# Spacetime Algebra and Octonions: Links with the groups used in particle physics and computer graphics

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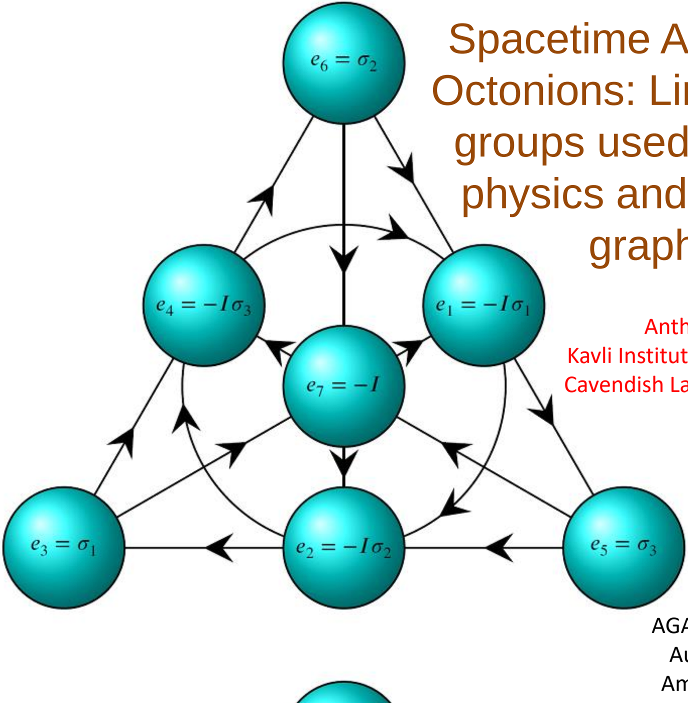
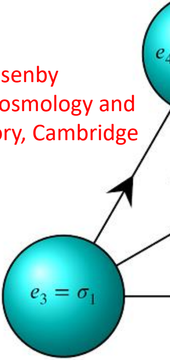



AGACSE 2024
August 27
Amsterdam

# Overview



- The STA is the geometric algebra of spacetime
- Given the success of the STA in dealing with electromagnetism and gravity then this suggests the question:
- Is the STA itself is enough to get to all the laws of physics?
- This would have the overwhelming 'good feature' that we only know about 4 dimensions existing, and we know what metric ($(1,3)$ or $(3,1)$) they satisfy
- So the GA of such a space seems a truly minimal way of approaching physical laws
- Separately, Cohl Furey and others have been showing how the Octonions, the last of the four division algebras after the reals, the complex numbers and quaternions, have some interesting properties as regards the gauge groups of particle physics, and may be linked with the symmetries that nature has chosen
- Over the last 3 years, I've been able to show how octonions can be successfully implemented and understood in the STA and have made progress on how the application of the STA to particle physics this link with octonions brings about

# Overview II

- It also has effects for other groups, of interest in geometry and computer graphics, and I will discussing some perhaps surprising aspects of this
- Not much time to cover a lot of ground, so excuse the brevity of introductions to various topics!
- For a summary of where things stood about 2 years ago, see Anthony Lasenby, Some recent results for $SU(3)$ and octonions within the geometric algebra approach to the fundamental forces of nature, Mathematical Methods in the Applied Sciences, (2022), https://doi.org/10.1002/mma.8934, arXiv:2202.06733
- For a summary talk on this given online in October 2023 see `https://www.youtube.com/watch?v=0m__fhtkMzg`
- And there's another paper underway with a good deal more, hopefully ready shortly!

# The STA I

- Have four vectors $\{e_0, e_i\}, i = 1 \ldots 3$ with properties

$$e_0{}^2 = 1, \qquad e_i{}^2 = -1 \qquad e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}$$

- Have 6 bivectors, 3 each squaring to +1 or -1
- Spacelike (Euclidean) bivectors satisfy

$$(e_i \wedge e_j)^2 = -e_i{}^2 e_j{}^2 = -1$$

  and generate rotations in a plane
- Timelike (Lorentz) bivectors satisfy

$$(e_i \wedge e_0)^2 = -e_i{}^2 e_0{}^2 = 1$$

  and generate hyperbolic geometry e.g.:

$$\begin{aligned} e^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \alpha^2/2! + \alpha^3/3!\, e_1 e_0 + \cdots \\ &= \cosh\alpha + \sinh\alpha e_1 e_0 \end{aligned}$$

## The Pseudoscalar

- Define the pseudoscalar $I$

$$I = e_0 e_1 e_2 e_3$$

- Reverses to itself, and squares to -1
- NB $I$ anticommutes with vectors and trivectors. (In space of even dimensions). $I$ always commutes with even-grade.
- Now settle on a given fixed Cartesian frame of vectors in which to do our physics — can think of this as the laboratory frame, and rename our $e_\mu$ to be $\gamma_\mu$
- Now have the basic tool for relativistic physics — the STA

| $1$ | $\{\gamma_\mu\}$ | $\{\gamma_\mu \wedge \gamma_\nu\}$ | $\{I\gamma_\mu\}$ | $I = \gamma_0\gamma_1\gamma_2\gamma_3$ |
|---|---|---|---|---|
| 1 | 4 | 6 | 4 | 1 |
| scalar | vectors | bivectors | trivectors | pseudoscalar |

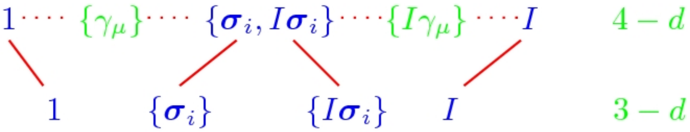
# The STA III

- We drop down to 3d by defining $\sigma_i = \gamma_i \gamma_0$
- These are actually spacetime bivectors, but can function as spatial vectors in the frame orthogonal to $\gamma_0$
- A nice feature is that the volume element is

$$\sigma_1 \sigma_2 \sigma_3 = (\gamma_1 \gamma_0)(\gamma_2 \gamma_0)(\gamma_3 \gamma_0) = -\gamma_1 \gamma_0 \gamma_2 \gamma_3 = I$$

so the 3-d subalgebra shares same pseudoscalar as spacetime!

- So projected onto the even subalgebra of the STA we have the following picture:




Combinations of the elements on the second line here gives us what we will call Dirac spinors, and later Octonions!

# Octonions I

- Octonions are generalisations of quaternions, and have 8 elements, $e_0$ through $e_7$, instead of 4
- $e_1$ through $e_7$ all mutually anticommute and square to -1
- The key features of Octonions is that they form a 'normed division algebra'
- Any product of octonions has a norm which is the product of the individual norms, and two non-zero octonions always multiply to produce a further non-zero octonion
- As is well known, the only normed division algebras are the real numbers, complex numbers, quaternions and octonions
- Among these, the complex numbers are commutative and associative, the quaternions are non-commutative but still associative, while famously the octonions are neither commutative nor associative
- In particular, there is no general rule that

$$(ab)c = a(bc) \tag{1}$$

for general elements $a$, $b$ and $c$

- So e.g. differs from the geometric product of GA in this regard
- Why then might it they be useful in Physics, and how are we going to be able to represent them within GA?
- We are going to use the STA!
- The key is that (in the STA) the Dirac spinors $\psi$ can be divided into parts that commute or anticommute with the unit timelike vector $\gamma_0$
- Can use these two parts (the 'Pauli' and 'non-Pauli' parts of the Dirac spinor), in a way which defines the octonionic product of two Dirac spinors in terms of these sub-parts.
- An STA spinor has 8 real degrees of freedom, so we are able to identify an octonion directly with a spinor.
- For an 'octonion' $\psi$, we define

$$\psi_+ = \tfrac{1}{2}\left(\psi + \gamma_0\psi\gamma_0\right), \quad \psi_- = \tfrac{1}{2}\left(\psi - \gamma_0\psi\gamma_0\right) \tag{2}$$

as the two sub-parts of $\psi$.

# Octonions III

- These will correspond to the even and odd parts of the full 3d Pauli algebra, in the usual 'spacetime split' correspondence between the Pauli and Dirac geometric algebras, given by multiplication by $\gamma_0$
- Then given two octonions, $\psi$ and $\phi$, the octonionic product between them, which we will denote '$\star$', is the Dirac spinor $\theta$ given by

$$\boxed{\theta = \psi \star \phi = \psi_+ \phi_+ + \tilde{\phi}_- \psi_- + \phi_- \psi_+ + \psi_- \tilde{\phi}_+} \qquad (3)$$

- Note carefully that the four individual products on the right hand side of this equation are all usual *geometric products* taking place within the ordinary STA, the 'reverse' is just an ordinary STA reverse, and the spinors involved are just ordinary STA spinors
- Hence our claim about being able to compute everything entirely within the STA!

- From the form of this we can see that such a product is highly unlikely to be associative, and indeed in general it is not
- The property it *does* have, however, comes from the (essentially) defining property of the octonions, already discussed, that they form a normed division algebra
- To define the norm, we need to define a conjugate element. For us this is (using $^*$ to denote conjugation)

$$\boxed{\psi^* = \tilde{\psi}_+ - \psi_-} \quad (4)$$

where (as said above) the tilde over the first term on the r.h.s. denotes the usual GA reversion.

- This yields the following norm:

$$\boxed{||\psi|| \equiv \psi \star \psi^* = \frac{1}{2}\left(\gamma_0\psi\gamma_0\tilde{\psi} + \psi\gamma_0\tilde{\psi}\gamma_0\right) = J\cdot\gamma_0} \quad (5)$$

where $J = \psi\gamma_0\tilde{\psi}$ is the Dirac current!

- We know from its interpretation as a probability current, that $J$ is always non-zero and future pointing as long as $\psi$ is non-zero
- This, along with its scalar nature (in the sense of grades present) coincides perfectly with properties a norm should have
- The other property we need for our product and norm to be representing a normed division algebra is the fundamental relation that the norm respects multiplication, so we require

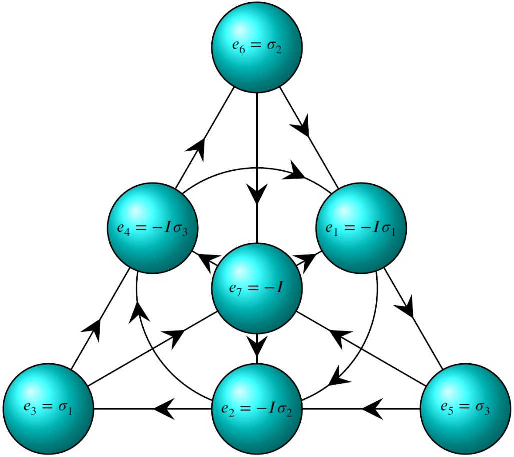
$$||\psi \star \phi|| = ||\psi|| \, ||\phi|| \tag{6}$$

  for all Dirac spinors $\psi$ and $\phi$.
- Given our definitions so far, this is a matter of computation, and one can find that this does indeed work and with $\theta = \psi \star \phi$ it corresponds to the (ordinary STA) result

$$||\theta|| = \left(\theta\gamma_0\tilde{\theta}\right) \cdot \gamma_0 = \left[\left(\psi\gamma_0\tilde{\psi}\right) \cdot \gamma_0\right] \left[\left(\phi\gamma_0\tilde{\phi}\right) \cdot \gamma_0\right] \tag{7}$$

# Assigment of STA elements to unit Octonions

- On this diagram, which shows the Fano plane construction for Octonion multiplication, we explicitly display the assignment of Octonion units to STA elements
- Lot of ways of doing this — we have settled on one which gives a central position to the pseudoscalar $I$ and treats the Pauli even and odd elements symmetrically



# Octonion chains I

- Cohl Furey (and others) have particularly emphasised the role of *chains* of octonion operators
- This is a clever technique to do with using sequences of octonions to pack in a great deal of 'information' despite the relatively small size of the spaces involved
- The non-associativity of the octonions is crucial to this, since then e.g.

  $A(B\phi)$ for some octonions $A$ and $B$ acting on a state $\phi$

  is *different* from the sequence

  $(AB)\phi$ also acting on $\phi$
- The possibilities increase as we go to longer sequences, rather like a code, though find that if we consider purely one-sided multiplications, then we should consider sequences of a maximum of 6 of these, and Furey and others show that this is isomorphic to the 6 dimensional anti-Euclidean Clifford algebra $Cl(0,6)$
- This is the algebra that has all grade 1 objects squaring to $-1$, and anticommuting

# Octonion chains II

- That this happens we can understand quite well from the STA viewpoint
- Have recently understood that stucture of our product makes it easiest to translate into STA if consider chains on the right instead
- Find the translations then are very simple:

$$\boxed{\psi \star e_i \leftrightarrow \gamma_0 \psi \gamma_0 e_i}, \quad i = 1, 2, 4, \quad \text{i.e. for } -I\sigma_1, -I\sigma_2, -I\sigma_3$$

$$\boxed{\psi \star e_i \leftrightarrow e_i \gamma_0 \psi \gamma_0}, \quad i = 3, 5, 6, \quad \text{i.e. for } \sigma_1, \sigma_3, \sigma_2$$

$$\boxed{\psi \star e_7 \leftrightarrow -I\psi = -\psi I}, \quad i = 7$$

- We call them ‘sandwich forms’
- From these we read off the properties we want more or less by eye!
- E.g. squaring to minus 1 for the $I\sigma_i$ is because they square to -1 themselves but commute with $\gamma_0$

- Squaring to minus 1 for the $\sigma_i$ is because they square to 1 themselves but anticommute with $\gamma_0$
- Proving anti-commutativity goes similarly
- Summarising, we really do have

$$(\phi \star e_i) \star e_j = \begin{cases} -\phi & i = j \\ -(\phi \star e_j) \star e_i & i \neq j \end{cases} \tag{8}$$

which if we define a 'vector' as the map $\phi \mapsto \phi \star e_i$ are the defining relations for the unit vectors in a Clifford algebra

- As we build up the chains of Octonionic multiplications, we are building up blades in the algebra of $Cl(0,6)$
- And each of these is a defined map of an STA even-grade element (a spinor) to new spinor!
- Thus by a 4d Clifford algebra acting on itself (the STA, with a total of 16 elements), we have generated a 6d Clifford algebra, $Cl(0,6)$, with 64 elements

# Octonion chains 

- If build up the 64 elements corresponding to chains of right multiplication, then we are going to number the resulting chains as $R_0$ (the identity) to $R_{63}$ (the pseudoscalar of the space)
- Similarly for left multiplications, which we'll call $L_i$, $i = 0, \ldots, 63$
- Note we choose the numbering such that e.g. $R_i$ translates as $\psi * e_i$, $i = 0, \ldots, 7$
- This defines what we are going to work with — each of these is effectively map in the STA from a Dirac spinor to Dirac spinor
- Worth thinking a little about the non-associativity of the octonions versus the maps we have introduced
- Let's work with an example of non-associativity
- Start with pure octonions

$$
\begin{array}{ccc}
(e_1 e_2) e_3 & \neq & e_1 (e_2 e_3) \\
\downarrow & & \downarrow \\
e_4 e_3 & & e_1 e_5 \\
\downarrow & & \downarrow \\
-e_6 & & e_6
\end{array}
$$

- So could write this e.g. as

$$\begin{aligned} & R_3(R_2 e_1) = -e_6, \quad L_1(L_2 e_3) = e_6 \\ \text{or} \quad & (R_3 R_2) e_1 = -e_6, \quad (L_1 L_2) e_3 = e_6 \\ \text{or} \quad & R_{12} e_1 = -e_6, \quad L_7 e_3 = e_6 \end{aligned}$$

  since the maps *are* associative. (In the last line we are using our numbering system for chain states.)
- So can begin to see how the chain states can be pulled away and treated as operators in their own right
- Also, can see how different maps are involved when comparing the non-associative expressions, so the fact that the maps themselves are associative is not an issue
- Ok, have got enough in place that we can start exploring the use of these octonionic maps in some unexpected contexts

# Dirac's $SO(3,2)$ space 

- Going to look at a first example, where we have a natural imaginary present
- In a 5d Clifford algebra, the pseudoscalar always commutes with all elements of the algebra ($e_i e_1 e_2 e_3 e_4 e_5 = e_1 e_2 e_3 e_4 e_5 e_i, \forall i$)
- We are going to ask (a) that the traditional $I\sigma_3$ on the right constitutes the 'imaginary'
- and (b) That this same state functions as the pseudoscalar of the resulting Clifford Algebra, and so automatically commutes with everything
- Turns out this immediately pins down the algebra to $Cl(3,2)$ or $Cl(4,1)$ (or their inverted signature equivalents)
- These are quite interesting
- $SO(3,2)$ is the group which Dirac concentrated on in two papers from the 1960's and 1970's which have recently started to get more attention

# Dirac's $SO(3,2)$ space II

- These are *A Remarkable Representation of the 3 + 2 de Sitter Group*, P. A. M. Dirac, J. Math. Phys., Vol. **4**, pp. 901–909 (1963) and *A Positive-Energy Relativistic Wave Equation*, P. A. M. Dirac, Proceedings of the Royal Society of London. Series A, Vol. **322**, No. 1551, pp. 435-445 (1971)
- The second paper uses the $SO(3,2)$ representation of the first to construct a new relativistically covariant wave equation for a particle that has positive energy only — at this stage Dirac felt that having negative energy solutions was not a good thing, despite of the success of his first relativistic particle equation (from the 1920's) in predicting anti-particles!
- It turns out these new particles cannot interact with anything electromagnetically, and recently they have been discussed as possible dark matter candidates (e.g. *Positive-Energy Dirac Particles and Dark Matter*, E. Bogomolny, arXiv:2406.01654 (2024))
- What we do here generalises $SO(3,2)$ to the whole $Cl(3,2)$

# Dirac's $SO(3,2)$ space III

- We will skip over some central aspects of what Dirac did, in particular that his wavefunction is a complex function of two 'internal' variables at each point, as well as spacetime position, and just look at the matrices he uses
- These are new versions of the previous $\gamma$ matrices
- Find out by looking at the action on a Dirac column spinor, that in the STA these are all of the 'sandwich' form, and we translate them as follows:

$$\begin{array}{ccccc} \hat{\gamma}_0 & \leftrightarrow & \gamma_0 \psi \gamma_3 & \leftrightarrow & R_{59} \\ \hat{\gamma}_1 & \leftrightarrow & \gamma_3 \psi \gamma_3 & \leftrightarrow & R_{39} \\ \hat{\gamma}_2 & \leftrightarrow & -\gamma_1 \psi \gamma_3 & \leftrightarrow & R_{41} \\ \hat{\gamma}_3 & \leftrightarrow & -\psi \sigma_3 & \leftrightarrow & R_{56} \end{array}$$

- So Dirac here was using octonions, without knowing it!
- The implied signature is $(R_{59}^2, R_{39}^2, R_{41}^2, R_{56}^2) = (-1, 1, 1, 1)$, i.e. $(3, 1)$

# Dirac's $SO(3,2)$ space 

- We then take commutator products of these to give the 6 bivectors of the space, which Dirac then adds to the vectors we have just found to give a set of 10 quantities, which he interprets as the generators (bivectors) of $SO(3,2)$
- We can instead seek the underlying vectors which produce these bivectors, and thence construct a whole $Cl(3,2)$ space upon them
- The particular numbers won't mean anything, but we find these are the set

$$R_6, R_{18}, R_{21}, R_{38}, R_{46}$$

- Starting afresh with these as vectors, then leads, as stated, to entire $Cl(3,2)$,plus a commutative imaginary, $R_7$, which is also the pseudoscalar of the space
- So this is good for understanding (a part of) what Dirac was attempting in the 1971 paper, but why is any of this interesting as regards geometry and computer graphics?

# Dirac's $SO(3,2)$ space V

- Point is that the commutative imaginary means that we can multiply it into any of these 'vectors' and thereby change their signature
- So actually we have decoded all of $Cl(3,2)$, $Cl(2,3)$, $Cl(1,4)$ and $Cl(4,1)$ in octonion/STA terms
- Looking first at the last, we recognise this in the Conformal Geometric Algebra (CGA) of 3d Euclidean space
- The vectors in this case are

  $$R_6, R_{38}, R_{44}, R_{46}, R_{47}$$

  and all the rest of the elements of the $Cl(4,1)$ 32d algebra are in 1-1 correspondence with unique elements of our 'right chain' space
- This means we can implement all CGA operations using the underlying STA software we use for the octonion chain mappings

# Dirac's $SO(3,2)$ space VI

- Will say a bit more about this below — the octonion mapping method seems quite efficient, and is probably worth investigating as regards implementions of CGA and PGA ('Look Ma: Octonions!)
- Note that indeed we can get to the PGA via the method that I laid out in a paper in the Amsterdam proceedings from 2010 (see Lasenby, A. (2011). Rigid Body Dynamics in a Constant Curvature Space and the '1D-up' Approach to Conformal Geometric Algebra. In: Dorst, L., Lasenby, J. (eds) Guide to Geometric Algebra in Practice. Springer, London. https://doi.org/10.1007/978-0-85729-811-9_18)
- Let's look at some code:

# PGA via Octonions

```
tmp_trans_PGA:=proc(multi,xx,yy,zz)

    local tmp0,tmp_t,tmp;

    tmp0:=multi;

    tmp_t:=ds(xx*tmp_bas_e1+yy*tmp_bas_e2+zz*tmp_bas_e3);

    go_op(tmp_bas_n,tmp0);

    go_op(tmp_t,%);

    tmp:=ds(tmp0-1/2*%);

    go_op(%,tmp_bas_n);

    go_op(%,tmp_t);

    ds(tmp-1/2*%);

end:
```

```
# Ok, let's translate I_3 using this.

tmp_trans_PGA(tmp_cliff_right64,x,y,z);

find_right_states_new_with_print_and_chain(%);

# Compare with what we expect.

ds(1/2*tmp_bas_nb-(x*tmp_bas_e1+y*tmp_bas_e2+z*tmp_bas_e3)):
go_op(%,tmp_Wstar_ps);

find_right_states_new_with_print_and_chain(%);

# So if project this to trivector part, these agree!
```

# PGA via Octonions II

What's going on here as regards the algebra is explained in the following extract:

18 Rigid Body Dynamics in a Constant Curvature Space 385

Firstly, we can see that since the Euclidean points are represented by their duals (in the Gunn approach), then the object in $\hat{W}^*$ representing the origin will be $\hat{J}(\bar{n}/2) = e_1 e_2 e_3$, which we can denote as $I_3$.

The rotors used for moving points around in $\hat{W}^*$ are exactly the same as used in Chap. 1 in the five-dimensional approach, i.e. a combination of the usual spatial rotors, built out of even combinations of 1 and the $e_i$, and translation rotors of the form $R_t = 1 - \frac{1}{2} t n_\infty$, where $t$ is the 3D spatial vector through which we translate. Starting from the 'origin', therefore, and applying a translation, we reach:

$$
\begin{aligned}
R_t I_3 \tilde{R}_t &= R_t \hat{J}(e^0) \tilde{R}_t = R_t \left\langle \frac{1}{2} \bar{n} I_d \right\rangle_3 \tilde{R}_t \\
&= R_t \frac{1}{2} \bar{n} \cdot I_d \tilde{R}_t = \left( R_t \frac{1}{2} \bar{n} \tilde{R}_t \right) \cdot I_d
\end{aligned}
\tag{18.40}
$$

with the last equality following since $I_d$, containing a factor $n_\infty$, is left invariant under $R_t$.

# The Hestenes-Dirac case I

- There's another algebra which is of interest to us in 4d
- An obvious question which arises is about the Hestenes form of the Dirac equation!
- What does this look like in this approach?
- This is a somewhat 'meta' proceeding, since we can already do the Dirac equation in the STA of course!
- But what does it look like when we make the link with octonions?
- Is there an Octonionic form of the Dirac equation for the electron? (People have searched for this, and proposed some answers, for some while.)

# The Hestenes-Dirac case II

- Come back to this, but let's go straight to the translation of the $\gamma$ matrices. Here we already know the first two columns of the following

$$\begin{array}{ccccc} \hat{\gamma}_0 & \leftrightarrow & \gamma_0\psi\gamma_0 & \leftrightarrow & -R_{23} \\ \hat{\gamma}_1 & \leftrightarrow & \gamma_1\psi\gamma_0 & \leftrightarrow & R_3 \\ \hat{\gamma}_2 & \leftrightarrow & \gamma_2\psi\gamma_0 & \leftrightarrow & R_6 \\ \hat{\gamma}_3 & \leftrightarrow & \gamma_3\psi\gamma_0 & \leftrightarrow & R_5 \end{array}$$

  but the third column is new to us. Again, we've in fact been doing octonions all the time, in the STA version of the Dirac algebra!
- (Note a difference with the (2,3) case is that here it ends up that the pseudoscalar of the space is different from the 'imaginary' — former is $R_{63}$, latter is still $R_7$.)
- Useful to think about these actions in terms of the idea of preservation of the time component of the Dirac current, which as we said is equivalent to 'conservation of norm' in multiplying octonions
- So consider $\langle\gamma_0\psi\gamma_0\tilde{\psi}\rangle$ where we let $\psi \mapsto \gamma_\mu\psi\gamma_0$

# The Hestenes-Dirac case III

- We get

$$\langle \gamma_0 \psi \gamma_0 \tilde{\psi} \rangle \mapsto \langle \gamma_0 \gamma_\mu \psi \gamma_0 \gamma_0 \gamma_0 \tilde{\psi} \gamma_\mu \rangle = \gamma_0^2 \langle \gamma_\mu \gamma_0 \gamma_\mu \psi \gamma_0 \tilde{\psi} \rangle$$

  by the cyclic property of taking the scalar part
- So we see that we preserve the norm provided $\gamma_i^2 = -\gamma_0^2$ for $i = 1, 2, 3$
- This gives an interesting connection between the metric and the octonionic norm!
- We can think about this more generally
- Let $E_i$ and $E_j$, $i, j = 1, \ldots, 16$ be any two of the basis elements of the STA and use these to sandwich $\psi$, i.e. we let

$$\psi \mapsto E_i \psi E_j$$

# The Hestenes-Dirac case IV

- We now get (no sum on repeated indices)

$$\langle \gamma_0 \psi \gamma_0 \tilde{\psi} \rangle \mapsto \langle \gamma_0 E_i \psi E_j \gamma_0 \tilde{E}_j \tilde{\psi} \tilde{E}_i \rangle$$

- Now it is a fact, easy to verify, that

  $\gamma_0$ commutes with $1$, $I\sigma_k$, $I\gamma_k$, $\gamma_0$, $\quad k = 1, 2, 3$
  and each of these satisfies $E_i \tilde{E}_i = +1$

  while

  $\gamma_0$ anticommutes with $I$, $\sigma_k$, $\gamma_k$, $I\gamma_0$, $\quad k = 1, 2, 3$
  and each of these satisfies $E_i \tilde{E}_i = -1$

- From the cyclic reordering property of $\langle \dots \rangle$ we thus get that the signs always cancel and the octonionic norm is always preserved by the sandwiching
- Note we didn't have to say that $E_i$ and $E_j$ were either both even or both odd, but we'll need this currently to keep $\psi$ still being even
- So what can we get by trying these for all possible elements?

# The Hestenes-Dirac case



- Counting such that we only include linearly independent elements is not immediately obvious, since 1 and $I$ commute with (even) $\psi$
- But what we in fact get is precisely the set of all 64 'right chain' states, which we've also established are the same as $Cl(0,6)$!
- Moreover, one can show that each 'right chain' state can be expressed as a sum of either 1 or 4 'left chain' states, and vice versa (not sure that others know this, by the way), and this means we've got the whole chain space covered
- Returning to the question about an octonionic form of the Dirac equation, here's our answer:
- With the current assignment of octonion units to STA elements this is

$$-\partial_t\psi \star e_4 + ((\partial_x\psi \star e_1) \star e_2) \star e_3 + ((\partial_y\psi \star e_1) \star e_2) \star e_6 \\ + ((\partial_z\psi \star e_1) \star e_2) \star e_5 - m\psi = 0$$

- Looks rather clumsy, but this is because we haven't optimised the order of assignment of STA elements to octonions
- However, have verified that it actually works, and e.g. starting from the equation for an electron at rest can then use covariance to get the 'wavefunction' (=octonion) for a particle in motion, which works out to be the right thing
- Finally, on the issue of what Clifford algebras we can get to in the 64 element chain space, if we forego having a state available for use as an 'i' equivalent, we can successfully step up from a (4,1) space, the CGA, to a (4,2) space, the CSTA
- Then applying the same method as in the Amsterdam paper just discussed, can drop down from this to 2 copies of the PSTA (or STAP), which Martin, Steven, David and others are working on
- The 64d chain space as a whole, i.e. our set of elements $R_0$ through $R_{63}$, is in 1-1 correspondence with $CL(0,6)$, hence this is another space which we can represent all operations in

# The Hestenes-Dirac case VII

- Thus in starting to talk about $SU(3)$ etc. our answers with the STA, and Cohl Furey's with pure octonions, can be compared in detail with those from Martin *et al.* for their own approaches to e.g. $SU(3)$ using $Cl(6)$
- Lot's of interesting comparisons to come!