Geometric Algebra and symmetry in crystallography and physics

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Bible, John 3:8:

The wind blows wherever it pleases.
You hear its sound, but you cannot tell where it comes from or where it is going. So it is with everyone born of the Spirit.

Note: The Greek for *Spirit* is the same as that for *wind*.

- My family
- C. Perwass, D. Proserpio, S. Sangwine
- Organizers of the AGACSE 2024

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Dedicated to: Andreas Soennichsen, Andreas Schoefbeck, Takayuki Miyazawa

Outline + adverts

1. Introduction

2. Symmetries of $Cl(3,1)$ and $Cl(1,3)$ multivectors generated by space inversion, reversion and principal reverse

3. On symmetries of $Cl(3,1)$ and $Cl(1,3)$ related to elementary particles: charge conjugation, parity reversal and time reversal

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Multivector symmetry in crystallography and physics

- [7] and [6] classified multivectors by symmetries under space inversion (grade involution) $\hat{1}$, time reversal $1'$ (no algebraic expression) and reversion $\tilde{1}$.
- [7] says *One could perhaps explore charge reversal ($\hat{C}$), parity reversal ($\hat{P}$) and time reversal ($\hat{T}$) in the relativistic context [8]*.
- In the standard model of elementary particles violation of parity (space inversion) symmetry by weak interaction, and of $\hat{C}\hat{P}$ symmetry. Yet, strong interactions preserve $\hat{C}\hat{P}$ symmetry [4].
- Here: $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries [5] on multivectors of $Cl(3,1)$, a (geometric) algebra for space-time physics. First effect on 16 basis blades of $Cl(3,1)$, then including functional dependence of coefficients in linear combinations that might express spinors or other physical quantities.
- [8], [5] use $Cl(1,3)$, we start first with $Cl(3,1)$ because its volume-time subalgebra $\{1, e_0, e_{123}, e_{0123}\}$ isomorphic to quaternions, e_0 expresses the time direction, important for space-time Fourier transforms [10].

Rational use of Grassmann and Clifford algebra

- [7], [6] use Clifford algebra to generalize cross product of 3D.
- J. G. Grassmann (father of Hermann) originally introduced the characterization of crystal planes by orthogonal vectors, now called Miller indices (see E. Scholz [16]).
- J. G. Grassmann's work, including his mathematical school textbooks, provided H. G. Grassmann with fertile ideas for his new concepts of algebra (including exterior algebra), solely defined by the relations of its elements, from which G. Peano distilled the modern concept of vectors.
- Grassmann so far ahead that only few bright minds (like R. W. Hamilton, F. Klein and S. Lie) recognized his genius late in Grassmann's life.
- But the young W. K. Clifford elegantly unified the earlier works of Hamilton on quaternions and Grassmann's metric-free algebra of extension to geometric algebras, adding the inner product (for measurements) and the outer product of Grassmann. [11]

Clifford algebra and involutions

- The unit blade basis for $Cl(3,1)$, metric (+,+,+,-):

$$\{1, e_0, e_1, e_2, e_3, e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}, e_{023}, e_{031}, e_{012}, e_{123}, I = e_{0123}\}, \quad (1)$$

- We then have for $j, k \in \{1,2,3\}$, $j \neq k$,

$$e_{0j}^2 = 1, \quad e_{jk}^2 = -1, \quad e_{0jk}^2 = 1, \quad e_{123}^2 = -1, \quad I^2 = -1. \quad (2)$$

- Main grade involution (space inversion), $\langle M \rangle_k$ grade k part.

$$\hat{1} M = \widehat{M} = \sum_{k=0}^{4} (-1)^k \langle M \rangle_k, \quad (3)$$

- Reversion: $\tilde{1} M = \widetilde{M} = \sum_{k=0}^{4} (-1)^{\frac{1}{2}k(k-1)} \langle M \rangle_k$.
- Clifford conjugation: $\bar{1} M = \overline{M} = \hat{1}\tilde{1} M = \tilde{1}\hat{1} M$.
- Principal reverse $1' M = M'$ is reversion and $e_0 \to -e_0$.

Abelian group of involutions

- Composition gives Abelian group of involutions with 8 elements

$$G = \{1, \hat{1}, \tilde{1}, \bar{1}, 1', \hat{1}', \tilde{1}', \bar{1}'\}. \quad (4)$$

Table: Action of group G on $Cl(3,1)$ basis: Scalar S; spatial vector V, bivector B and trivector T. Time vector V_0, bivector B_0, trivector T_0, pseudoscalar quadvector Q, all with e_0 factor. o = sign change, e = no sign change.

Type	Basis element	$\hat{1}$	$\tilde{1}$	$\bar{1}$	$1'$	$\hat{1}'$	$\tilde{1}'$	$\bar{1}'$
S	1	e	e	e	e	e	e	e
V_0	e_0	o	e	o	o	e	o	e
V	e_1	o	e	o	e	o	e	o
	e_2	o	e	o	e	o	e	o
	e_3	o	e	o	e	o	e	o
B_0	e_{01}	e	o	o	e	e	o	o
	e_{02}	e	o	o	e	e	o	o
	e_{03}	e	o	o	e	e	o	o
B	e_{23}	e	o	o	o	o	e	e
	e_{31}	e	o	o	o	o	e	e
	e_{12}	e	o	o	o	o	e	e
T_0	e_{023}	o	o	e	e	o	o	e
	e_{031}	o	o	e	e	o	o	e
	e_{012}	o	o	e	e	o	o	e
T	e_{123}	o	o	e	o	e	e	o
Q	e_{0123}	e	e	e	o	o	o	o

Group action yields 51 multivector types

- 8 principal types $S, V_0, V, B_0, B, T_0, T$ and Q in Table 1 uniquely characterized by action of the group of involutions G.
- Like in [7], [6] linear combinations of principal types give 43 new types, with *mixed* m of group G.
- For example, scalars plus quadvectors type $SQ = S + Q$ with 7 group action entries (like in table)

$$e \quad e \quad e \quad m \quad m \quad m \quad m. \tag{5}$$

- Or combination of SB_0B or of SB_0BQ with 7 group action entries

$$e \quad m \quad m \quad m \quad m \quad m \quad m. \tag{6}$$

- Like Table 3 of [6] this leads to exactly 51 types of multivectors characterized by the action of the group G.
- NB: Analogous results can be shown in $Cl(1,3)$.

Full transfer of 51 multivector types [6] to $Cl(3,1)$

- 8 principal- and 43 further multivector types [6] transfer to $Cl(3,1)$. Index 31 for $Cl(3,1)$, index GF for authors [7, 6].
- Map of seven involutions

$$\bar{1}_{GF} \to \hat{1}_{31}, \quad 1'_{GF} \to \tilde{1}'_{31}, \quad 1^{\dagger}_{GF} \to \tilde{1}_{31}, \quad 1'^{\dagger}_{GF} \to 1'_{31},$$
$$\bar{1}'_{GF} \to \bar{1}'_{31}, \quad \bar{1}^{\dagger}_{GF} \to \bar{1}_{31}, \quad \bar{1}'^{\dagger}_{GF} \to \hat{1}'_{31}. \qquad (7)$$

- Map for multivector type labels

$$S'_{GF} \to S_{31}, \quad V_{GF} \to V_{0\,31}, \quad V'_{GF} \to V_{31}, \quad B_{GF} \to B_{0\,31},$$
$$B'_{GF} \to B_{31}, \quad T_{GF} \to T_{0\,31}, \quad T'_{GF} \to T_{31}, \quad S_{GF} \to Q_{31}. \qquad (8)$$

- These 2 maps transfer all results of Table 3 in [6] to a classification of $Cl(3,1)$ multivectors into 51 types, including 8 principal types (Table 1, first column). The grades in Table 3 of [6] are restricted to $\{0,1,2,3,4\}$, S has grade 0, Q has grade 4. NB: Similar maps work for $Cl(1,3)$.
- For example, the label $S'VBT'$ of No. 43 in Table 3 of [6] is mapped to SV_0B_0T, etc.

Charge conjugation, parity reversal and time reversal

- Symmetry operations of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ in $Cl(3,1)$ for space-time [5], p. 283. There for spinors (even grade valued multivector functions $\mathbb{R}^{1,3} \to Cl_+(1,3)$) including reflection at time axis e_0 of the argument of the spinor.
- First, we study action of $\hat{C}$, $\hat{P}$, $\hat{T}$ on basis of $Cl(3,1)$.
- For general $M \in Cl(3,1)$ define

$$\hat{C}M = Me_1e_0, \qquad \hat{P}M = e_0Me_0, \qquad \hat{T}M = Ie_0Me_1, \tag{9}$$

- Composition of $\hat{C}$, $\hat{P}$, $\hat{T}$ is associative (so we drop brackets), e.g.,

$$\hat{C}(\hat{P}(\hat{T}M)) = (\hat{C}\hat{P})\hat{T}M = \hat{C}(\hat{P}\hat{T}M) = \hat{C}\hat{P}\hat{T}M, \tag{10}$$

- We further find that

$$\hat{C}\hat{C}M = Me_{10}e_{10} = M, \qquad \hat{P}\hat{P}M = e_0^2Me_0^2 = M,$$
$$\hat{T}\hat{T}M = Ie_0Ie_0Me_1^2 = e_{123}^2M = -M, \tag{11}$$

Combining $\hat{C}$, $\hat{P}$, $\hat{T}$ symmetries

- Applying each symmetry twice gives

$$\hat{C}^2 = 1, \qquad \hat{P}^2 = 1, \qquad \hat{T}^2 = -1. \tag{12}$$

- We obtain the following commutation relations

$$\begin{aligned} &\hat{P}\hat{T}M = \hat{T}\hat{P}M = I\hat{C}M, \qquad \hat{T}\hat{C}M = -\hat{C}\hat{T}M = -I\hat{P}M, \\ &\hat{C}\hat{P}M = -\hat{P}\hat{C}M = -I\hat{T}M = e_0 M e_1 \quad \Rightarrow \quad \hat{T}M = I\hat{C}\hat{P}M. \end{aligned} \tag{13}$$

- Moreover,

$$\hat{C}\hat{P}\hat{T}M = IM, \qquad \hat{C}\hat{P}\hat{T}\hat{C}\hat{P}\hat{T}M = -M. \tag{14}$$

- and applying the above commutation relations leads to

$$\hat{C}\hat{P}\hat{T} = \hat{P}\hat{T}\hat{C} = -\hat{P}\hat{C}\hat{T} = \hat{C}\hat{T}\hat{P} = -\hat{T}\hat{C}\hat{P} = \hat{T}\hat{P}\hat{C}, \tag{15}$$

- and

$$\hat{C}\hat{C}\hat{P}\hat{T}M = \hat{P}\hat{T}M, \quad \hat{P}\hat{C}\hat{P}\hat{T}M = -\hat{C}\hat{T}M, \quad \hat{T}\hat{C}\hat{P}\hat{T}M = \hat{C}\hat{P}M.$$

Compositions of symmetry operators $\hat{C}$, $\hat{P}$ and $\hat{T}$

Table: Table of all compositions of symmetry operators $\hat{C}$, $\hat{P}$ and $\hat{T}$, where operations in the top row are applied first to M, followed by an operation from the first column. For example: combining $\hat{T}\hat{C}$ (top row) with $\hat{C}\hat{P}$ (first column, 6th row) gives $\hat{C}\hat{P}\,\hat{T}\hat{C}\,M = \hat{P}\hat{T}M$.

1st op.: / 2nd op.:	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
$\hat{C}$	$\hat{C}$	1	$\hat{C}\hat{P}$	$-\hat{T}$	$\hat{P}$	$-\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$
$\hat{P}$	$\hat{P}$	$-\hat{C}\hat{P}$	1	$\hat{C}\hat{P}\hat{T}$	$-\hat{C}$	$\hat{P}\hat{T}$	$\hat{T}\hat{C}$	$\hat{T}$
$\hat{T}\hat{C}$	$\hat{T}\hat{C}$	$\hat{T}$	$-\hat{C}\hat{P}\hat{T}$	1	$\hat{P}\hat{T}$	$\hat{C}$	$-\hat{P}$	$\hat{C}\hat{P}$
$\hat{C}\hat{P}$	$\hat{C}\hat{P}$	$-\hat{P}$	$\hat{C}$	$\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}\hat{T}$	$-\hat{T}$	$-\hat{T}\hat{C}$
$\hat{T}$	$\hat{T}$	$\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$-\hat{C}$	$-\hat{C}\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}$	$-\hat{P}$
$\hat{C}\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$-\hat{T}\hat{C}$	$\hat{P}$	$\hat{T}$	$-\hat{C}\hat{P}$	-1	$-\hat{C}$
$\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{T}$	$\hat{C}\hat{P}$	$-\hat{T}\hat{C}$	$-\hat{P}$	$-\hat{C}$	-1

Algebra of symmetries $\hat{C}$, $\hat{P}$, $\hat{T}$ isomorphic to $Cl(3,0)$

- Inspection: under the following map from the three symmetry operations and their compositions to the elements of the geometric algebra of space $Cl(3,0) \cong Cl_+(3,1)$ (even subalgebra), the $\hat{C}$, $\hat{P}$, $\hat{T}$ composition table Table 2 is isomorphic to the multiplication table of $Cl(3,0)$ itself.

$$\hat{C} \to e_1, \quad \hat{P} \to e_2, \quad \hat{T}\hat{C} \to e_3,$$
$$\hat{C}\hat{P} \to e_{12}, \quad \hat{T} \to e_{31}, \quad \hat{C}\hat{P}\hat{T} \to e_{23}, \quad \hat{P}\hat{T} \to e_{123}. \tag{16}$$

- NB: composition of $\hat{C}$, $\hat{P}$ and $\hat{T}$ forms a non-Abelian group isomorphic to 16 element group $\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_{23}, \pm e_{31}, \pm e_{12}, \pm e_{123}\}$ of products the basis elements of $Cl(3,0) \cong Cl_+(3,1)$.

Table: Application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ (top row) to basis of $Cl(3,1)$.

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}

$\hat{C}$, $\hat{P}$, $\hat{T}$ applied to $Cl(3,1)$ basis (I)

- The maps $\hat{C}$, $\hat{P}$ and $\hat{T}$ preserve even and odd grades, of multivector subspaces of $Cl(3,1)$.
- The map $\hat{P}$ only changes signs.
- The rows for the even basis elements $\{1, e_{01}, e_{23}, e_{0123}\}$ all contain these four elements twice, they form a commutative subalgebra generated by $\{e_{01}, e_{23}\}$.
- The operators $\hat{C}$, $\hat{T}$ and $\hat{T}\hat{C}$, applied to any of the four elements $\{1, e_{01}, e_{23}, e_{0123}\}$, generate the other three.
- The rows for the other four bivectors $\{e_{02}, e_{03}, e_{31}, e_{12}\}$ all contain these bivectors twice, i.e., they exclude the bivectors $\{e_{01}, e_{23}\}$, and do not form a subalgebra.
- The operators $\hat{C}$, $\hat{T}$ and $\hat{T}\hat{C}$, applied to any one of the four bivectors $\{e_{02}, e_{03}, e_{31}, e_{12}\}$, generate the other three.
- Similar observations apply to the two sets of odd basis blades $\{e_0, e_1, e_{023}, e_{123}\}$ and $\{e_2, e_3, e_{031}, e_{012}\}$.

$\hat{C}$, $\hat{P}$, $\hat{T}$ applied to $Cl(3, 1)$ basis (II)

- The table has 4 groups (2 with even blades, 2 with odd blades) of 4 rows, inside each group each of the 4 rows contains the same set of elements twice in different positions.
- Within each group of 4, the operators $\hat{C}$, $\hat{T}$ and $\hat{T}\hat{C}$, applied to any of the 4 elements present in that group, generate the other 3.
- The 4 groups can be clustered together by reordering, see next table. There each group of 4 contains 2 pairs of dual elements (multiplication with $\pm I$), where duality is element wise from left to right in each pair of rows.
- The reordered table also reveals that (up to ± 1) every row can be obtained from the first row (starting with 1) by multiplication with the first element of each row.
- The same applies to the relation of the first column with every other column (using multiplication of the first column with the elements in the top row of each column).

Table: Reordered table of application of $\hat{C}$,$\hat{P}$ and $\hat{T}$ (top row) to $Cl(3,1)$ basis. Double rows contain vertical pairs of dual elements. Top: even grades, bottom: odd grades.

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$

Full $\hat{C}$, $\hat{P}$, $\hat{T}$ symmetries on multivector functions

- Here we apply the full symmetries $\hat{C}$, $\hat{P}$, $\hat{T}$ to multivector-valued functions $x \in \mathbb{R}^{3,1} \to M(x) \in Cl(3,1)$, including scalars and spinors, etc.
- The full symmetries are defined in [5], p. 283, as

$$\hat{C}M(x) = M(x)e_1e_0, \quad \hat{P}M(x) = e_0M(e_0xe_0)e_0, \quad \hat{T}M(x) = Ie_0M(-e_0xe_0)e_1,$$

  where $x \to e_0xe_0$ maps $e_0 \to -e_0$ (reflection at space hyperplane), and $x \to -e_0xe_0$ preserves e_0, it is the $\mathbb{R}^3$ space inversion.
- In the compositions of $\hat{C}$, $\hat{P}$, $\hat{T}$ the transformation of the space-time argument x of M(x) has to be taken into account, e.g.

$$\hat{C}\hat{P}\hat{T}M(x) = IM(-x), \qquad \text{etc.} \tag{17}$$

- NB: composition Table 2 same for full $\hat{C}$, $\hat{P}$, $\hat{T}$ symmetries and $\hat{C}$, $\hat{P}$ and $\hat{T}$ still forms a non-Abelian group isomorphic to 16 element group $\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_{23}, \pm e_{31}, \pm e_{12}, \pm e_{123}\}$ of products of basis elements of $Cl(3,0) \cong Cl_+(3,1)$.

Working with $Cl(1,3)$

- [5, 8] prefer to work with space-time algebra $Cl(1,3)$.
- We define in $Cl(1,3)$ the same symmetry operators

$$\hat{C}M(x) = M(x)e_1e_0, \quad \hat{P}M(x) = e_0M(e_0xe_0)e_0, \quad \hat{T}M(x) = Ie_0M(-e_0xe_0)e_1.$$

- Detailed computation shows that the composition of $\hat{C}$, $\hat{P}$, and $\hat{T}$ results in exactly the same Table 2, as for $Cl(1,3)$.
- Therefore, for $Cl(1,3)$, the composition of $\hat{C}$, $\hat{P}$, and $\hat{T}$ also forms a non-Abelian group isomorphic to 16 element group $\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_{23}, \pm e_{31}, \pm e_{12}, \pm e_{123}\}$ of products of basis elements of $Cl(3,0) \cong Cl_+(3,1)$!
- Key result robust against signature change.

Conclusions

- Application of elementary symmetries of $Cl(3,1)$ and $Cl(1,3)$ that both describe space-time. Inspired by [7], [6], we chose 3 involutions of space inversion, reverse and *principal reverse* and studied their Abelian group and its action on all multivectors.
- We found similar to [6], a classification in 8 principal and further 43 types of multivectors, i.e., a total of 51 types.
- The composition of the symmetry operations $\hat{C}$, $\hat{P}$ and $\hat{T}$ forms a non-Abelian group isomorphic to 16 element group $\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_{23}, \pm e_{31}, \pm e_{12}, \pm e_{123}\}$ of products of basis elements of $Cl(3,0) \cong Cl_+(3,1) \cong Cl_+(1,3)$.
- New: Algebraic aspects of applying charge conjugation, parity reversal and time reversal to multivector basis and multivector functions of $Cl(3,1)$ and $Cl(1,3)$.
- Structures found when $\hat{C}$, $\hat{P}$ and $\hat{T}$ are applied to the complete set of basis blades of $Cl(3,1)$ and $Cl(1,3)$.
- Interesting to apply both approaches in Clifford space gravity [2], and elementary particles using a new embedding of octonions in geometric algebra [14, 12].

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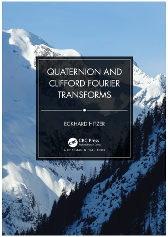

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