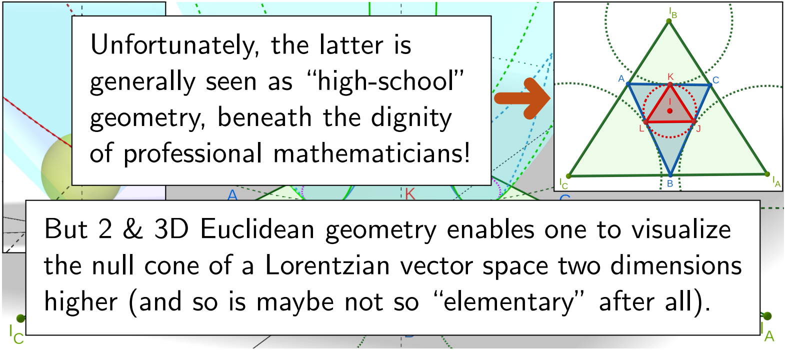
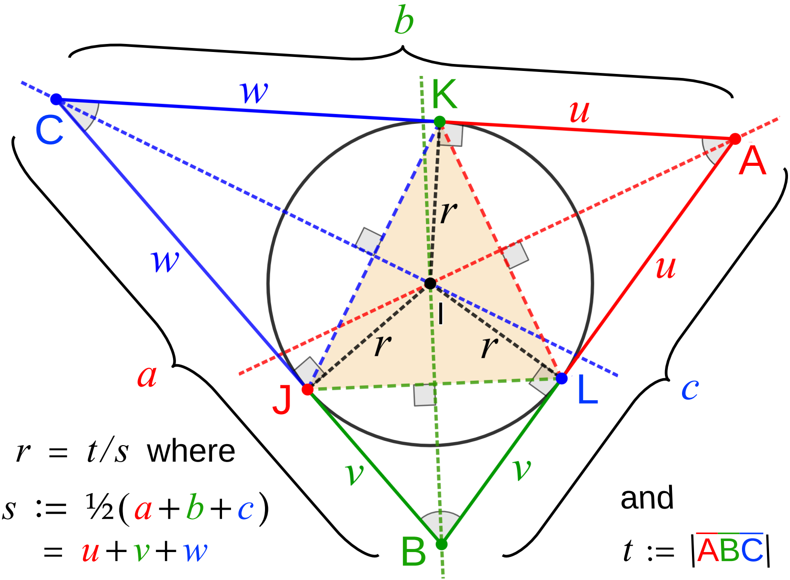


Unfortunately, the latter is generally seen as "high-school" geometry, beneath the dignity of professional mathematicians!

But 2 & 3D Euclidean geometry enables one to visualize the null cone of a Lorentzian vector space two dimensions higher (and so is maybe not so "elementary" after all).

Case in Point: Heron's Formula for the Area of a Triangle



$r = t/s$ where

$$s := \tfrac{1}{2}(a+b+c) = u+v+w$$

and

$$t := |ABC|$$

Case in Point: Heron's Formula for the Area of a Triangle

Although better known as a means of calculating the area of a triangle $\overline{ABC}$, Heron's formula is essentially a relation between the edge lengths a, b, c and the <u>squared</u> radius r of its incircle:

$$(2r)^2 \;=\; (-a+b+c)(a-b+c)(a+b-c)/(a+b+c)$$

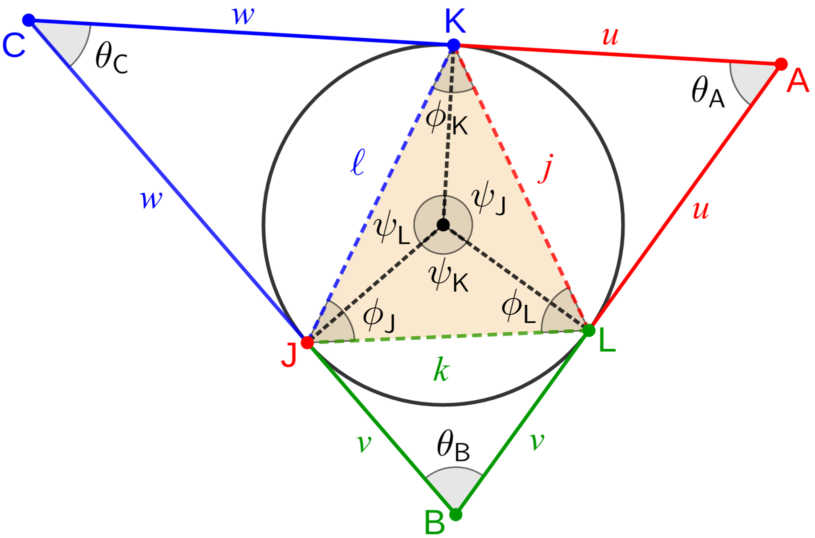
Squared area $t^2 = r^2 s^2$ with *semi-perimeter* s defined as shown.

Since $u = (-a+b+c)/2)$, $v = (a-b+c)/2)$, $w = (a+b-c)/2)$ are the lengths of the segments shown in the figure, this may be written more compactly as

$$r^2 \;=\; \frac{1}{2s}\,\Omega \;\Leftrightarrow\; t^2 \;=\; \frac{s}{2}\,\Omega\,, \quad \Omega(u,v,w) := \det\begin{bmatrix} 0 & u & v \\ u & 0 & w \\ v & w & 0 \end{bmatrix},$$

wherein $\Omega = 2\,uvw$. Since $a = v + w$, $b = u + w$, $c = u + v$, the **Heron parameters** u, v, w determine the triangle up to isometry.

Trigonometric Versions of the Heron Parameters & In-touch Triangle Edges



Trigonometric Versions of the Heron Parameters & In-touch Triangle Edges

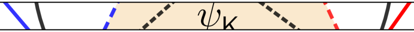
Basic trigonometry yields the following relations among the angles & distances in the triangle and its *in-touch triangle* $\overline{\mathsf{JKL}}$:

$$\psi_{\mathsf{J}} \;=\; 2\,\phi_{\mathsf{J}}\,, \quad \psi_{\mathsf{K}} \;=\; 2\,\phi_{\mathsf{K}}\,, \quad \psi_{\mathsf{L}} \;=\; 2\,\phi_{\mathsf{L}}\;;$$

$$u = r\cot(\theta_{\mathsf{A}}/2) = r\tan(\phi_{\mathsf{J}})\,, \quad j = 2r\cos(\theta_{\mathsf{A}}/2) = 2r\sin(\phi_{\mathsf{J}})\,;$$

$$v = r\cot(\theta_{\mathsf{B}}/2) = r\tan(\phi_{\mathsf{K}})\,, \quad k = 2r\cos(\theta_{\mathsf{B}}/2) = 2r\sin(\phi_{\mathsf{K}})\,;$$

$$w = r\cot(\theta_{\mathsf{C}}/2) = r\tan(\phi_{\mathsf{L}})\,, \quad \ell = 2r\cos(\theta_{\mathsf{C}}/2) = 2r\sin(\phi_{\mathsf{L}})\,.$$



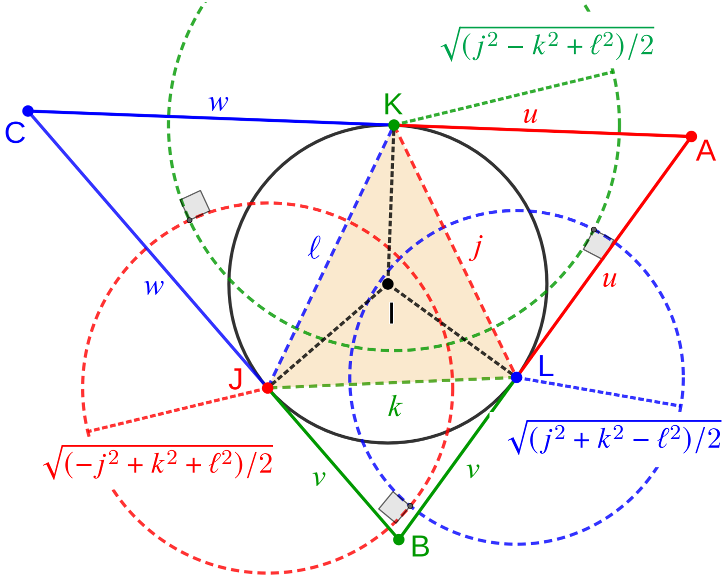
Eliminating r and applying the laws of sines & cosines then yields the algebraic relations between u, v, w and j, k, ℓ:

$$u \;=\; jk\ell \,/\, (-j^2 + k^2 + \ell^2)\,, \quad j \;=\; 4\,u^2vw \,/\, (bc)\;;$$

$$v \;=\; jk\ell \,/\, (j^2 - k^2 + \ell^2)\,, \quad k \;=\; 4\,uv^2w \,/\, (ac)\;;$$

$$w \;=\; jk\ell \,/\, (j^2 + k^2 - \ell^2)\,, \quad \ell \;=\; 4\,uvw^2 \,/\, (ab)$$

Getting the In-touch Triangle's Area without Using Heron's Formula



Getting the In-touch Triangle's Area without Using Heron's Formula

Since the vectors $\mathbf{j}, \mathbf{k}, \mathbf{l} \in \mathbb{R}^2$ representing the *in-touch points* $\bar{\mathsf{J}}$, $\bar{\mathsf{K}}$, $\bar{\mathsf{L}}$ are barycentric sums of the triangle's vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$, i.e.

$$\mathbf{j} \;=\; \frac{w}{a}\,\mathbf{b} + \frac{v}{a}\,\mathbf{c}\,, \quad \mathbf{k} \;=\; \frac{w}{b}\,\mathbf{a} + \frac{u}{b}\,\mathbf{c}\,, \quad \mathbf{l} \;=\; \frac{v}{c}\,\mathbf{a} + \frac{u}{c}\,\mathbf{b}\,,$$

the conformal blades of the triangle & in-touch triangle satisfy

$$abc\,\boldsymbol{n}_\infty \wedge \boldsymbol{j} \wedge \boldsymbol{k} \wedge \boldsymbol{l} \;=\; \Omega(u,v,w)\,\boldsymbol{n}_\infty \wedge \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}\;,$$

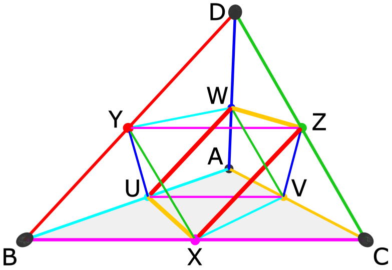
wherein $\boldsymbol{a} := \boldsymbol{n}_0 + \mathbf{a} + \boldsymbol{n}_\infty\,\mathbf{a}^2/2 \in \mathcal{G}_{3,1}$ etc. are <u>conformal</u> points.

Replacing the in-touch points $\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l}$ by <u>orthogonal</u> circles $\boldsymbol{j}'$, $\boldsymbol{k}'$, $\boldsymbol{l}'$ centered upon them (i.e. $\boldsymbol{j}' = \boldsymbol{j} - \boldsymbol{n}_\infty J/2$ etc.) "diagonalizes" their *Cayley-Menger determinant* <u>without</u> changing its value:

$$\|\boldsymbol{n}_\infty \wedge \boldsymbol{j}' \wedge \boldsymbol{k}' \wedge \boldsymbol{l}'\,\|^2 \;:=\; \det\begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & J & 0 & 0 \\ -1 & 0 & K & 0 \\ -1 & 0 & 0 & L \end{bmatrix} \quad \text{with} \quad \begin{cases} J := \frac{1}{2}\left(k^2 + \ell^2 - j^2\right)\;; \\ K := \frac{1}{2}\left(\ell^2 + j^2 - k^2\right); \\ L := \frac{1}{2}\left(j^2 + k^2 - \ell^2\right)\,. \end{cases}$$

It follows that $-\|\boldsymbol{n}_\infty \wedge \boldsymbol{j} \wedge \boldsymbol{k} \wedge \boldsymbol{l}\,\|^2 \;=\; JK + JL + KL.$

Heron for Tetrahedra, Concept #1: Medial Parallelograms & Octahedron

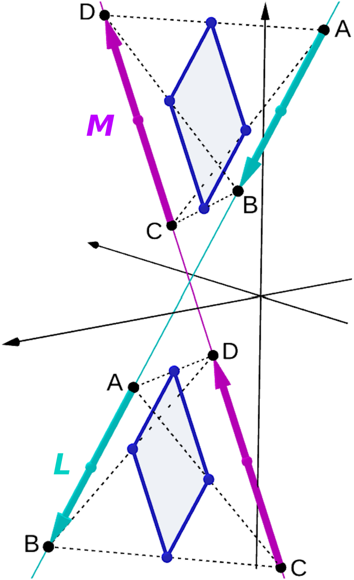


NB: parallel line segments in space have been drawn in the same colors, and the medial parallelogram $\overline{XUWZ}$ is drawn in **bold**; the diagonals of the octahedron $\overline{UZ}$, $\overline{VY}$, $\overline{WX}$, or the *bimedians* of the tetrahedron, are omitted for simplicity.

The **medial octahedron** of a tetrahedron $\overline{ABCD}$ is spanned by the midpoints $\overline{U}, \ldots, \overline{Z}$ of its edges. This octahedron's edges are pairwise parallel to those of the tetrahedron (in same color) but only half as long.

The octahedron's 3 diagonals intersect at its centroid $\overline{G}$, and any two of these diagonals are the diagonals of one of the tetrahedron's three **medial parallelograms** (also called "interior faces" in following).

Concept #1 (cont): Representing Medial Parallelograms in Geometric Algebra



In $\mathcal{G}_3$, the bivector of a medial parallelogram of $\overline{ABCD}$ is e.g.

$$\begin{aligned}\mathbf{P} \;:=\; & \left(\tfrac{1}{2}(\mathbf{b}+\mathbf{c}) - \tfrac{1}{2}(\mathbf{a}+\mathbf{c})\right) \\ & \wedge\; \left(\tfrac{1}{2}(\mathbf{d}+\mathbf{a}) - \tfrac{1}{2}(\mathbf{c}+\mathbf{a})\right) \\ =\; & \tfrac{1}{4}\,(\mathbf{b}-\mathbf{a})\wedge(\mathbf{d}-\mathbf{c})\end{aligned}$$

In the conformal model $\mathcal{G}_{4,1}$, its plane-bound bivector is related to the commutator product $\bowtie$ of line-bound vectors of opposite pairs of edges, e.g. $\boldsymbol{L} := \boldsymbol{n}_\infty \wedge \boldsymbol{a} \wedge \boldsymbol{b}$ and $\boldsymbol{M} := \boldsymbol{n}_\infty \wedge \boldsymbol{c} \wedge \boldsymbol{d}$, as

$$\boldsymbol{n}_\infty \wedge \boldsymbol{g} \wedge \mathbf{P} \;=\; \boldsymbol{n}_\infty \wedge \boldsymbol{g} \wedge (\boldsymbol{L} \bowtie \boldsymbol{M})$$

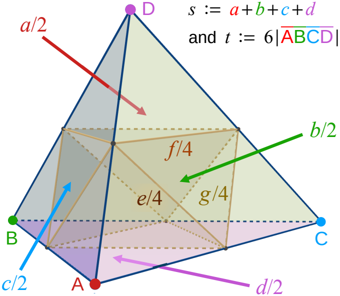
with $\boldsymbol{g}$ any point in the mid-plane.

Concept #2: Tetrahedron Inequalities Among the Seven Facial Areas

$r \;=\; t/s$ where

$s \;:=\; a+b+c+d$

and $t \;:=\; 6|\overline{\mathsf{ABCD}}|$



Much as in Euler's triangle notation,

$$d \;:=\; \|(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a})\|$$

will be twice the area of the exterior face opposite $\overline{\mathsf{D}}$, and similarly for the areas c, b, a opposite $\overline{\mathsf{C}}$, $\overline{\mathsf{B}}$, $\overline{\mathsf{A}}$, while

$$e \;:=\; \|(\mathbf{b}-\mathbf{a})\wedge(\mathbf{d}-\mathbf{c})\|$$

will be 4 times that of the indicated interior face, and similarly for f, g.

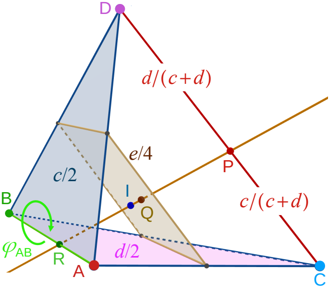
Then by the triangle inequality for Euclidean (bi)vectors, we have

$$(\mathbf{b}-\mathbf{a})\wedge(\mathbf{d}-\mathbf{c}) \;=\; (\mathbf{b}-\mathbf{a})\wedge(\mathbf{d}-\mathbf{a}) \;-\; (\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{a})$$

$$\Longrightarrow \quad e \;\le\; c+d \quad \text{as well as} \quad c \;\le\; d+e\,, \quad d \;\le\; c+e\,.$$

All in all, we get 18 such **tetrahedron inequalities**, which are stronger than the better-known areal inequalities $a \le b+c+d$ etc.

Concept #3: The Areal Laws of Sines & Cosines; McConnell's Rigidity Theorem



Dihedral Angle Bisectors

The plane bisecting the dihedral angle φ_{AB} divides $\overline{\mathsf{CD}}$ in the ratios shown, and similarly for the other dihedrals.

The Areal Law of Cosines

The cosine of φ_{AB} (etc.) satisfies:

$$c\,d\,\cos\left(\varphi_{\mathsf{AB}}\right) \;=\; \tfrac{1}{2}\left(c^2 + d^2 - e^2\right)$$

The Areal Law of Sines

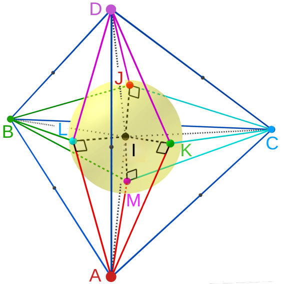
The squared sine of φ_{AB} (etc) is given by the Heron-like formula:

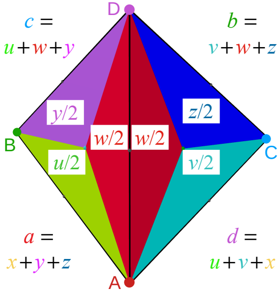
$$c^2\,d^2\,\sin\left(\varphi_{\mathsf{AB}}\right)^2 \;=\; \tfrac{1}{4}\,(c+d+e)(c+d-e)(c-d+e)(-c+d+e)$$

McConnell's Rigidity Theorem

The seven areas determine a non-degenerate tetrahedron up to isometry (proof: $cd\sin(\varphi_{\mathsf{AB}}) = \|\mathbf{a}-\mathbf{b}\|\,t^2$ & $t^4 = T(a,\ldots,g)$).

Concept #4: The Natural (Analogs of the Heron) Parameters





The insphere "touches" the exterior faces at points $\bar{\mathsf{J}}$, $\bar{\mathsf{K}}$, $\bar{\mathsf{L}}$, $\bar{\mathsf{M}}$ all at a distance r (the inradius) from its center $\bar{\mathsf{I}}$. Thus the distances from each vertex to its three adjacent **in-touch points** are equal.

It follows that pairs of **contact triangles** sharing a common edge are congruent. The **natural parameters** $u, \ldots, z$ of a tetrahedron are defined as twice the areas of these 6 congruent pairs (1 per edge).

Clearly the natural parameters determine the areas of the exterior faces (as seen on the left). It can be shown they also determine those of the interior, and hence the tetrahedron itself up to isometry.

Formulae for the Natural Parameters, and the Inverse Parameters

With $s := a + b + c + d$ as twice the exterior surface area, we have:

$$
\begin{aligned}
u &= r\,\|\mathbf{a}-\mathbf{b}\|\cot\big(\varphi_{\mathsf{AB}}/2\big) = (c+d+e)(c+d-e)\;/\;(2\,s)\\
v &= r\,\|\mathbf{a}-\mathbf{c}\|\cot\big(\varphi_{\mathsf{AC}}/2\big) = (b+d+f)(b+d-f)\;/\;(2\,s)\\
w &= r\,\|\mathbf{a}-\mathbf{d}\|\cot\big(\varphi_{\mathsf{AD}}/2\big) = (b+c+g)(b+c-g)\;/\;(2\,s)\\
x &= r\,\|\mathbf{b}-\mathbf{c}\|\cot\big(\varphi_{\mathsf{BC}}/2\big) = (a+d+g)(a+d-g)\;/\;(2\,s)\\
y &= r\,\|\mathbf{b}-\mathbf{d}\|\cot\big(\varphi_{\mathsf{BD}}/2\big) = (a+c+f)(a+c-f)\;/\;(2\,s)\\
z &= r\,\|\mathbf{c}-\mathbf{d}\|\cot\big(\varphi_{\mathsf{CD}}/2\big) = (a+b+e)(a+b-e)\;/\;(2\,s)
\end{aligned}
$$

We also **define** the corresponding "inverse" parameters as:

$$
\begin{aligned}
\tilde{u} &= r\,\|\mathbf{a}-\mathbf{b}\|\tan\big(\varphi_{\mathsf{AB}}/2\big) = (e+d-c)(e-d+c)\;/\;(2\,s)\\
\tilde{v} &= r\,\|\mathbf{a}-\mathbf{c}\|\tan\big(\varphi_{\mathsf{AC}}/2\big) = (f+d-b)(f-d+b)\;/\;(2\,s)\\
\tilde{w} &= r\,\|\mathbf{a}-\mathbf{d}\|\tan\big(\varphi_{\mathsf{AD}}/2\big) = (g+c-b)(g-c+b)\;/\;(2\,s)\\
\tilde{x} &= r\,\|\mathbf{b}-\mathbf{c}\|\tan\big(\varphi_{\mathsf{BC}}/2\big) = (g+d-a)(g-d+a)\;/\;(2\,s)\\
\tilde{y} &= r\,\|\mathbf{b}-\mathbf{d}\|\tan\big(\varphi_{\mathsf{BD}}/2\big) = (f+c-a)(f-c+a)\;/\;(2\,s)\\
\tilde{z} &= r\,\|\mathbf{c}-\mathbf{d}\|\tan\big(\varphi_{\mathsf{CD}}/2\big) = (e+b-a)(e-b+a)\;/\;(2\,s)
\end{aligned}
$$

Heron's Formula for Tetrahedra

The inverse parameters are related to the areas of the triangles into which the exterior faces are divided by their "ex-touch" points. They may be expressed in terms of the natural parameters as:

$$\begin{aligned}
\tilde{u} &= 2\left((v+x)(w+y) - uz\right)/s\,, & \tilde{z} &= 2\left((v+w)(x+y) - uz\right)/s \\
\tilde{v} &= 2\left((u+x)(w+z) - vy\right)/s\,, & \tilde{y} &= 2\left((u+w)(x+z) - vy\right)/s \\
\tilde{w} &= 2\left((u+y)(v+z) - wx\right)/s\,, & \tilde{x} &= 2\left((u+v)(y+z) - wx\right)/s
\end{aligned}$$

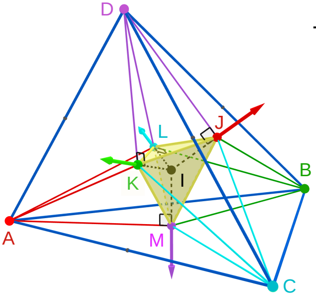
Theorem:

The volume $t := 6|\overline{\mathsf{ABCD}}|$ & inradius $r = t/s$ of a tetrahedron are given in terms of the natural parameters and $s = 2\,(u + \cdots + z)$ by

$$t^4 \;=\; s^2\,\Omega \quad \& \quad r^4 \;=\; \Omega/s^2\,, \qquad \text{wherein}$$

$$\Omega \;=\; \Omega(u,v,w,x,y,z) \;:=\; -\det\begin{bmatrix} 0 & u & v & w \\ u & 0 & x & y \\ v & x & 0 & z \\ w & y & z & 0 \end{bmatrix}.$$

Proofs of the Theorem, the In-Touch Tetrahedron, and the Areal Vectors



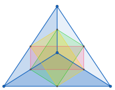
The theorem can be proven by either:

1. substituting $D_{\mathrm{AB}} \leftarrow u\tilde{u}/r^2$ etc. in the 4-point Cayley-Menger determinant $\Delta_D[\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}]$ and simplifying (a lot)
2. expressing the **in-touch tetrahedron** $\overline{\mathrm{JKLM}}$ volume in terms of the natural parameters & using the relation:

$$abcd\,\boldsymbol{n}_\infty \wedge \boldsymbol{j} \wedge \boldsymbol{k} \wedge \boldsymbol{l} \wedge \boldsymbol{m} \;=\; -\Omega(u,v,w,x,y,z)\,\boldsymbol{n}_\infty \wedge \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c} \wedge \boldsymbol{d}$$

(which follows as in 2D from the barycentric representations of $\boldsymbol{j},\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}$ together with the determinant multiplication theorem).

Yet another proof can be given based on the fact that the determinant of the Gram matrix of the **areal vectors** (facet normals weighted by the areas, as above) of any 3 facets also equals t^4.

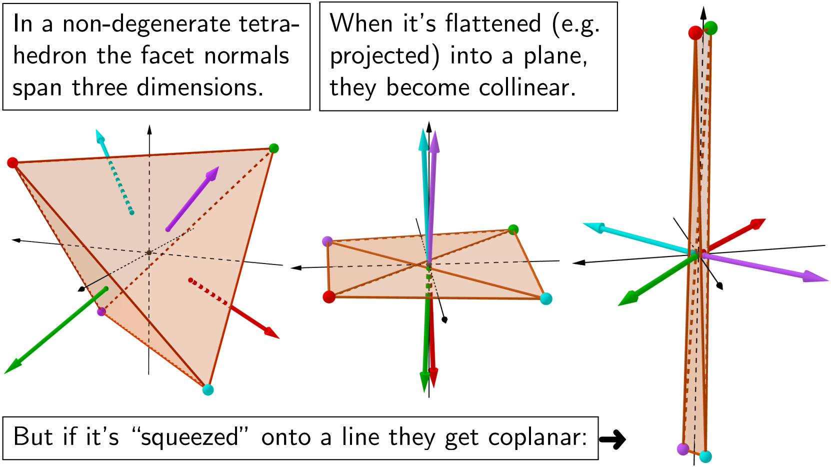


So I will now address the question everyone is probably asking:

Why should I care about yet-another way of computing tetrahedron volumes (and a convoluted one at that)?

In a non-degenerate tetrahedron the facet normals span three dimensions.

When it's flattened (e.g. projected) into a plane, they become collinear.



But if it's "squeezed" onto a line they get coplanar: ➜

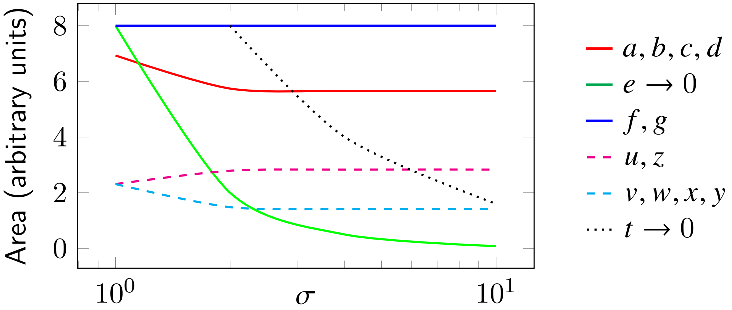
The Zeros as Limits of Affine Squeeze (and Stretch) Transformations

Analytically, the zeros of Ω are the limits of a sequence of affine transformations of the form

$$\mathcal{A}_\sigma \;:=\; \begin{bmatrix} \sigma^{-1} & 0 & 0 \\ 0 & \sigma^{-1} & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

applied to generic tetrahedra in space as $\sigma \to \infty$.

Areas & N.P.s vs. squeeze factor σ

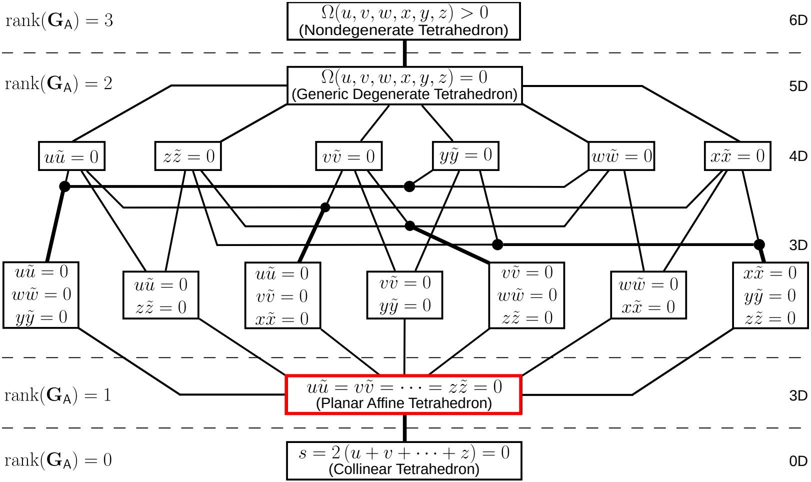


These Zeros Constitute a Profound Difference between 2 & 3 Dimensions

- Triangles with any given area and vertices at infinity also exist, but two of their Heron parameters become infinite.
- In contrast, tetrahedra with finite natural parameters and with $\Omega = 0$ do not correspond to quadruples of points in the Euclidean plane (in any obvious way), even though the set of planar quadruples also has $4 \cdot 2 - 3 = 5$ degrees of freedom.
- The ratios of their squared distances $u\tilde{u}/v\tilde{v}, \ldots, y\tilde{y}/z\tilde{z}$ are generically finite, but the zeros of Ω are not quadruples on a line in the projective completion of Euclidean space, because such lines have only one point at infinity.
- These zeros are clearly non-physical, but they are perfectly well-defined mathematically and full of geometric structure.
- And it seems no one's ever before noticed that such a novel "completion" of the Euclidean symmetric product $\mathbb{E}^{3\otimes 4}$ exists!

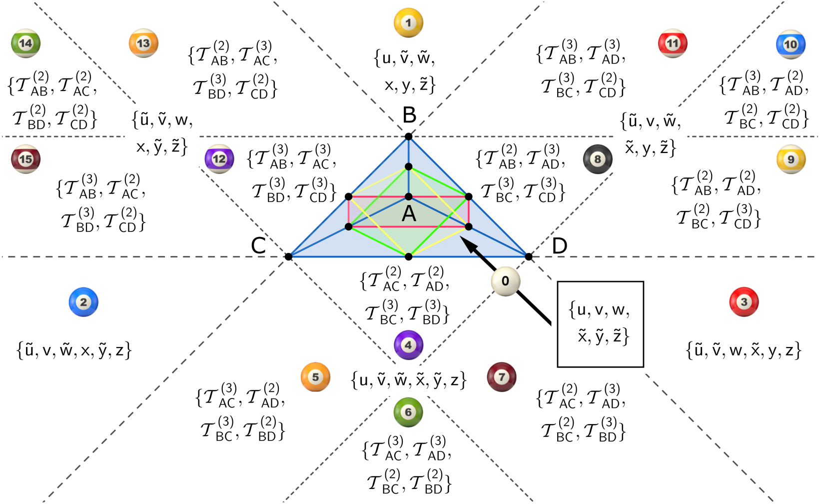
The Structure of the 5D Set of Zeros as a Combinatorial Lattice (PO-Set)

Hasse diagram of lattice structure indudced by $\Omega = 0$ *and all independent combinations of complementary products* $u\tilde{u}, v\tilde{v}$ *etc. vanishing (ranks on left are dimension of span of facet normals).*



Rank 1 Zeros Correspond to 4-Point Configurations in Equi-Affine Plane

Plane divided into 16 regions by lines along sides of triangle $\overline{ABC}$ and their parallels thru its vertices (labeled by pool ball icons ✌).



Rank 1 Zeros Correspond to 4-Point Configurations in Equi-Affine Plane

Plane divided into 16 regions by lines along sides of triangle $\overline{\text{BCD}}$ and their parallels thru its vertices (labeled by pool ball icons ✌).

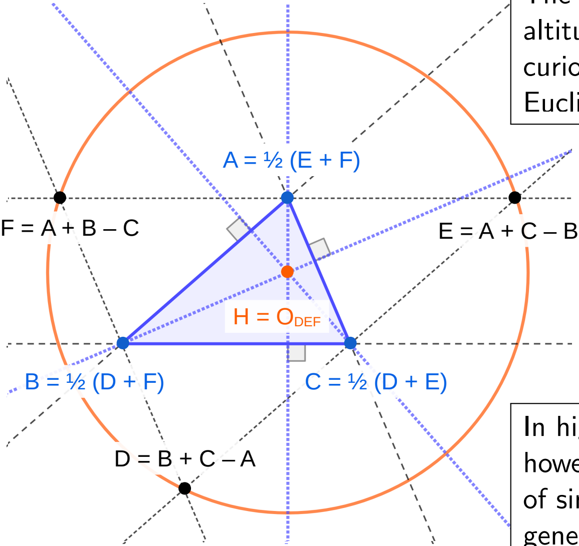
Moving $\overline{\text{A}}$ into various regions while keeping $\overline{\text{BCD}}$ fixed makes the sets of natural & inverse parameters indicated vanish.

The 7 regions separated by dashed lines thru sides of $\overline{\text{BCD}}$ have differing "chirotopes" (aka affine oriented matroids).

Dotted lines thru the vertices divide 3 of those regions into 4 subregions each, distinguished by which tetrahedron inequalities saturate, e.g. $\mathcal{T}^{(2)}_{\text{AB}} \equiv e + d = c$, $\mathcal{T}^{(3)}_{\text{AB}} \equiv e + c = d$, etc.

This classification of affine point configurations was studied intensively in the late 20^{th} century by Goodman & Pollack, without knowledge of the tetrahedron inequalities.

What Euclid Missed, Gauss Got Right, and the Tetrahedron Left Behind

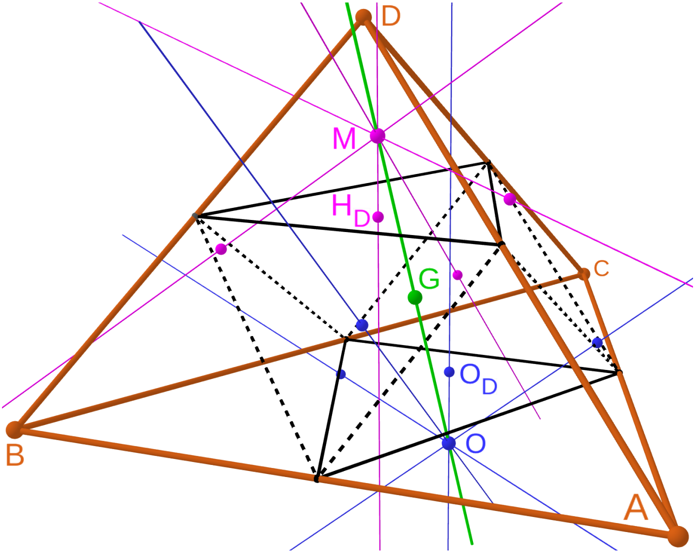


The concurrence of the altitudes of a triangle is curiously missing from Euclid's Elements.

Gauss proved it by noting that the orthocenter $\overline{\mathrm{H}}$ is the circum-center $\overline{\mathrm{O}}_{\mathrm{DEF}}$ of the "anti-medial" triangle $\overline{\mathrm{DEF}}$.

In higher dimensions, however, the altitudes of simplices are <u>not</u> generally concurrent!

The Monge Point and the Medial Octahedron's Facial Orthocenters

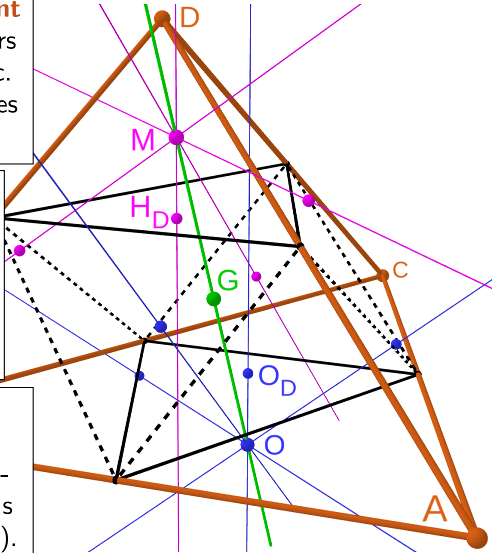


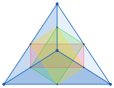
The Monge Point and the Medial Octahedron's Facial Orthocenters

A tetrahedron's **Monge point** $\overline{\mathrm{M}}$ is where the perpendiculars thru the orthocenters $\overline{\mathrm{H}}_{\mathrm{D}}$ etc. of its medial octahedron faces <u>inside</u> the tetrahedron meet.

When the altitudes likewise concur, they do so at $\overline{\mathrm{M}}$, which is also the reflection of the circumcenter $\overline{\mathrm{O}}$ in the centroid $\overline{\mathrm{G}}$, and they're all on the Euler line (green).

The perpendiculars thru the orthocenters of the octahedron's "surface" faces, or circumcenters $\overline{\mathrm{O}}_{\mathrm{D}}$ etc. of $\overline{\mathrm{ABCD}}$'s facets, meet at $\overline{\mathrm{O}}$ (blue lines).





Orthocentric Tetrahedra: The True Generalization of Triangles to 3D?


Orthocentric Simplices as the True Generalizations of Triangles

Mowaffaq Hajja & Horst Martini

The Mathematical Intelligencer
2013

Simplicies where the altitudes do concur are termed **orthocentric**, and behave more like triangles, e.g. they are equilateral iff the incenter and centroid coincide.

In particular, a tetrahedron is orthocentric if & only if either:

- $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{c}) \;=\; (\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{b}) \;=\; (\mathbf{c}-\mathbf{b})\cdot(\mathbf{d}-\mathbf{a}) \;=\; 0$
- $\|\mathbf{b}-\mathbf{a}\|^2+\|\mathbf{d}-\mathbf{c}\|^2 = \|\mathbf{c}-\mathbf{a}\|^2+\|\mathbf{d}-\mathbf{b}\|^2 = \|\mathbf{c}-\mathbf{b}\|^2+\|\mathbf{d}-\mathbf{a}\|^2$
- there exist $D_\mathsf{A}, D_\mathsf{B}, D_\mathsf{C}, D_\mathsf{D} \in \mathbb{R}$ such that their pairwise sums equal the squared inter-vertex distances, i.e.

$$\|\mathbf{b}-\mathbf{a}\|^2 \;=\; D_\mathsf{A}+D_\mathsf{B}\,,\;\ldots\,,\;\|\mathbf{d}-\mathbf{c}\|^2 \;=\; D_\mathsf{C}+D_\mathsf{D}$$

Note that triangles always satisfy this last condition with

$$D_\mathsf{A} \coloneqq \tfrac{1}{2}\left(-a^2+b^2+c^2\right),\; D_\mathsf{B} \coloneqq \tfrac{1}{2}\left(a^2-b^2+c^2\right),\; D_\mathsf{C} \coloneqq \tfrac{1}{2}\left(a^2+b^2-c^2\right).$$

The Amazingly Simple Distance Geometry of Orthocentric Tetrahedra

The Cayley-Menger determinants and other squared distance-area-volume relations simplify amazingly in the orthocentric case:

- $a^2 \;=\; D_{\mathsf{B}} D_{\mathsf{C}} + D_{\mathsf{B}} D_{\mathsf{D}} + D_{\mathsf{C}} D_{\mathsf{D}}$ (and similarly b^2, c^2, d^2)
- $e^2 \;=\; (D_{\mathsf{A}} + D_{\mathsf{B}})(D_{\mathsf{C}} + D_{\mathsf{D}})$ (and similarly f^2, g^2)
- $(a^2 + b^2 - e^2)/2 \;=\; D_{\mathsf{C}} D_{\mathsf{D}}$ (and similarly for the rest)
- $t^2 \;=\; D_{\mathsf{A}} D_{\mathsf{B}} D_{\mathsf{C}} + D_{\mathsf{A}} D_{\mathsf{B}} D_{\mathsf{D}} + D_{\mathsf{A}} D_{\mathsf{C}} D_{\mathsf{D}} + D_{\mathsf{B}} D_{\mathsf{C}} D_{\mathsf{D}}$
- $R^2 \;=\; \left(a^2 D_{\mathsf{A}}^2 + b^2 D_{\mathsf{B}}^2 + c^2 D_{\mathsf{C}}^2 + d^2 D_{\mathsf{D}}^2\right) \Big/ (4\,t^2)$ (circum-radius)

Unfortunately we cannot get the squared inradius $r^2 = t^2/s^2$ (or the natural parameters) from the D's without taking square roots; the exterior surface area $s = a + b + c + d$ involves four of those!

It **is** possible to go the other way, e.g. $D_{\mathsf{A}} = (u\tilde{u} + v\tilde{v} - x\tilde{x})/(2\,r^2)$; moreover, a tetrahedron is orthocentric if & only if

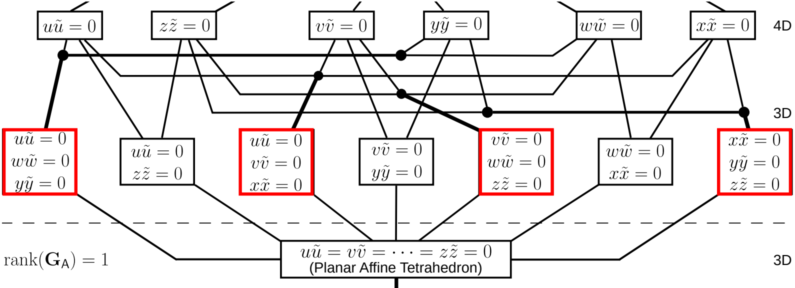
$$u\tilde{u} + z\tilde{z} \;=\; v\tilde{v} + y\tilde{y} \;=\; w\tilde{w} + x\tilde{x} \quad \left(=\; (D_{\mathsf{A}} + D_{\mathsf{B}} + D_{\mathsf{C}} + D_{\mathsf{D}})\,r^2\,\right).$$

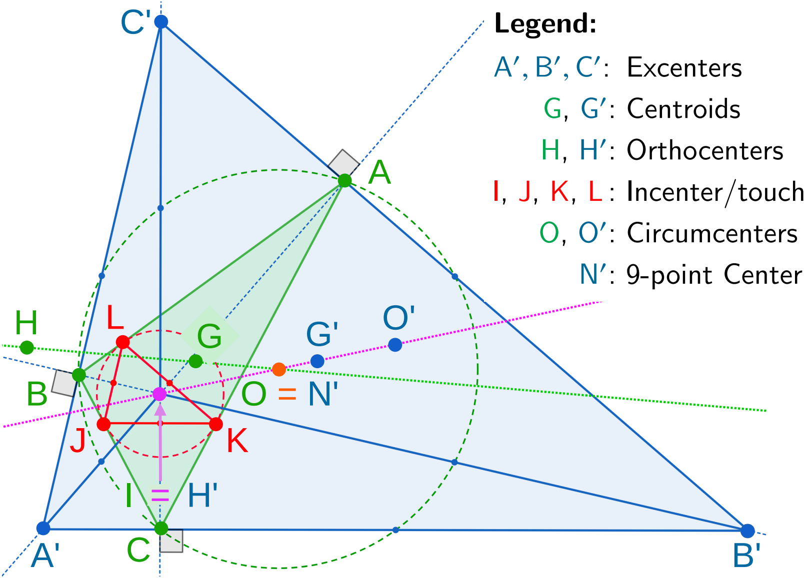
Can Orthocentric Tetrahedra Have Facetal Vectors which Span a Plane?

These latter conditions don't involve r^2, so they apply to degenerate tetrahedra, i.e. to the zeros of Ω, which leads to the question:

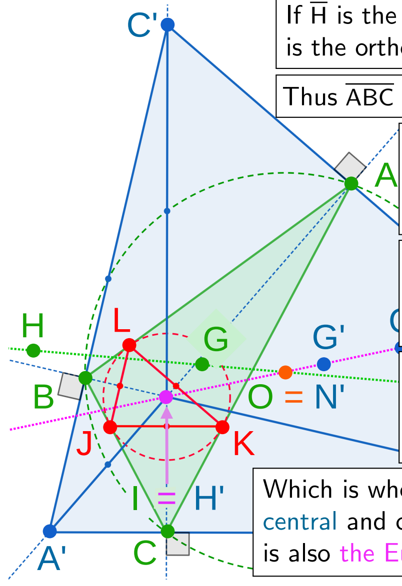
> *Can the squared distances in orthocentric tetrahedra become infinite while all the areas & natural parameters stay finite, or do they also behave like triangles in that way?*

A detailed analysis of the equations shows that while most of the rank 2 zeros of Ω are eliminated by the orthocentricity constraints, one kind is **not**; in the previous lattice diagram, they are (in red):




C'
A
B
C
A'
B'
H
L
G
G'
O'
O = N'
J
K
I = H'
Legend:
A′, B′, C′: Excenters
G, G′: Centroids
H, H′: Orthocenters
I, J, K, L: Incenter/touch
O, O′: Circumcenters
N′: 9-point Center

Orthocentric Systems & In/Excenter Geometry in the Plane



If $\overline{\mathrm{H}}$ is the orthocenter of a triangle $\overline{\mathrm{ABC}}$, then $\overline{\mathrm{A}}$ is the orthocenter of $\overline{\mathrm{BCH}}$, and likewise for $\overline{\mathrm{B}}$, $\overline{\mathrm{C}}$.

Thus $\overline{\mathrm{ABC}}$ & $\overline{\mathrm{H}}$ is called an **orthocentric system**.

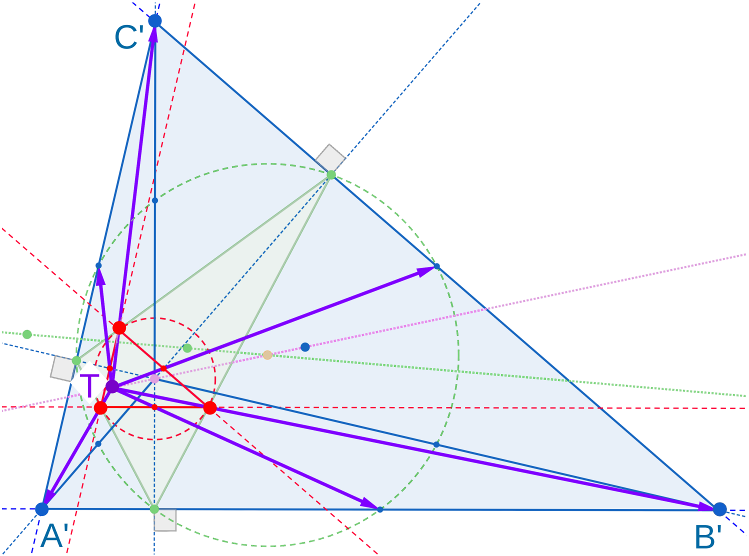
If $\overline{\mathrm{A'B'C'}}$ is the **excentral triangle** of $\overline{\mathrm{ABC}}$, then $\overline{\mathrm{A'B'C'}}$ & $\overline{\mathrm{I}}$ constitute an orthocentric system with $\overline{\mathrm{I}} = \overline{\mathrm{H'}}$.

The triangle $\overline{\mathrm{ABC}}$ is called the **orthic triangle** of the system; its vertices are the feet of the altitudes of $\overline{\mathrm{A'B'C'}}$.

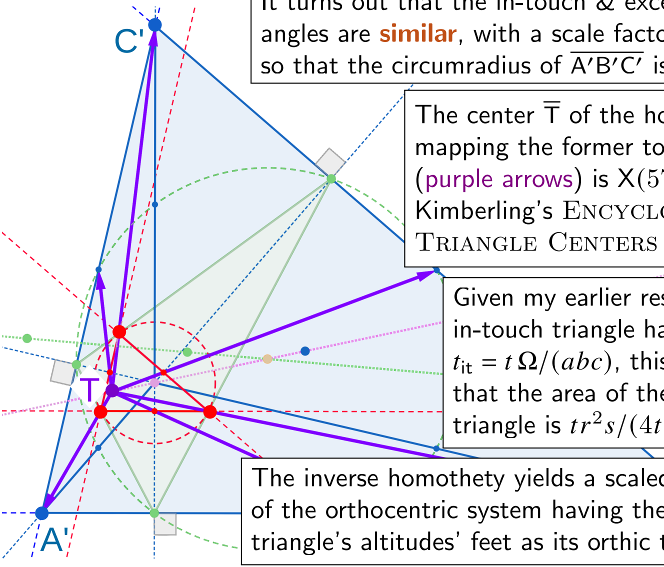
The circumcircle of $\overline{\mathrm{ABC}}$ is the nine-point circle of all four triangles in the system, and their centers are $\overline{\mathrm{O}} = \overline{\mathrm{N'}}$.

Which is where the Euler lines (dotted) of the excentral and of its orthic triangle meet; the former is also the Euler line of the in-touch triangle $\overline{\mathrm{JKL}}$.

The Similarity of the In-Touch and Excentral Triangles



The Similarity of the In-Touch and Excentral Triangles



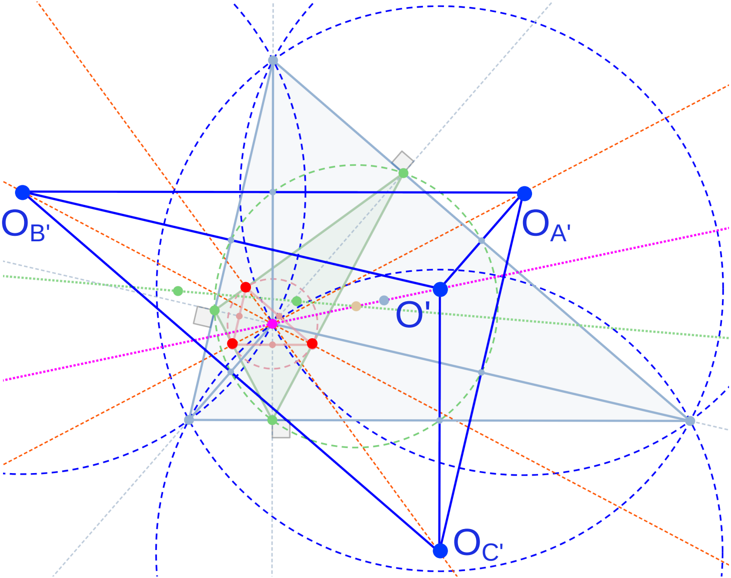
It turns out that the in-touch & excentral triangles are **similar**, with a scale factor of $2\,R/r$, so that the circumradius of $\overline{\mathrm{A'B'C'}}$ is $2R$.

The center $\overline{\mathrm{T}}$ of the homothety mapping the former to the latter (purple arrows) is X(57) in Clark Kimberling's Encyclopedia of Triangle Centers (of 70,000).

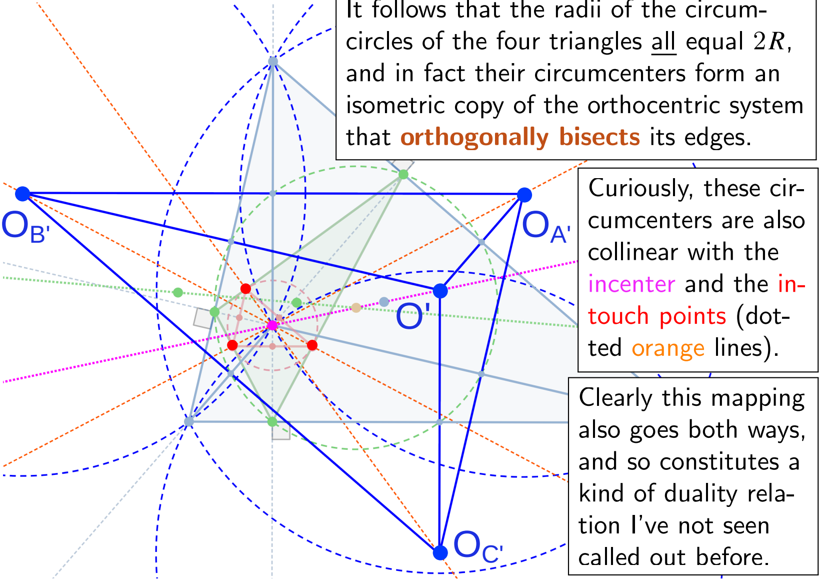
Given my earlier result that the in-touch triangle has an area of $t_{\mathrm{it}} = t\,\Omega/(abc)$, this also shows that the area of the excentral triangle is $tr^2s/(4tR) = 2Rs$.

The inverse homothety yields a scaled-down copy of the orthocentric system having the in-touch triangle's altitudes' feet as its orthic triangle.

The Equality of the Circum-Radii in an Orthocentric System



The Equality of All Four Circum-Radii in an Orthocentric System

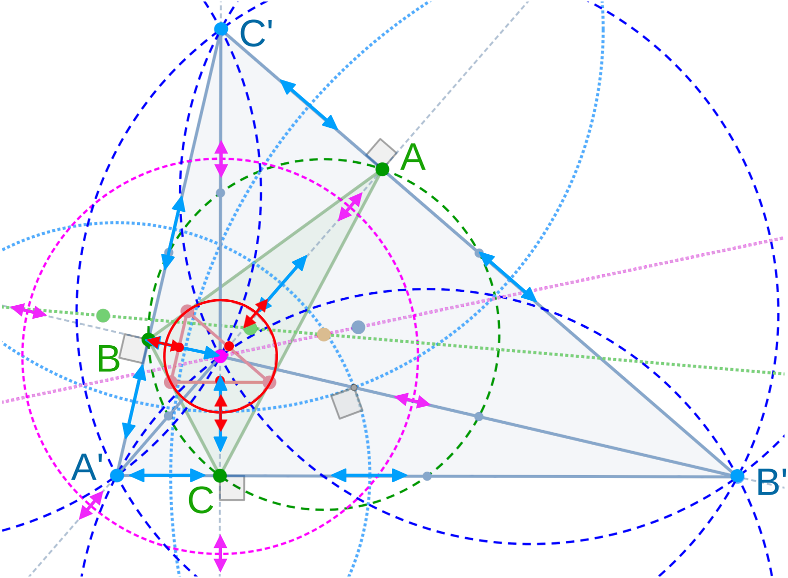


It follows that the radii of the circumcircles of the four triangles <u>all</u> equal $2R$, and in fact their circumcenters form an isometric copy of the orthocentric system that **orthogonally bisects** its edges.

Curiously, these circumcenters are also collinear with the incenter and the in-touch points (dotted orange lines).

Clearly this mapping also goes both ways, and so constitutes a kind of duality relation I've not seen called out before.

Inversion in the Incircle, the Polar Circles, and Conformal Duality



Inversion in the Incircle, the Polar Circles, and Conformal Duality

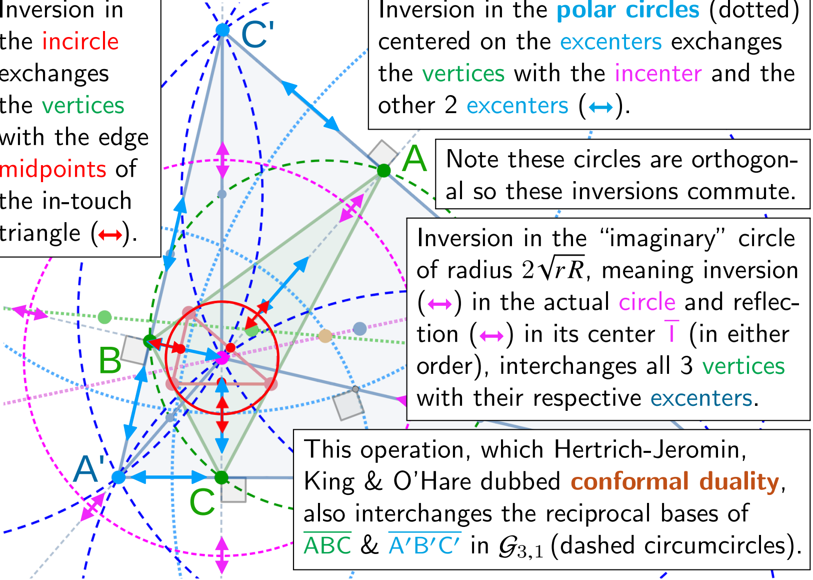
Inversion in the incircle exchanges the vertices with the edge midpoints of the in-touch triangle (↔).

Inversion in the **polar circles** (dotted) centered on the excenters exchanges the vertices with the incenter and the other 2 excenters (↔).

Note these circles are orthogonal so these inversions commute.

Inversion in the "imaginary" circle of radius $2\sqrt{rR}$, meaning inversion (↔) in the actual circle and reflection (↔) in its center $\overline{\mathrm{I}}$ (in either order), interchanges all 3 vertices with their respective excenters.

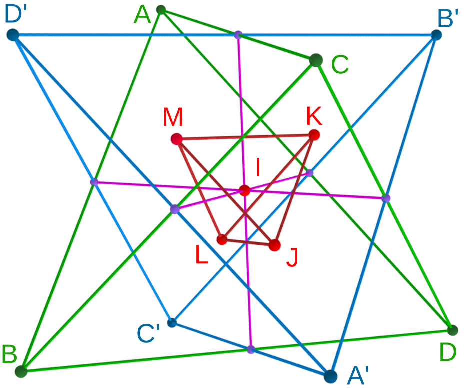
This operation, which Hertrich-Jeromin, King & O'Hare dubbed **conformal duality**, also interchanges the reciprocal bases of $\overline{\mathrm{ABC}}$ & $\overline{\mathrm{A'B'C'}}$ in $\mathcal{G}_{3,1}$ (dashed circumcircles).



But Almost None of This Generalizes to Three (or Higher) Dimensions!

If one constructs the in- & excenters of a tetrahedron (green) one finds, even if the tetrahedron is orthocentric, that its in-touch (red) and excentral (blue) tetrahedra:

- Are not similar to one another;
- Are not orthocentric, so their altitudes are not concurrent;
- And that the vertices of $\overline{ABCD}$ are not the altitudes' feet on the facets of $\overline{A'B'C'D'}$.

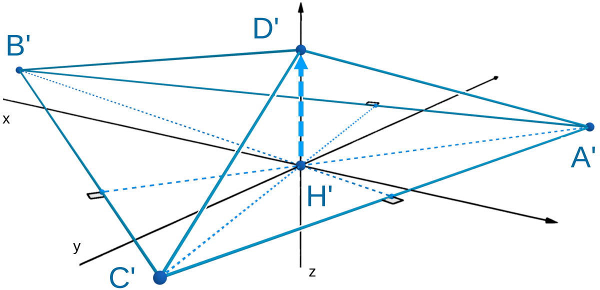


But the green edges **do** intersect the blue edges in the vertices of an octahedron (not the medial) that bisect the dihedral angles **and** span diagonals meeting at $\overline{I}$.

The Trick Is to Make the Excentral Tetrahedron Be Orthocentric!

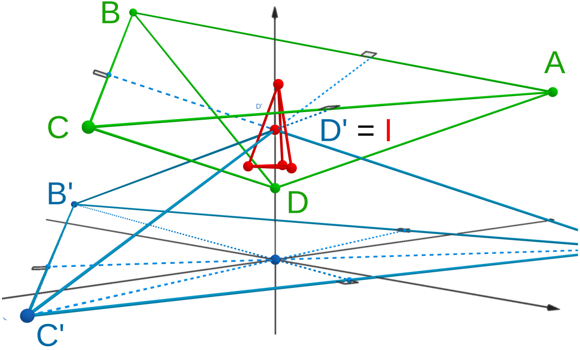
➤ One way to construct an orthocentric tetrahedron is to take a planar orthocentric system and lift its orthocenter out of the plane.

➤ I have also found formulae that take an orthocentric tetrahedron's parameters $D_{\mathrm{A'}}, D_{\mathrm{B'}} D_{\mathrm{C'}}, D_{\mathrm{D'}}$ and returns the vertices $\overline{\mathrm{A}}$, $\overline{\mathrm{B}}$, $\overline{\mathrm{C}}$, $\overline{\mathrm{D}}$ of the tetrahedron with it as its excentral tetrahedron, but ...



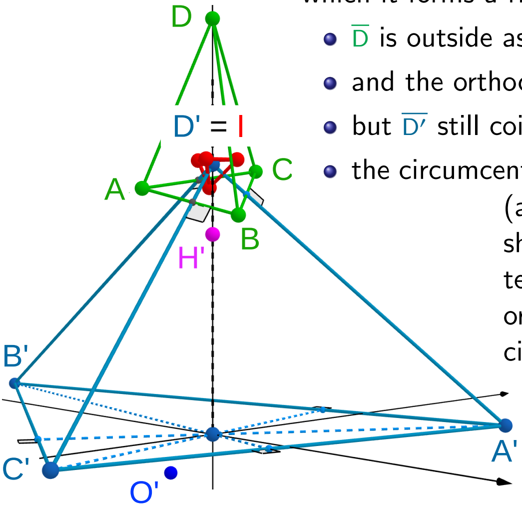
The Trick Is to Make the Excentral Tetrahedron Be Orthocentric!

- One way to construct an orthocentric tetrahedron is to take a planar orthocentric system and lift its orthocenter out of the plane.
- I have also found formulae that take an orthocentric tetrahedron's parameters $D_{A'}, D_{B'}, D_{C'}, D_{D'}$ and return the vertices $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ of the tetrahedron with it as its excentral tetrahedron, but . . .



- for small lifts the incenter $\overline{I}$ coincides with the lifted vertex $\overline{D'}$, while the new tetrahedron's vertex $\overline{D}$ is inside of $\overline{A'B'C'D'}$, and that is obviously not right!

The Trick Is to Make the Excentral Tetrahedron Be Orthocentric!



- Once $\overline{D'}$ is gets above the height at which it forms a right prism with $\overline{A'B'C'}$:
  - $\overline{D}$ is outside as it should be,
  - and the orthocenter $\overline{H'}$ is inside,
  - but $\overline{D'}$ still coincides with $\overline{I}$, while
  - the circumcenter $\overline{O'}$ is outside $\overline{A'B'C'D'}$ (although it can be shown that any excentral tetrahedron, orthocentric or otherwise, contains its circumcenter).

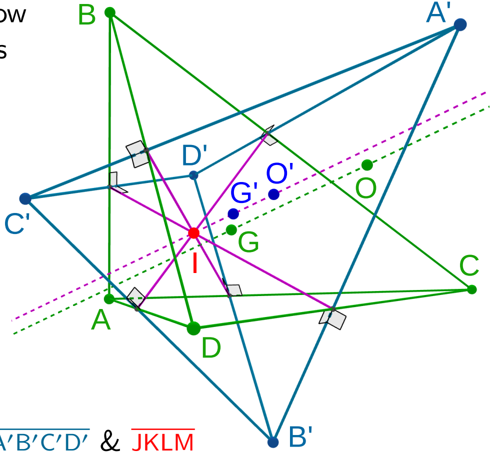
Analogies with Triangles When the Ex-Central Tetrahedron Is Orthocentric

- Happily, on lifting $\overline{D'}$ a bit more, $\overline{O'}$ passes inside, $\overline{ABCD}$ changes **discontinuously**, and everything pops miraculously into place.

- First and foremost, $\overline{I}$ now coincides with $\overline{H'}$, and $\overline{D'}$ is outside of $\overline{ABCD}$.

Second, the edges of $\overline{ABCD}$ & $\overline{A'B'C'D'}$ become orthogonal.

Third, the diagonals of the excentral octahedron also become orthogonal to its edges (but not to those of $\overline{ABCD}$).

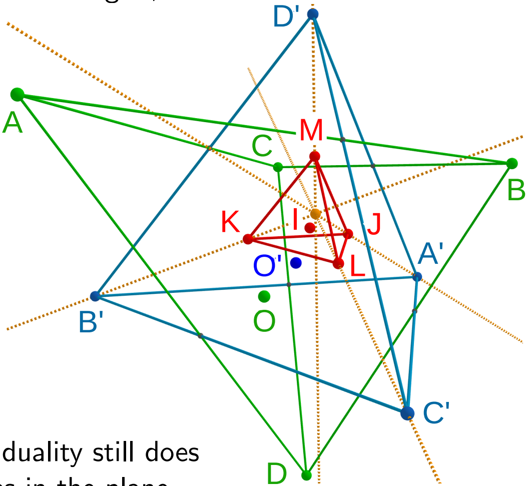


Fourth, the Euler lines of $\overline{A'B'C'D'}$ & $\overline{JKLM}$ (not shown) coincide and are parallel that of $\overline{ABCD}$ (dashed lines).

Analogies with Triangles When the Ex-Central Tetrahedron Is Orthocentric

Last but not least: The in-touch $\overline{\mathrm{JKLM}}$ and excentral $\overline{\mathrm{A'B'C'D'}}$ tetrahedra are similar as in triangles, but:

- The scale factor is not $2\,R/r$ as it is in triangles, nor any simple rational multiple thereof.
- The homothetic center (orange) of $\overline{\mathrm{JKLM}}$ & $\overline{\mathrm{A'B'C'D'}}$ is on their Euler line, but I haven't otherwise characterized it.



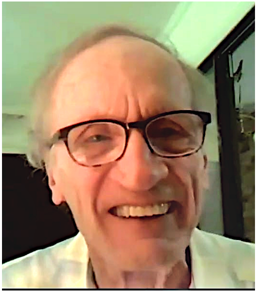
- And alas, conformal duality still does not work the way it does in the plane.

Open Questions about these Orthocentric Zeros and in General

- Because the (orthocentric) excentral tetrahedron depends on four parameters and uniquely determines the base tetrahedron, the latter also constitutes a four-parameter manifold of some kind: **can it be independently characterized?**
- Although the excentral tetrahedron can't be "squashed" into a plane (or line) without its circumcenter getting outside it, you can squash its base tetrahedron into a plane by sending one or more excentral vertices to infinity: **can you reach all the zeros with an orthocentric excentral tetrahedron?**
- There is no (finite) quadruple of null vectors in $\mathcal{G}_{4,1}$ that corresponds to the rank 2 zeros of Ω: **what is the "simplest" geometric algebra containing $\mathcal{G}_{4,1}$ that can represent all the zeros explicitly?**

On that last question: I suspect it is $\mathcal{G}_{4,2}$, the geometric algebra of **Lie sphere** (or "contact") geometry.

A Couple of Acknowledgments



I would first like to acknowledge my co-author Garret Sobczyk (who was unable to attend for personal reasons), and in particular for the matrix representation of $\mathcal{G}_{4,1}$ that I used to validate all my calculations. As we all know, his contributions to the field of geometric algebra are second to none!

And since we're in The Netherlands, I'd also like to acknowledge Johan Jacob (Jaap) Seidel (1919-2001), whose remarks during my circa 1985 visit to him in Eindhoven eventually led me to see the connection between Hestenes' work on $\mathcal{G}_{n+1,1}$ and distance geometry.