From Null Monomials to Versors in Conformal Geometry

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Outline

- Basics of CGA
- Null Monomials in CGA
- Null Versors in CGA

Basics of CGA

- Basic geometric entities in conformal geometry represented by Grassmannians/blades:
  - Point:

    $$\mathbf{p} \in \mathbb{R}^n \mapsto \acute{\mathbf{p}} := \mathbf{e}_0 + \mathbf{p} + \frac{\mathbf{p}^2}{2}\mathbf{e}_\infty \in \mathcal{N}(\mathbb{R}^{n+1,1}),$$

    called the *standard nullification* of $\mathbf{p} \in (\mathbf{e}_0 \wedge \mathbf{e}_\infty)^\perp$ wrt $\mathbf{e}_0, \mathbf{e}_\infty$:

    $$\acute{\mathbf{p}} \cdot \mathbf{e}_\infty = -1.$$

    For all $\lambda \neq 0$, $\lambda\acute{\mathbf{p}}$ represents the same point $\mathbf{p} \in \mathbb{R}^n$.
  - 0-d circle (a pair of points, one can be at infinity): $\mathbf{a} \wedge \mathbf{b}$.
  - 1-d circle (including line): $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$. ......

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- Conformal transformation group represented by pin group:

  $$\mathrm{pin}(n+1,1) \longrightarrow \text{ (onto homomorphism) } \mathrm{conf}(\mathbb{R}^n).$$

  Quadruple covering with kernel $\{\pm 1, \pm \mathbf{I}_{n+1,1}\}$.

Conformal transformations induced by geometric entities

For versor $\mathbf{V} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_k$ where the $\mathbf{u}_i$ are invertible,

$$Ad^*_{\mathbf{V}}(\mathbf{x}) = (-1)^k \mathbf{V}\mathbf{x}\mathbf{V}^{-1}, \text{ for all } \mathbf{x} \in \mathbb{R}^{n+1,1}.$$

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$$\text{either } \mathbf{a} \equiv \acute{\mathbf{c}} - \frac{\rho^2}{2}\mathbf{e}_\infty, \text{ or } \mathbf{a} \equiv \mathbf{n} + \delta\mathbf{e}_\infty.$$

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- Negative inversion wrt a hypersphere: $\mathbf{V} = \acute{\mathbf{c}} + \frac{\rho^2}{2}\mathbf{e}_\infty$ (negative vector).

  $$\mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{x}' \in \mathbb{R}^n, \text{ s.t. } (\mathbf{x} - \mathbf{c})(\mathbf{x}' - \mathbf{c}) = -\rho^2.$$

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- Inversion wrt a circle $(\mathbf{c}, \rho) \cap \Pi$ in high dimensions (composition of inversion wrt hypersphere $(\mathbf{c}, \rho)$ and reflection wrt supporting plane Π): $\mathbf{V} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$.

- Translation by a vector $\mathbf{t} \in \mathbb{R}^n$:

$$\mathbf{V} = 1 + \mathbf{e}_\infty \frac{\mathbf{t}}{2} = (\frac{\mathbf{t}}{\mathbf{t}^2} + \frac{\mathbf{e}_\infty}{2})\mathbf{t}.$$

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- Dilation about center $\mathbf{c} \in \mathbb{R}^n$ with ratio λ:

$$\mathbf{V} = \mathbf{e}_\infty \acute{\mathbf{c}} + \lambda^{-1} \acute{\mathbf{c}} \mathbf{e}_\infty = -(1 + \lambda^{-1}) + (1 - \lambda^{-1}) \mathbf{e}_\infty \wedge \acute{\mathbf{c}}.$$

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- 2-d rotation of angle θ about a center $\mathbf{c} \in \mathbb{R}^n$ in a plane spanned by two directions $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$: let $\angle(\mathbf{a}, \mathbf{b}) = \theta/2$.

$$\begin{aligned}
\mathbf{V} &= \big(\acute{\mathbf{c}} \cdot (\mathbf{e}_\infty \wedge \mathbf{b})\big)\big(\acute{\mathbf{c}} \cdot (\mathbf{e}_\infty \wedge \mathbf{a})\big) \\
&\equiv (\acute{\mathbf{c}}\mathbf{e}_\infty \mathbf{b} - \mathbf{e}_\infty \mathbf{b} \acute{\mathbf{c}})(\acute{\mathbf{c}}\mathbf{e}_\infty \mathbf{a} - \mathbf{e}_\infty \mathbf{a} \acute{\mathbf{c}}) \\
&= \acute{\mathbf{c}}\mathbf{e}_\infty \mathbf{b} \acute{\mathbf{c}} \mathbf{e}_\infty \mathbf{a} + \mathbf{e}_\infty \mathbf{b} \acute{\mathbf{c}} \mathbf{e}_\infty \mathbf{a} \acute{\mathbf{c}} \\
&\equiv \acute{\mathbf{c}}\mathbf{e}_\infty \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a}\mathbf{e}_\infty \acute{\mathbf{c}}.
\end{aligned}$$

Note: $\big(\acute{\mathbf{c}} \cdot (\mathbf{e}_\infty \wedge \mathbf{b})\big) \equiv \mathbf{b} + (\acute{\mathbf{c}} \cdot \mathbf{b})\mathbf{e}_\infty$ represents the hyperplane normal to $\mathbf{b}$ and passing through point $\mathbf{c}$.

- Conformal transformation in the connected component of the identity: either

$$
\begin{aligned}
\mathbf{V} &= \exp(\lambda_1 \mathbf{B}_1) \exp(\lambda_1 \mathbf{B}_2) \cdots \exp(\lambda_l \mathbf{B}_l) \\
&= (\mathbf{u}_1 \mathbf{u}_2)(\mathbf{u}_3 \mathbf{u}_4) \cdots (\mathbf{u}_{2l-1} \mathbf{u}_{2l}),
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  where the 2-blades $\mathbf{B}_i$ are completely orthogonal to each other [M. Riesz, 1958].

- Versor compression: for any versor $\mathbf{V}$ in $\mathcal{C}l(\mathbb{R}^{n+1,1})$, if k is the maximal grade such that $\langle \mathbf{V}\rangle_k \neq 0$, then $\mathbf{V} \equiv \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k$ for some $\mathbf{v}_i \in \langle \mathbf{V}\rangle_k$.

  If $\mathbf{V} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_k$ where $\mathbf{v}_1 \wedge\cdots\wedge \mathbf{v}_k$, the invertible monomial form is said to be tight.

Null polynomials

Null monomial: geometric product of a finite sequence of null vectors and a scalar, where no two adjacent vectors are $\equiv$.

Degree: length of the senquence. Note:

$$\mathbf{aba} = 2(\mathbf{a} \cdot \mathbf{b})\mathbf{a},$$

left side: degree 3, right side: degree 1.

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- Grassmannians by null polynomials: e.g., $1 \equiv \mathbf{ab} + \mathbf{ba}$, and

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- Invertible vectors by null binomials: e.g., for any invertible vector $\mathbf{u}$, any null vector $\mathbf{q}$ s.t. $\mathbf{q} \cdot \mathbf{u} \neq 0$,

$$\mathbf{u} \equiv \mathbf{q} - Ad^*_{\mathbf{u}}(\mathbf{q}) =: \mathbf{q} + \mathbf{p},$$

where $\mathbf{p}$ represents the point of 0-d circle $\mathbf{u} \wedge \mathbf{q}$ other than $\mathbf{q}$.

Versors by null binomials

If versor $\mathbf{V}$ equals a null binomial

$$\underline{\mathbf{a}_1}\mathbf{a}_2 \cdots \underbrace{\mathbf{a}_k + \mathbf{b}_1}\mathbf{b}_2 \cdots \underline{\mathbf{b}_l},$$

where $k, l \geq 1$, then (map every tail to the other's head)

$$Ad^*_{\mathbf{V}}(\mathbf{a}_k) \equiv \mathbf{b}_1, \quad Ad^*_{\mathbf{V}}(\mathbf{b}_l) \equiv \mathbf{a}_1.$$

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Proof.

$$\begin{aligned} \mathbf{V}\mathbf{b}_l\mathbf{V}^\dagger &= \mathbf{a}_1 \cdots \mathbf{a}_{k-1}\underline{\mathbf{a}_k\mathbf{b}_l\mathbf{a}_k}\mathbf{a}_{k-1} \cdots \mathbf{a}_1 \\ &= 2(\mathbf{a}_k \cdot \mathbf{b}_l)\mathbf{a}_1 \cdots \mathbf{a}_{k-1}\underline{\mathbf{a}_k}\mathbf{a}_{k-1} \cdots \mathbf{a}_1 \\ &= \ldots\ldots \\ &= 2^k(\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{a}_2 \cdot \mathbf{a}_3) \cdots (\mathbf{a}_{k-1} \cdot \mathbf{a}_k)(\mathbf{a}_k \cdot \mathbf{b}_l)\mathbf{a}_1. \end{aligned}$$

Nullification of invertible vectors

- Let $\mathbf{q}$ be a null vector. For any invertible vector $\mathbf{u}$ s.t. $\mathbf{u} \cdot \mathbf{q} \neq 0$,
$$\acute{\mathbf{u}}_{\mathbf{q}} := \mathbf{u} - \frac{\mathbf{u}^2}{2\mathbf{u} \cdot \mathbf{q}}\mathbf{q} \equiv Ad^*_{\mathbf{u}}(\mathbf{q})$$
is the *nullification* wrt base point $\mathbf{q}$.

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  Geometrically, $\mathbf{u}$ is mapped to the point of 0-d circle $\mathbf{q} \wedge \mathbf{u}$ other than $\mathbf{q}$.

- Let $\mathbf{a} \not\equiv \mathbf{b}$ be two null vectors. For any invertible vector $\mathbf{u} \notin \mathbf{a} \wedge \mathbf{b}$,

$$\acute{\mathbf{u}}_{\mathbf{a},\mathbf{b}} := \mathbf{u} + \lambda\mathbf{a} + \mu\mathbf{b}, \text{ where } \mu = -\frac{\mathbf{u}^2 + 2\lambda\mathbf{u} \cdot \mathbf{a}}{2\mathbf{b} \cdot (\mathbf{u} + \lambda\mathbf{a})},$$

  is the *nullification* wrt base points $\mathbf{a}, \mathbf{b}$ and parameter $\lambda \neq -\frac{\mathbf{u} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{b}}$. Geometrically, $\mathbf{u}$ is mapped to a point other than $\mathbf{a}, \mathbf{b}$ on 1-d circle $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{u}$.

Unstructured null binomials for versors

For any versor $\mathbf{V} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_k$, if $k = 1$, recall: for any null vector $\mathbf{q}$ s.t. $\mathbf{q}\cdot\mathbf{u}_k \neq 0$,

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If $k = 2$, then if $\mathbf{u}_{k-1}\cdot\mathbf{q} \neq 0$ and $\mathbf{u}_{k-1}\cdot\mathbf{p} = -Ad^*_{\mathbf{u}_k}(\mathbf{u}_{k-1})\cdot\mathbf{q} \neq 0$,

$$\mathbf{u}_{k-1}\mathbf{u}_k \equiv \mathbf{u}_{k-1}\mathbf{p} + \mathbf{u}_{k-1}\mathbf{q} = (\acute{\mathbf{u}}_{k-1})_{\mathbf{p}}\mathbf{p} + (\acute{\mathbf{u}}_{k-1})_{\mathbf{q}}\mathbf{q}.$$

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$$\mathbf{u}_{k-1}\mathbf{u}_k \equiv \mathbf{u}_{k-1}\mathbf{p} + \mathbf{u}_{k-1}\mathbf{q} = {\color{blue}(\acute{\mathbf{u}}_{k-1})_{\mathbf{p}}}\mathbf{p} + {\color{red}(\acute{\mathbf{u}}_{k-1})_{\mathbf{q}}}\mathbf{q}.$$

If $k > 2$, then under a set of linear inequality constraints on $\mathbf{q}$,

$$\begin{aligned}
\mathbf{u}_1\cdots\mathbf{u}_{k-1}\mathbf{u}_k &\equiv \mathbf{u}_1\cdots\mathbf{u}_{k-2}{\color{blue}(\acute{\mathbf{u}}_{k-1})_{\mathbf{p}}}\mathbf{p} + \mathbf{u}_1\cdots\mathbf{u}_{k-2}{\color{red}(\acute{\mathbf{u}}_{k-1})_{\mathbf{q}}}\mathbf{q} \\
&= \mathbf{u}_1\cdots\mathbf{u}_{k-3}{\color{blue}(\acute{\mathbf{u}}_{k-2})_{(\acute{\mathbf{u}}_{k-1})_{\mathbf{p}}}(\acute{\mathbf{u}}_{k-1})_{\mathbf{p}}}\mathbf{p} \\
&\quad +\mathbf{u}_1\cdots\mathbf{u}_{k-3}{\color{red}(\acute{\mathbf{u}}_{k-2})_{(\acute{\mathbf{u}}_{k-1})_{\mathbf{q}}}(\acute{\mathbf{u}}_{k-1})_{\mathbf{q}}}\mathbf{q} \\
&= \ldots\ldots \\
&= {\color{blue}\acute{\mathbf{u}}_1\acute{\mathbf{u}}_2\cdots\acute{\mathbf{u}}_{k-1}}\mathbf{p} + {\color{red}\acute{\mathbf{u}}'_1\acute{\mathbf{u}}'_2\cdots\acute{\mathbf{u}}'_{k-1}}\mathbf{q}.
\end{aligned}$$

Structured null binomials for versors

For any versor $\mathbf{V} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_k$, if $k=1$, let $\mathbf{a} \not\equiv \mathbf{b}$ be null vectors such that $\mathbf{a}\cdot\mathbf{u}_k \neq 0$, then

$$\begin{aligned}\mathbf{u}_k &\equiv \mathbf{u}_k(\mathbf{ab}+\mathbf{ba})\\ &= (\mathbf{u}_k\mathbf{a})\mathbf{b} + \mathbf{u}_k\mathbf{b}\mathbf{u}_k^{-1}(\mathbf{u}_k\mathbf{a})\\ &= \acute{\mathbf{u}}_k\mathbf{a}\,\mathbf{b} - Ad^*_{\mathbf{u}_k}(\mathbf{b})\,\acute{\mathbf{u}}_k\mathbf{a}.\end{aligned}$$

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$$\begin{aligned}
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&= -Ad^*_{\mathbf{u}_{k-1}}(\acute{\mathbf{u}}_k)\underline{\mathbf{u}_{k-1}}\mathbf{ab} - Ad^*_{\mathbf{u}_{k-1}\mathbf{u}_k}(\mathbf{b})Ad^*_{\mathbf{u}_{k-1}}(\acute{\mathbf{u}}_k)\underline{\mathbf{u}_{k-1}}\mathbf{a} \\
&= -Ad^*_{\mathbf{u}_{k-1}}(\acute{\mathbf{u}}_k)\ \overline{(\acute{\mathbf{u}}_{k-1})_{Ad^*_{\mathbf{u}_{k-1}}(\acute{\mathbf{u}}_k),\mathbf{a}}}\ \mathbf{a}\ \mathbf{b} \\
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\end{aligned}$$

Similarly, for all $k \geq 2$,

$$\begin{aligned}
\mathbf{u}_1\cdots\mathbf{u}_{k-1}\mathbf{u}_k &\equiv Ad^*_{\mathbf{u}_1\cdots\mathbf{u}_{k-1}}(\acute{\mathbf{u}}_k)\acute{\mathbf{u}}_{k-1}\cdots\acute{\mathbf{u}}_1\mathbf{ab} \\
&\quad +(-1)^k Ad^*_{\mathbf{V}}(\mathbf{b})Ad^*_{\mathbf{u}_1\cdots\mathbf{u}_{k-1}}(\acute{\mathbf{u}}_k)\acute{\mathbf{u}}_{k-1}\cdots\acute{\mathbf{u}}_1\mathbf{a}.
\end{aligned}$$

Null versors

For versor $\mathbf{V} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_k$ where $k \geq 0$, if

$$\mathbf{V} \equiv \mathbf{p}(\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_l) + (\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_l)\mathbf{q},$$

where $l > 0$, the representation is called a centered null binomial with center $\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_l$ and ends $\mathbf{p}, \mathbf{q}$.

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Let $\mathbf{V} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_k$ be tight, i.e., $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k \neq 0$. Its null binomial representation is

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- Tight: if $l = k - 1$. Almost tight: if $l = k + 1$.
- In the tight case, if $k > 2$, then $\langle \mathbf{V} \rangle_k = \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{k-1} \wedge (\mathbf{q} + (-1)^{k-1}\mathbf{p})$ is Minkowski; if $k = 2$, then $\langle \mathbf{V} \rangle_k$ is either Minkowski or degenerate.

  Not every versor has tight centered null binomial form!

Main results

- Null monomials: geometric interpretation, DOF in the factorization.
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- Rewriting between invertible monomial form and centered null binomial form of versors.
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- Applications to symbolic geometric computation.

Outline

- Basics of CGA
- Null Monomials in CGA
- Null Versors in CGA

Null monomials

- Anisotropic: $\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k$ where $\mathbf{a}_1 \not\equiv \mathbf{a}_k$.
- Isotropic: $\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k\mathbf{a}_1$.

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In the anisotropic case, if $\mathbf{a}_i \not\equiv \mathbf{a}_1$ for all $i > 2$, then

$$\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k \equiv \underline{(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)(\mathbf{a}_1\mathbf{a}_3\mathbf{a}_4)\cdots(\mathbf{a}_1\mathbf{a}_{k-1}\mathbf{a}_k)\mathbf{a}_1}\mathbf{a}_k,$$

where the underlined part is isotropic; else for some $l \leq k$,

$$\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k \equiv (\mathbf{a}_1\mathbf{a}_2'\mathbf{a}_3')(\mathbf{a}_1\mathbf{a}_3'\mathbf{a}_4')\cdots(\mathbf{a}_1\mathbf{a}_{l-1}'\mathbf{a}_k)\mathbf{a}_1\mathbf{a}_k.$$

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Consider isotropic $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1$: for any scalars λ_1, λ_2,

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1 = \mathbf{a}_1\mathbf{a}_2(\mathbf{a}_3 + \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2)\mathbf{a}_1.$$

Choose $\mathbf{n} = \mathbf{a}_3 + \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2$ to be invertible, and $\mathbf{n}\cdot\mathbf{a}_1 = 0$.
Then

$$\mathbf{a}_1\mathbf{a}_2\mathbf{n}\mathbf{a}_1 = -\mathbf{a}_1\mathbf{a}_2\mathbf{a}_1\mathbf{n} = -2(\mathbf{a}_1\cdot\mathbf{a}_2)\mathbf{a}_1\mathbf{n} \equiv \mathbf{a}_1\mathbf{n}. \quad \text{(pseudo-versor form)}$$

Positive rejection (inverse procedure of nullification)

Let $\mathbf{a}, \mathbf{b}, \mathbf{d}$ be null vectors and $\mathbf{a} \not\equiv \mathbf{b}, \mathbf{d}$. For any null vector $\mathbf{c} \not\equiv \mathbf{a}, \mathbf{b}$, any scalars λ, μ, let $\mathbf{n} = \mathbf{c} + \lambda\mathbf{a} + \mu\mathbf{b}$ be positive.

- For any λ, there is a unique μ such that $\mathbf{n} \cdot \mathbf{a} = 0$. The corresponding $\mathbf{n}$, namely,

$$\grave{\mathbf{c}}_{\mathbf{b}}^{\mathbf{a}} := \mathbf{c} + \lambda\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\mathbf{b},$$

  is called the *rejection* of $\mathbf{c}$ from $\mathbf{a}$ wrt $\mathbf{b}$ and parameter λ.

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- There are unique parameters λ, μ such that $\mathbf{n} \cdot \mathbf{a} = 0$ and $\mathbf{n} \cdot \mathbf{d} = 0$. The corresponding $\mathbf{n}$, namely,

$$\grave{\mathbf{c}}_{\mathbf{b}}^{\mathbf{a},\mathbf{d}} := \mathbf{c} - \frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})}{(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{d})}\mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\mathbf{b},$$

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Note: $\grave{\mathbf{c}}_{\mathbf{b}}^{\mathbf{a},\mathbf{d}} \neq P^{\perp}_{\mathbf{a} \wedge \mathbf{d}}(\mathbf{c})$.

Pseudo-versor form of null monomials

Isotropic monomial: by

$$\mathbf{b}\mathbf{a}_1\mathbf{a}_2\mathbf{b} = 2(\mathbf{b}\cdot\mathbf{a}_1)(\grave{\mathbf{a}}_2)^{\mathbf{b}}_{\mathbf{a}_1}\mathbf{b} = -2(\mathbf{b}\cdot\mathbf{a}_2)(\grave{\mathbf{a}}_1)^{\mathbf{b}}_{\mathbf{a}_2}\mathbf{b},$$

where $(\grave{\mathbf{a}}_1)^{\mathbf{b}}_{\mathbf{a}_2} \in \mathbf{b}^{\perp}$ is invertible, we get

$$(\mathbf{b}\mathbf{a}_1\mathbf{a}_2)(\mathbf{b}\mathbf{a}_3\mathbf{a}_4)\cdots(\mathbf{b}\mathbf{a}_{2l-1}\mathbf{a}_{2l})\mathbf{b} \equiv (\grave{\mathbf{a}}_1)^{\mathbf{b}}_{\mathbf{a}_2}(\grave{\mathbf{a}}_3)^{\mathbf{b}}_{\mathbf{a}_4}\cdots(\grave{\mathbf{a}}_{2l-1})^{\mathbf{b}}_{\mathbf{a}_{2l}}\mathbf{b}.$$

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So

$$\mathbf{b}\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_k\mathbf{b} = \mathbf{V}^{\mathbf{b}}\mathbf{b} = (-1)^{|\mathbf{V}^{\mathbf{b}}|}\mathbf{b}\mathbf{V}^{\mathbf{b}},$$

where $\mathbf{V}^{\mathbf{b}}$ is a versor in $\mathcal{C}l(\mathbf{b}^{\perp})$ with length $|\mathbf{V}^{\mathbf{b}}|$.

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Pseudo-versor form of null monomials

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Corollary. Any null monomial is nonzero, no matter how the variables are specified.

Geometric interpretation of null monomials

Isotropic: $\mathbf{e}_\infty \acute{\mathbf{c}}_1 \acute{\mathbf{c}}_2 \cdots \acute{\mathbf{c}}_k \mathbf{e}_\infty \equiv \overrightarrow{\mathbf{c}_1\mathbf{c}_2}\ \overrightarrow{\mathbf{c}_2\mathbf{c}_3} \cdots \overrightarrow{\mathbf{c}_{k-1}\mathbf{c}_k}\ \mathbf{e}_\infty$,
where $\overrightarrow{\mathbf{c}_i\mathbf{c}_j} = \mathbf{c}_j - \mathbf{c}_i$. In general,

$$(\mathbf{e}_\infty \acute{\mathbf{c}}_1 \acute{\mathbf{c}}_2)(\mathbf{e}_\infty \acute{\mathbf{c}}_3 \acute{\mathbf{c}}_4) \cdots (\mathbf{e}_\infty \acute{\mathbf{c}}_{2l-1} \acute{\mathbf{c}}_{2l}) \mathbf{e}_\infty \equiv \overrightarrow{\mathbf{c}_1\mathbf{c}_2}\ \overrightarrow{\mathbf{c}_3\mathbf{c}_4} \cdots \overrightarrow{\mathbf{c}_{2l-1}\mathbf{c}_{2l}}\ \mathbf{e}_\infty.$$

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DOF in a null monomial = DOF of the versor factors. Examples:

- In $\mathbf{e}_\infty \acute{\mathbf{c}}_1 \acute{\mathbf{c}}_2 \mathbf{e}_\infty$, vectors $\acute{\mathbf{c}}_1, \acute{\mathbf{c}}_2$ can be replaced by any null vectors $\acute{\mathbf{d}}_1, \acute{\mathbf{d}}_2$ s.t. $(\mathbf{d}_1 - \mathbf{d}_2) \wedge (\mathbf{c}_1 - \mathbf{c}_2) = 0$.

  The isotropic null monomial represents a displacement vector!

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- In $\acute{\mathbf{c}}_1 \acute{\mathbf{c}}_2 \acute{\mathbf{c}}_3$, factors $\acute{\mathbf{c}}_1$ and $\acute{\mathbf{c}}_3$ have 0 DOF, factor $\acute{\mathbf{c}}_2$ has 1 DOF (any point on circle/line $\acute{\mathbf{c}}_1 \wedge \acute{\mathbf{c}}_2 \wedge \acute{\mathbf{c}}_3$ other than $\mathbf{c}_1, \mathbf{c}_3$).

  A ordered pair of points and the two open arcs between the two points on a circle!

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- Null Versors in CGA

Hybrid-versor form of null versors

Theorem 1. Any centered null binomial is a versor. Conversely, any versor $\mathbf{V}$ can be written as a centered null binomial of the form

$$\mathbf{V} = \mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1} + \mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}, \quad \text{where } \mathbf{p} = (-1)^k Ad^*_{\mathbf{V}}(\mathbf{q}).$$

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Hybrid-versor form of null versors:
Anisotropic: by pseudo-versor form of the center,

$$\mathbf{c}_1 \cdots \mathbf{c}_{k-1} \equiv \mathbf{V}^{\mathbf{c}_1,\mathbf{p}}\mathbf{c}_1\mathbf{c}_{k-1} = \mathbf{c}_1\mathbf{c}_{k-1}\mathbf{V}^{\mathbf{c}_{k-1},\mathbf{q}}.$$

So

$$\begin{aligned} \mathbf{V} &\equiv \mathbf{V}^{\mathbf{c}_1,\mathbf{p}}((-1)^{k-1}\mathbf{p}\mathbf{c}_1\mathbf{c}_{k-1} + \mathbf{c}_1\mathbf{c}_{k-1}\mathbf{q}) \\ &= (\mathbf{p}\mathbf{c}_1\mathbf{c}_{k-1} + (-1)^{k-1}\mathbf{c}_1\mathbf{c}_{k-1}\mathbf{q})\mathbf{V}^{\mathbf{c}_{k-1},\mathbf{q}}. \end{aligned}$$

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Isotropic: for $\mathbf{c}_{k-1} = \mathbf{c}_1$,

$$\mathbf{c}_1 \cdots \mathbf{c}_{k-2}\mathbf{c}_1 \equiv \mathbf{V}'^{\mathbf{c}_1,\mathbf{p}}\mathbf{c}_1 = \mathbf{c}_1\mathbf{V}'^{\mathbf{c}_1,\mathbf{q}}.$$

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DOF in a null versor

Let $\mathbf{V} = \mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1} + \mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}$, where $k > 1$. A null vector $\mathbf{r}$ is called an *alternate* of the center's right end $\mathbf{c}_{k-1}$, if there exist null vectors $\mathbf{c}'_2, \ldots, \mathbf{c}'_{k-2}$, s.t.

$$\mathbf{V} = \mathbf{p}(Ad^*_{\mathbf{V}}(\mathbf{r})\mathbf{c}'_2 \cdots \mathbf{c}'_{k-2}\mathbf{r}) + (Ad^*_{\mathbf{V}}(\mathbf{r})\mathbf{c}'_2 \cdots \mathbf{c}'_{k-2}\mathbf{r})\mathbf{q}.$$

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Theorem 2. [Freedom in versor ends] For any null vector $\mathbf{r} \not\equiv \mathbf{c}_{k-1}$,

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Theorem 3. [Freedom in center ends]

- If $\langle \mathbf{V} \rangle_k \neq 0$, then $\mathbf{r}$ is an alternate of the center's left/right end iff $\mathbf{r} \in \langle \mathbf{V} \rangle_k$ and $\mathbf{r} \not\equiv Ad^*_{\mathbf{V}}(\mathbf{r})$.

DOF in a null versor

Let $\mathbf{V} = \mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1} + \mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}$, where $k > 1$. A null vector $\mathbf{r}$ is called an *alternate* of the center's right end $\mathbf{c}_{k-1}$, if there exist null vectors $\mathbf{c}'_2, \ldots, \mathbf{c}'_{k-2}$, s.t.

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- If $\langle \mathbf{V} \rangle_k = 0$ but $\langle \mathbf{V} \rangle_{k-2} \neq 0$, then $\mathbf{r}$ is an alternate of the center's left/right end iff either $\mathbf{r} \cdot \langle \mathbf{V} \rangle_{k-2} \neq 0$, or $\langle \mathbf{V} \rangle_{k-2}$ is degenerate and $\mathbf{r} \in \langle \mathbf{V} \rangle_{k-2}$.

DOF in a null versor continued

- If $\langle \mathbf{V} \rangle_k = \langle \mathbf{V} \rangle_{k-2} = 0$, then any null vector of $\mathbb{R}^{n+1,1}$ is an alternate of the center's left/right end.

Remark: If the degree of the center is not restricted, then the center can always be trivially elongated to $\mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}$:

$$\mathbf{V} \equiv (\mathbf{q} \cdot \mathbf{c}_{k-1})\, \mathbf{c}_1\, (\mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}) + (\mathbf{p} \cdot \mathbf{c}_1)\, (\mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q})\, \mathbf{c}_{k-1}.$$

DOF in a null versor continued

- If $\langle \mathbf{V} \rangle_k = \langle \mathbf{V} \rangle_{k-2} = 0$, then any null vector of $\mathbb{R}^{n+1,1}$ is an alternate of the center's left/right end.

Remark: If the degree of the center is not restricted, then the center can always be trivially elongated to $\mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}$:

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Theorem 4. [Freedom in center interior] For fixed $\mathbf{p}, \mathbf{q}, \mathbf{c}_1, \mathbf{c}_{k-1}$, center $\mathbf{c}_1 \cdots \mathbf{c}_{k-1}$ is fixed, or equivalently, DOF of null vectors $\mathbf{c}_2, \ldots, \mathbf{c}_{k-2}$ = DOF of null monomial $\mathbf{c}_1\mathbf{c}_2 \cdots \mathbf{c}_{k-1}$.

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Theorem 5. [Tightness] Any versor has either tight or almost tight null versor form. Let $\mathbf{V} = \mathbf{u}_1 \cdots \mathbf{u}_k$ be a tight versor. Then $\mathbf{V}$ has tight null versor form iff $k \geq 2$, and

- when $k = 2$, then $\langle \mathbf{V} \rangle_k$ is Minkowski or degenerate;
- when $k > 2$, then $\langle \mathbf{V} \rangle_k$ is Minkowski and $\mathbf{V} \neq \langle \mathbf{V} \rangle_k$.

Multiplication of two null versors

To compute $\mathbf{V}\mathbf{V}'$, where
$$\begin{aligned} \mathbf{V} &= \mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{k-1} + \mathbf{c}_1 \cdots \mathbf{c}_{k-1}\mathbf{q}, \\ \mathbf{V}' &= \mathbf{p}'\mathbf{c}'_1 \cdots \mathbf{c}'_{l-1} + \mathbf{c}'_1 \cdots \mathbf{c}'_{l-1}\mathbf{q}', \end{aligned}$$

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(1) When $\mathbf{c}_{k-1}\not\equiv\mathbf{c}'_1$, choose new ends:

$$\begin{array}{ll} \mathbf{p}'=\mathbf{c}_{k-1}, & \mathbf{q}'=(-1)^l Ad^*_{\mathbf{V}'^{-1}}(\mathbf{c}_{k-1}); \\ \mathbf{q}=\mathbf{c}'_1, & \mathbf{p}=(-1)^k Ad^*_{\mathbf{V}}(\mathbf{c}'_1). \end{array}$$

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$$\begin{array}{rcl}\mathbf{V}\mathbf{V}'=\mathbf{p}\mathbf{C}\mathbf{q}'+2(\mathbf{c}'_1\cdot\mathbf{c}_{k-1})\mathbf{C} &=& (\mathbf{p}\mathbf{C})\mathbf{q}'+\dfrac{\mathbf{c}'_1\cdot\mathbf{c}_{k-1}}{\mathbf{p}\cdot\mathbf{c}_1}\mathbf{c}_1(\mathbf{p}\mathbf{C})\\ &=& \mathbf{p}(\mathbf{C}\mathbf{q}')+\dfrac{\mathbf{c}'_1\cdot\mathbf{c}_{k-1}}{\mathbf{q}'\cdot\mathbf{c}'_{l-1}}(\mathbf{C}\mathbf{q}')\mathbf{c}'_{l-1}.\end{array}$$

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Example 1: inversion, negative inversion and reflection

Recall: for any invertible vector $\mathbf{u}$, any null vector $\mathbf{q}$ s.t. $\mathbf{u} \cdot \mathbf{q} \neq 0$,

$$\mathbf{u} \equiv \mathbf{q} - Ad^*_{\mathbf{u}}(\mathbf{q}) := \mathbf{q} + \mathbf{p}.$$

Despite the DOF of $\mathbf{q}$, this null binomial has no center!

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Centered null binomial form:

$$\mathbf{q} + \mathbf{p} \equiv (\mathbf{q} + \mathbf{p})(\mathbf{pq} + \mathbf{qp}) = \mathbf{qpq} + \mathbf{pqp}.$$

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- Inversion induced by $\acute{\mathbf{c}} - (\rho^2/2)\mathbf{e}_\infty$:

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- Reflection induced by $\mathbf{n}+\delta\mathbf{e}_\infty$: for any point $\mathbf{c}\in\mathbb{R}^n$ not on the hyperplane $(\mathbf{n},\delta)$,

$$\mathbf{q}=\acute{\mathbf{c}},\quad \mathbf{p}=-Ad^*_{\mathbf{n}+\delta\mathbf{e}_\infty}(\acute{\mathbf{c}}).$$

Example 2: dilation and translation

(1) Dilation about center $\mathbf{c} \in \mathbb{R}^n$ with ratio λ:

$$\mathbf{V} = \lambda \mathbf{e}_\infty \acute{\mathbf{c}} + \acute{\mathbf{c}} \mathbf{e}_\infty .$$

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(2) Translation from point $\mathbf{c} \in \mathbb{R}^n$ to point $\mathbf{d} \in \mathbb{R}^n$ is induced by $\mathbf{V} = 1 + \mathbf{e}_\infty(\mathbf{d} - \mathbf{c})/2$:

$$\acute{\mathbf{d}} \mathbf{e}_\infty + \mathbf{e}_\infty \acute{\mathbf{c}} = -2 + \mathbf{e}_\infty \wedge (\acute{\mathbf{c}} - \acute{\mathbf{d}}) = -2\mathbf{V}.$$

Center of the null versor: $\mathbf{e}_\infty$.

Example 3: 2-d rotation

2-d rotation $\mathbf{ba}$, where $\mathbf{a}, \mathbf{b} \in \mathbf{e}_{\infty}{}^{\perp}$ are invertible.

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$$\begin{aligned}
\mathbf{ba} &\equiv \mathbf{b}(\mathbf{c}\mathbf{e}_\infty + \mathbf{e}_\infty\mathbf{c})\underline{\mathbf{a}} \\
&= \mathbf{b}(-\mathbf{c}\underline{\mathbf{a}}\mathbf{e}_\infty + \mathbf{e}_\infty\mathbf{c}\underline{\mathbf{a}}) \\
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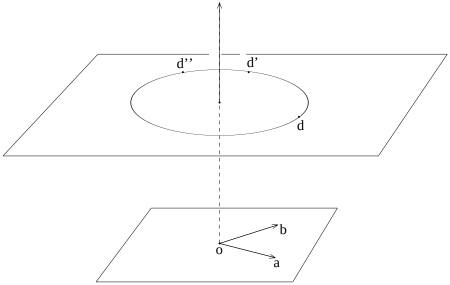
Let $\mathbf{d} = Ad^*_{\mathbf{a}}(\mathbf{c})$, then the center is

$$Ad^*_{\mathbf{ba}}(\mathbf{d})\,Ad^*_{\mathbf{a}}(\mathbf{d})\,\mathbf{d}.$$

In the null monomial, $Ad^*_{\mathbf{a}}(\mathbf{d})$ can be replaced by any null vector in $Ad^*_{\mathbf{ba}}(\mathbf{d}) \wedge Ad^*_{\mathbf{a}}(\mathbf{d}) \wedge \mathbf{d}$ that $\neq \mathbf{d}, Ad^*_{\mathbf{ba}}(\mathbf{d})$.

Geometric meaning of the center

Let $\mathbf{d}'' = Ad^*_{\mathbf{a}}(\mathbf{d})$, $\mathbf{d}' = Ad^*_{\mathbf{ba}}(\mathbf{d})$. Center: $\mathbf{d}'\mathbf{d}''\mathbf{d}$.



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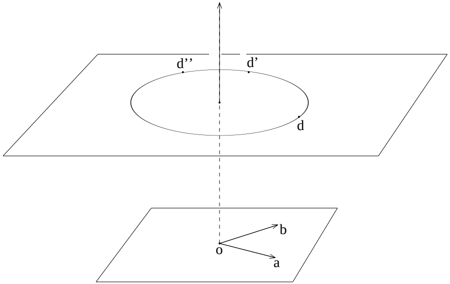

The center gives the full trajectory of a generic point $\mathbf{d}$ under the 2-d rotation induced by $\mathbf{ba}$:

- $\mathbf{d}$ is the starting point; $\mathbf{d}''$ is the image of $\mathbf{d}$ under reflection $\mathbf{a}$; ending point $\mathbf{d}'$ is the image of $\mathbf{d}''$ under reflection $\mathbf{b}$.
- $\mathbf{d}''$ can be replaced by any point on the circular trajectory formed by point $\mathbf{d}$, except for the starting and ending points.

Example 4: Inversion wrt circle in high dimensions

Inversion induced by $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ (null vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$):

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Let $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ where the $\mathbf{e}_i$ are mutually orthogonal. For any null vector $\mathbf{d}$ s.t. $\mathbf{d} \cdot \mathbf{e}_i \neq 0$ for $i = 1, 2, 3$,

$$\begin{aligned}
\mathbf{V} &\equiv \mathbf{e}_1\mathbf{e}_2\underline{\mathbf{e}_3}(\mathbf{cd} + \mathbf{dc}) \\
&= \mathbf{e}_1\mathbf{e}_2(-Ad^*_{\mathbf{e}_3}(\mathbf{c})\underline{\mathbf{e}_3}\mathbf{d} + \underline{\mathbf{e}_3}\mathbf{dc}) \\
&\equiv \mathbf{e}_1\mathbf{e}_2(-Ad^*_{\mathbf{e}_3}(\mathbf{c})\underline{Ad^*_{\mathbf{e}_3}(\mathbf{d})}\mathbf{d} + \underline{Ad^*_{\mathbf{e}_3}(\mathbf{d})}\mathbf{dc}) \\
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&= \mathbf{cD} + \mathbf{Dc}, \qquad \text{where } \mathbf{D} = Ad^*_{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3}(\mathbf{d})Ad^*_{\mathbf{e}_1\mathbf{e}_2}(\mathbf{d})Ad^*_{\mathbf{e}_1}(\mathbf{d}).
\end{aligned}$$

From null versors to invertible monomials

By the hybrid-versor form, only need to consider tight degree-2 and degree-3 cases.

Tight degree-2: $\mathbf{V} = \mathbf{qc} + \mathbf{cp}$.

- If $\langle \mathbf{V} \rangle_2$ is non-degenerate, then $\mathbf{V}$ is dilation-like.
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- Let $\mathbf{V}' = \mathbf{uV}$. Then $\mathbf{V} \equiv \mathbf{uV}'$, and

$$\mathbf{V}' = (\mathbf{q} + \lambda\mathbf{c}_1)(\mathbf{q}\mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_1\mathbf{c}_2\mathbf{c}_1) \equiv (\mathbf{c}_1 \cdot \mathbf{c}_2)\mathbf{q}\mathbf{c}_1 + (\mathbf{c}_1 \cdot \mathbf{q})\mathbf{c}_1\mathbf{c}_2$$

  is a degree-2 null versor: dilation-like or translation-like.

Null versor compression algorithm

Input: $\mathbf{V} = \mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{l-1} + \mathbf{c}_1 \cdots \mathbf{c}_{l-1}\mathbf{q}$, where $l \geq 4$.
Output: $\mathbf{V} = \mathbf{p}\mathbf{c}'_1 \cdots \mathbf{c}'_{k-1} + \mathbf{c}'_1 \cdots \mathbf{c}'_{k-1}\mathbf{q}$, where k is minimal.

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Step 3: Compute the hybrid-versor form of $\mathbf{V}$. If $\mathbf{c}'_1 \not\equiv \mathbf{c}'_{k-1}$, then $\mathbf{V} = \mathbf{V}^{\mathbf{c}'_1,\mathbf{p}}((-1)^{l-1}\mathbf{p}\mathbf{c}'_1\mathbf{c}'_{k-1} + \mathbf{c}'_1\mathbf{c}'_{k-1}\mathbf{q})$, else $\mathbf{V} = \mathbf{V}'^{\mathbf{c}'_1,\mathbf{p}}((-1)^{l}\mathbf{p}\mathbf{c}'_1 + \mathbf{c}'_1\mathbf{q})$.

Null versor compression algorithm

Input: $\mathbf{V} = \mathbf{p}\mathbf{c}_1 \cdots \mathbf{c}_{l-1} + \mathbf{c}_1 \cdots \mathbf{c}_{l-1}\mathbf{q}$, where $l \geq 4$.
Output: $\mathbf{V} = \mathbf{p}\mathbf{c}'_1 \cdots \mathbf{c}'_{k-1} + \mathbf{c}'_1 \cdots \mathbf{c}'_{k-1}\mathbf{q}$, where k is minimal.

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Step 4: In $\mathcal{C}l((\mathbf{p} \wedge \mathbf{c}'_1)^{\perp})$, compress $\mathbf{V}^{\mathbf{c}'_1,\mathbf{p}}$ (or $\mathbf{V}'^{\mathbf{c}'_1,\mathbf{p}}$) to $\mathbf{W}^{\mathbf{c}'_1,\mathbf{p}}$ (or $\mathbf{W}'^{\mathbf{c}'_1,\mathbf{p}}$).

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Step 5: Rewrite $\mathbf{W}^{\mathbf{c}'_1,\mathbf{p}}((-1)^{l-1}\mathbf{p}\mathbf{c}'_1\mathbf{c}'_{k-1} + \mathbf{c}'_1\mathbf{c}'_{k-1}\mathbf{q})$ (or $\mathbf{W}'^{\mathbf{c}'_1,\mathbf{p}}((-1)^{l}\mathbf{p}\mathbf{c}'_1 + \mathbf{c}'_1\mathbf{q})$) to null versor form.

Summary

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3. Nice properties of null versors:
   - DOF of the center and ends
   - Multiplication
   - Rewriting between null binomials and invertible monomials
   - Compression to tight/almost null versor tight form
   - Symbolic geometric computation applications

Thanks!