The Algebra of Geometry

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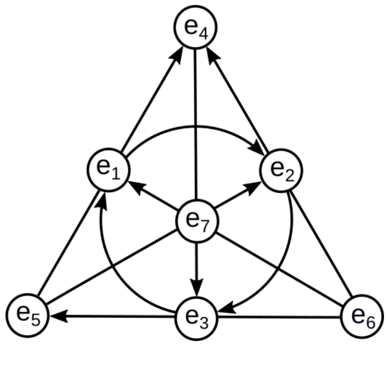
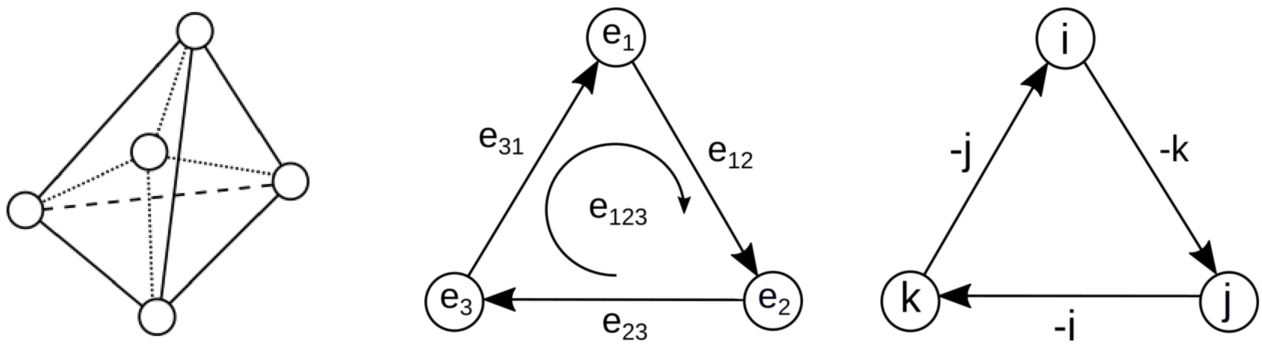
Simplices and Geometric algebra

Pascal's Triangle

N	V	E	F	T	⋯			
1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱



Fano Plane Diagram



(a) **4-simplex** (b) **2-simplex to GA(3)** (c) **Quaternions**

Pfaffian connection to simplices

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\ldots\mathbf{a}_n = \sum_{i=0}^{[\frac{n}{2}]}\sum_{\mu\in\mathcal{C}^n_{2i}} (-1)^k \backslash \mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2},\ldots,\mathbf{a}_{\mu_{2i-1}}\cdot\mathbf{a}_{\mu_{2i}}|\ \mathbf{a}_{\mu_{2i+1}}\wedge\ldots\wedge\mathbf{a}_{\mu_n}$$

Pfaffian derivations in geometric algebra

Pfaffian expansion $\mathbf{abc} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})e_{123} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$

Derivation

$$\begin{aligned}\nabla\mathbf{ab} &= \nabla(\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b})\\ &= \nabla(|\mathbf{a}||\mathbf{b}|(\cos(\phi)+I_{\mathbf{ab}}\sin(\phi)))\\ &= \nabla(\mathbf{a}\cdot\mathbf{b}) - \nabla\times(\mathbf{a}\times\mathbf{b}) + \nabla\cdot(\mathbf{a}\times\mathbf{b})e_{123}\end{aligned}$$

Maxwell's 8 equations

$$\nabla_0(e_0\mathbf{E} - e_{321}\mathbf{B}) = q + e_{0123}\mu$$

where μ is the monopole 4-current, $e_0\mathbf{E} = |\mathbf{E}|(\cosh(\varphi) + e_0\frac{\mathbf{E}}{|\mathbf{E}|}\sinh(\varphi))$ and $e_0^2 = -1$, $e_0e_i = -e_ie_0$