Construction of Special Conics in Bundles of Conics Using Geometric Algebras

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Contents

GAC...Clifford algebra $\mathcal{C}l(5,3)$ with embedding $C : \mathbb{R}^2 \to \mathbb{R}^{5,3}$ of point (x, y) defined as

$$C(x,y) = \bar{n}_+ + xe_1 + ye_2 + \frac{1}{2}(x^2+y^2)n_+ + \frac{1}{2}(x^2-y^2)n_- + xyn_\times . \quad (1)$$

Conic section representations in GAC:

- IPNS representation

$$Q_I = \bar{v}^\times \bar{n}_\times + \bar{v}^- \bar{n}_- + \bar{v}^+ \bar{n}_+ + v^1 e_1 + v^2 e_2 + v^+ n_+ \quad (2)$$

- OPNS representation

$$Q_O = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5 \quad (3)$$

- matrix representation

$$M = \begin{pmatrix} -\frac{1}{2}(\bar{v}^{+} + \bar{v}^{-}) & -\frac{1}{2}\bar{v}^{\times} & \frac{1}{2}v^{1} \\ -\frac{1}{2}\bar{v}^{\times} & -\frac{1}{2}(\bar{v}^{+} - \bar{v}^{-}) & \frac{1}{2}v^{2} \\ \frac{1}{2}v^{1} & \frac{1}{2}v^{2} & -v^{+} \end{pmatrix} \tag{4}$$

$$\bar{M} = \begin{pmatrix} -\frac{1}{2}(\bar{v}^{+} + \bar{v}^{-}) & -\frac{1}{2}\bar{v}^{\times} \\ -\frac{1}{2}\bar{v}^{\times} & -\frac{1}{2}(\bar{v}^{+} - \bar{v}^{-}) \end{pmatrix} \tag{5}$$

Real projective plane $\mathbb{RP}^2 = \mathbb{R}^2 \cup l_\infty$:

- $\mathbb{R}^2$...real plane, set of *proper* points
- l_∞...line at infinity, set of *improper* points

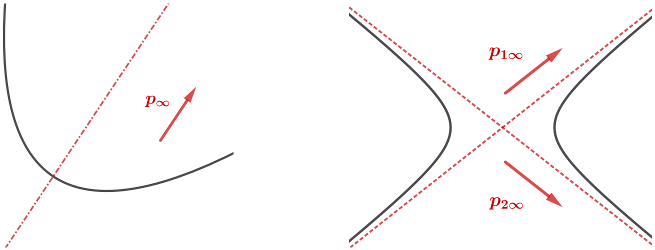


Figure 1: Improper points of parabola and hyperbola

Distinguishing proper & improper points:

$$(a,b) \mapsto \begin{cases} k(a,b,1),\ k \neq 0, & \text{if } (a,b) \text{ is proper} \\ k(a,b,0),\ k \neq 0, & \text{if } (a,b) \text{ is improper} \end{cases} \tag{6}$$

Embedding $C\mathbb{P} : \mathbb{RP}^2 \to \mathbb{R}^{5,3}$:

$$C\mathbb{P}(a,b,c) = c^2\bar{n}_+ + ace_1 + bce_2 + \frac{1}{2}(a^2+b^2)n_+ + \frac{1}{2}(a^2-b^2)n_- + abn_\times \tag{7}$$

Corollary:

$$C\mathbb{P}(x,y,1) \equiv C(x,y),$$
$$C\mathbb{P}(s,t,0) = \frac{1}{2}(s^2+t^2)n_+ + \frac{1}{2}(s^2-t^2)n_- + stn_\times .$$

Bundle of conics generated by conics Q^1 and Q^2

- $\{\lambda Q^1 + \mu Q^2 : (\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}\}$
- set of all conics passing through the intersection points of Q^1 and Q^2

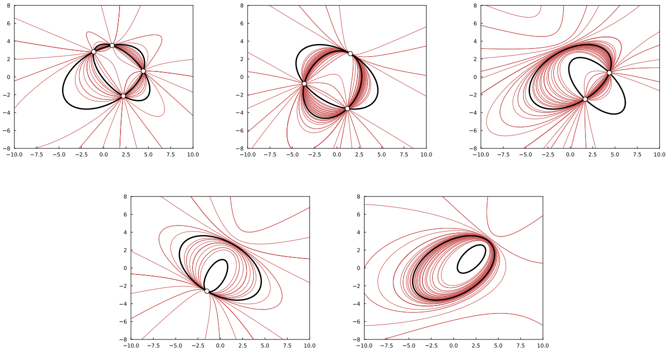


Figure 2: Bundles of conics generated by two conics, 4 to 0 real points of intersection

Four-point

- GAC object representing intersection of two conics
- IPNS four-point:

$$\left(Q^1 \cap Q^2\right)_I = Q^1_I \wedge Q^2_I. \tag{8}$$

Wedge of a four-point and another point $\implies$ OPNS conic:

$$Q_O = \left(Q^1_I \wedge Q^2_I\right)^* \wedge P_I \tag{9}$$

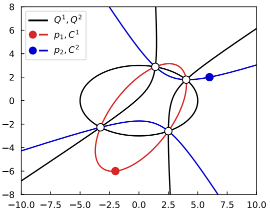


Figure 3: Four-point obtained as an intersection of conics Q^1, Q^2. Conics C^1 and C^2 were constructed as wedge of the four-point with points p_1 and p_2, respectively.

- A bundle of conics generally contains 3 degenerate conics (*line-pairs*)

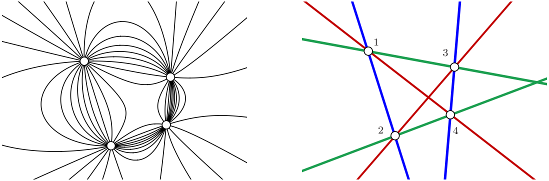


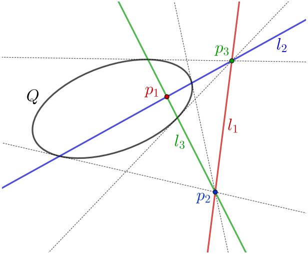
Figure 4: Bundle of conics, its four-point and three line-pairs, [11]

Wedge construction of the line-pairs

- the line-pairs pass through the four-point *and* the *double points* p_1, p_2, p_3:

$$LP_O^i = \left(Q_I^1 \wedge Q_I^2\right)^* \wedge C\mathbb{P}(p_i). \qquad (10)$$

- Double points can be found as the vertices of the *common self-polar triangle* of generating conics Q^1 and Q^2:



(a) self-polar triangle of conic Q

(b) common self-polar triangle of conics Q^1 and Q^2

Figure 5

- Computation of the double points using conic matrices M_1 and M_2 and a generalised eigenproblem:

$$M_1 p = \lambda M_2 p. \tag{11}$$

Example 1

- Ellipses E^1, E^2 with IPNS representations

$$E_I^1 = 16\bar{n}_- - 34\bar{n}_+ + 225n_+,$$
$$E_I^2 = -3\bar{n}_- - 5\bar{n}_+ - 16e_1 - 2e_2 - n_+,$$

and the associated matrices

$$M_1 = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -225 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 4 & 0 & -8 \\ 0 & 1 & -1 \\ -8 & -1 & 1 \end{pmatrix}.$$

- Solution to generalised eigenproblem (11):

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \approx \begin{pmatrix} 13.0501 \\ 21.6502 \\ 2.7997 \end{pmatrix}, \quad \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \approx \begin{pmatrix} 2.4167 & 2.2320 & 10.1865 \\ -1.0921 & -6.4631 & -0.1261 \\ 1 & 1 & 1 \end{pmatrix}$$

- Computation of the line-pairs:

$$LP_O^i = \left(Q_I^1 \wedge Q_I^2\right)^* \wedge C\mathbb{P}(p_i), \qquad i = 1, 2, 3.$$

Example 1

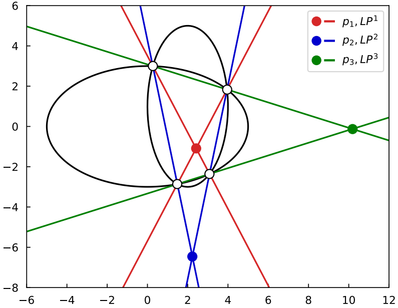


Figure 6: Four-point obtained as an intersection of two conics from Example 1. Each of the three line-pairs was constructed by wedging the four-point and the corresponding double point.

Example 2

- Concentric conics $\implies$ easier computation of the-line-pairs
- Possible use of improper points in the construction

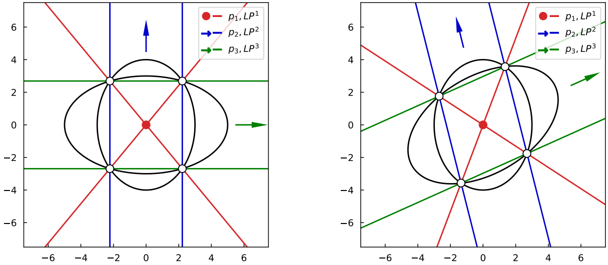
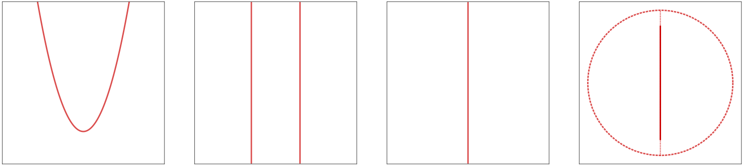


Figure 7: Four-point as an intersection of two concentric conics, three associated line-pairs

- *Generalised parabola*...conic with singular principal submatrix of form (5)
- Bundles generally contains 2 generalised parabolas
- Generalised parabolas include also geometrically degenerated parabolas



(a) ordinary parabola
(b) parallel lines
(c) double line
(d) union of ordinary line & line at infinity

Figure 8: Types of generalised parabolas

Wedge construction of the generalised parabolas

- The generalised parabolas pass through the four-point *and* their *improper points* $p_{\infty 1}, p_{\infty 2}$:

$$P_O^j = \left(Q_I^1 \wedge Q_I^2\right)^* \wedge C\mathbb{P}(p_{\infty j}). \tag{12}$$

- The directions $\bar{p}_{\infty 1}, \bar{p}_{\infty 2}$ of improper points $p_{\infty 1}, p_{\infty 2}$ are the *common conjugate directions* of generating conics Q^1 and Q^2
- The common conjugate directions can be computed using conic submatrices $\bar{M}_1, \bar{M}_2$ and a generalised eigenproblem:

$$\bar{M}_1 \bar{p}_\infty = \lambda \bar{M}_2 \bar{p}_\infty. \tag{13}$$

Example 3

- Ellipses E^1, E^2 with IPNS representations

$$E_I^1 = -8\sqrt{3}\bar{n}_\times - 8\bar{n}_- + 34\bar{n}_+ - 225n_+,$$
$$E_I^2 = 24\bar{n}_\times + 40\bar{n}_+ + 92e_1 + 68e_2 - 19n_+,$$

and the associated principal submatrices

$$\bar{M}_1 = \begin{pmatrix} -13 & 4\sqrt{3} \\ 4\sqrt{3} & -21 \end{pmatrix}, \quad \bar{M}_2 = \begin{pmatrix} -20 & -12 \\ -12 & -20 \end{pmatrix}.$$

- Solution to generalised eigenproblem (13):

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \approx \begin{pmatrix} 0.2916 \\ 3.0142 \end{pmatrix}, \quad \begin{pmatrix} \bar{p}_{\infty 1} & \bar{p}_{\infty 2} \end{pmatrix} \approx \begin{pmatrix} 1.4547 & -0.9115 \\ 1 & 1 \end{pmatrix},$$

- Computation of the generalised parabolas:

$$P_O^j = \left(E_I^1 \wedge E_I^2\right)^* \wedge C\mathbb{P}\left(p_{\infty j}\right), \qquad j = 1, 2.$$

Example 3

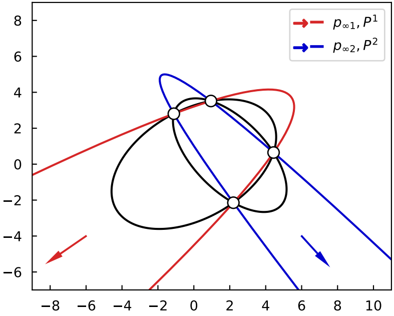


Figure 9: Four-point obtained as an intersection of two conics from Example 3, associated generalised parabolas and their improper points

Example 4

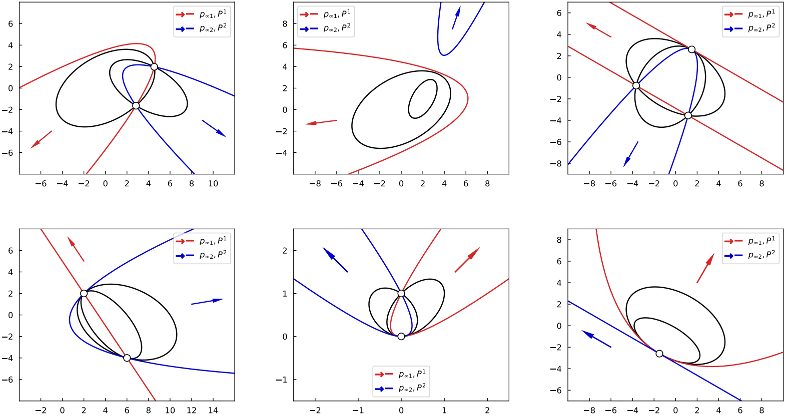


Figure 10: More generalised parabolas of bundles

Thank you for your attention!

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