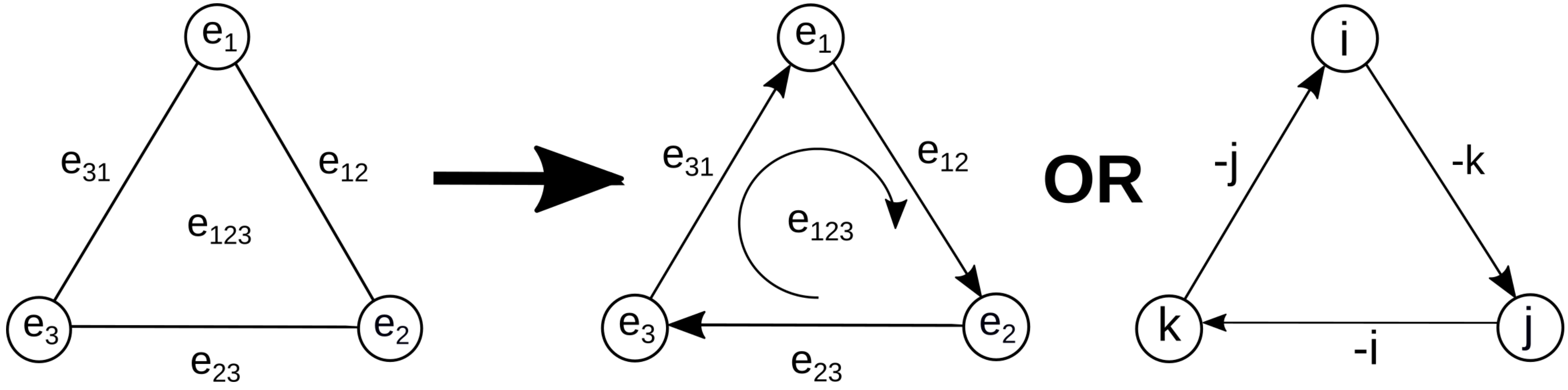
Construction of exceptional Lie algebra G2 and non-associative algebras using geometric algebra

G. P. Wilmot

August 27, 2024

Geometric and Cayley-Dickson algebras



The 2-simplex

Geometric algebra $\mathbb{G} = \mathbb{C} \otimes \mathbb{H}$

3-cycles $\{\mathrm{e}_{12}, \mathrm{e}_{23}, \mathrm{e}_{31}\}, \mathrm{e}_{12}\mathrm{e}_{23}\mathrm{e}_{31} = 1$

$\mathbf{a} \times \mathbf{b} = \mathrm{e}_{321}\mathbf{a} \wedge \mathbf{b}, \;\; \mathrm{e}_{123}^2 = -1$

$\mathbb{Q}^2$	**i**	**j**	**k**
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

$$\mathrm{e}_i\mathrm{e}_j = \delta_{ij} + \mathrm{e}_i \wedge \mathrm{e}_j = \delta_{i,j} + \mathrm{e}_{ij}, \quad i, j \in \mathbb{N}_1^n, \mathbb{N}_1^n = \{1, 2, \ldots, n\}$$

Pseudoscalar "i" $= \pm\mathrm{e}_{123}$ swaps quaternions and vectors but is confusing in higher dimensions

$-i$, $-j$, $-k$ is an antiisomorpism and all arrow combinations are isomorphic to it or i,j,k

$\mathrm{GA}(3)$ construction generalises to $\mathrm{GA}(7)$ with octonions and $\mathrm{GA}(15)$ for sedonions

Calibrations in $\mathrm{GA}(7)$

Pascal's Triangle for 6-simplex $\{1, 7, 21, 35, 35, 21, 7, 1\}$ edges and faces both divisible by 3 and 7

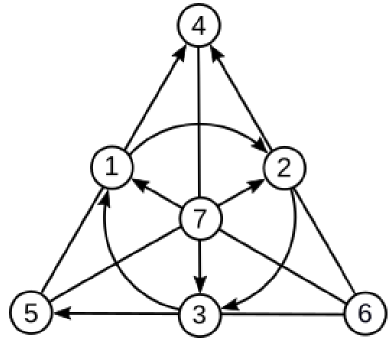
$$\begin{aligned}\Phi_1 &= \mathrm{e}_{123} + \mathrm{e}_{145} + \mathrm{e}_{167} + \mathrm{e}_{246} + \mathrm{e}_{257} + \mathrm{e}_{347} + \mathrm{e}_{356} \\ \Phi_1^* &= \mathrm{e}_{1247} + \mathrm{e}_{1256} + \mathrm{e}_{1346} + \mathrm{e}_{1357} + \mathrm{e}_{2345} + e_{2367} + \mathrm{e}_{4567}\end{aligned}$$

Complement $\Phi^* = -\mathrm{e}_{1234567}\,\Phi$ and natural or primary cross product $\mathbf{a} \times \mathbf{b} = \Phi_1\;\mathbf{a} \wedge \mathbf{b}$

Lemma The terms of $\{1 + \Phi_i^*\}$ and $\{1 + \Phi_i + \Phi_i^* + \mathrm{e}_{1234567}\}$, $i \in \mathbb{N}_1^{30}$, are closed

Call the first "enabling" algebra which is a subalgebra of Spin(7)

There are 30 ways to select 7 primary faces, Φ_i, $i \in \mathbb{N}_1^{30}$
Fano plane or 6-simplex shown for Φ_1 is not a calibration
Change sign of e_{246} to get octonion representation, Φ_O



Label permutations $7! = 30 \times 168$ where Φ_i^* is the $168 = 7 \times 4! = 8 \times 7 \times 3$ orders of labels for each primary, $\mathrm{PSL}(2, \mathbb{Z}_7)$, but 8 sign changes in Φ_i^* are covered by all sign changes $2^7 = 128$, so all label and arrow combinations of the 6-simplex is $(7!/168) \times 2^7 = 3840$

Classification Theorem

Definition Φ_i, for $i \in \mathbb{N}_1^{30}$ are the primaries and $\Phi_{i,j}$, $j \in \mathbb{N}_1^{128}$ are all sign combinations. Note that negation $\Phi_{i,j+64} = -\Phi_{i,j}$, $\sigma_{j+64} = 7 - \sigma_j$, σ_j is the number of minus signs giving parity $\mathbb{O}^+$ and $\mathbb{O}^-$

Octonion exterior form $\Phi_O \wedge \Phi_O^* = 7\mathrm{e}_{1234567} \;\rightarrow\;$ GA(7) form $\Phi_O^2 + 7 = (-1)^{\sigma_j} 6\mathrm{e}_{1234567}\Phi_O$

Theorem

$$\rho_{i,j} = \frac{1}{4}(3\mathrm{e}_{1234567} - (-1)^{\sigma}\Phi_{i,j}),$$

$$2\mathrm{e}_{1234567}(\rho_{i,j}^2 + 1) = \begin{cases} 0, & \text{or} \\ \pm\phi_{i,k} + (-1)^{\sigma}\Phi_{i,j} & \end{cases}$$

If $\Phi_{i,j} = \Phi_O$ then $\rho_{i,j} = \rho_O$ and $\rho_O^2 = -1$, so $\rho_O^{-1} = -\rho_O$

Remainder $\phi_{i,k}$, $k \in \mathbb{N}_1^7$, one of the terms of Φ_i, classifies one of $\mathbb{S}_m$, $m \in \{4, 8, 10, 12, 14, 16\}$

where $\mathbb{S}_m$ has m non-associative triple products. Call these sub-octonion algebras

Classification Theorem classes

$\mathbf{i}$	$\Phi_{\mathbf{i}}$	**Classes**	**Remainder**
1	$e_{123}+e_{145}+e_{167}+e_{246}+e_{257}+e_{347}+e_{356}$	$(\mathbb{S}_4, 2\mathbb{S}_{12}, 4\mathbb{S}_{14})$	$-e_{246}$
2	$e_{123}+e_{145}+e_{167}+e_{247}+e_{256}+e_{346}+e_{357}$	$(\mathbb{S}_4, 2\mathbb{S}_{12}, 4\mathbb{S}_{14})$	$-e_{357}$
3	$e_{123}+e_{146}+e_{157}+e_{245}+e_{267}+e_{347}+e_{356}$	$(\mathbb{S}_4, 2\mathbb{S}_{14}, 2\mathbb{S}_{12}, 2\mathbb{S}_{14})$	$-e_{157}$
4	$e_{123}+e_{146}+e_{157}+e_{247}+e_{256}+e_{345}+e_{367}$	$(\mathbb{S}_4, 4\mathbb{S}_{14}, 2\mathbb{S}_{12})$	$-e_{146}$
5	$e_{123}+e_{147}+e_{156}+e_{245}+e_{267}+e_{346}+e_{357}$	$(\mathbb{S}_4, 2\mathbb{S}_{14}, 2\mathbb{S}_{12}, 2\mathbb{S}_{14})$	$-e_{346}$
6	$e_{123}+e_{147}+e_{156}+e_{246}+e_{257}+e_{345}+e_{367}$	$(\mathbb{S}_4, 4\mathbb{S}_{14}, 2\mathbb{S}_{12})$	$-e_{257}$
7	$e_{124}+e_{135}+e_{167}+e_{236}+e_{257}+e_{347}+e_{456}$	$(\mathbb{S}_8, 2\mathbb{S}_{12}, 3\mathbb{S}_{14}, \mathbb{S}_{10})$	$-e_{135}$
8	$e_{124}+e_{135}+e_{167}+e_{237}+e_{256}+e_{346}+e_{457}$	$(\mathbb{S}_8, 2\mathbb{S}_{12}, 3\mathbb{S}_{14}, \mathbb{S}_{10})$	$-e_{124}$
9	$e_{124}+e_{136}+e_{157}+e_{235}+e_{267}+e_{347}+e_{456}$	$(\mathbb{S}_8, 2\mathbb{S}_{14}, 2\mathbb{S}_{12}, \mathbb{S}_{14}, \mathbb{S}_{10})$	$-e_{267}$
10	$e_{124}+e_{136}+e_{157}+e_{237}+e_{256}+e_{345}+e_{467}$	$(\mathbb{S}_8, 4\mathbb{S}_{14}, \mathbb{S}_{12}, \mathbb{S}_8)$	$-e_{237}$
11	$e_{124}+e_{137}+e_{156}+e_{235}+e_{267}+e_{346}+e_{457}$	$(\mathbb{S}_8, 2\mathbb{S}_{14}, 2\mathbb{S}_{12}, \mathbb{S}_{14}, \mathbb{S}_{10})$	0
12	$e_{124}+e_{137}+e_{156}+e_{236}+e_{257}+e_{345}+e_{467}$	$(\mathbb{S}_8, 4\mathbb{S}_{14}, \mathbb{S}_{12}, \mathbb{S}_8)$	$-e_{156}$
⋮	⋮	⋮	⋮
26	$e_{127}+e_{134}+e_{156}+e_{236}+e_{245}+e_{357}+e_{467}$	$(\mathbb{S}_{12}, \mathbb{S}_{10}, 3\mathbb{S}_{14}, \mathbb{S}_{12}, \mathbb{S}_8)$	$-e_{467}$
27	$e_{127}+e_{135}+e_{146}+e_{234}+e_{256}+e_{367}+e_{457}$	$(2\mathbb{S}_{12}, \mathbb{S}_{16}, \mathbb{S}_{10}, \mathbb{S}_{14}, 2\mathbb{S}_{10})$	$-e_{146}$
28	$e_{127}+e_{135}+e_{146}+e_{236}+e_{245}+e_{347}+e_{567}$	$(\mathbb{S}_8, 4\mathbb{S}_{14}, \mathbb{S}_{12}, \mathbb{S}_4)$	$-e_{135}$
29	$e_{127}+e_{136}+e_{145}+e_{234}+e_{256}+e_{357}+e_{467}$	$(\mathbb{S}_{12}, 2\mathbb{S}_{14}, \mathbb{S}_{10}, \mathbb{S}_{12}, \mathbb{S}_8)$	$-e_{357}$
30	$e_{127}+e_{136}+e_{145}+e_{235}+e_{246}+e_{347}+e_{567}$	$(\mathbb{S}_{12}, 4\mathbb{S}_{14}, \mathbb{S}_{12}, \mathbb{S}_4)$	$-e_{246}$

Classification Theorem proof 1

Φ_1 First 8 Signs with Transformations

j	$\Phi_{1,j}$	Remainder	Permutation	Rotation	Reflection
1	$e_{123} + e_{145} + e_{167} + e_{246} + e_{257} + e_{347} + e_{356}$	$-e_{246}$	$(-1)(45)(67)$	$e_{45}e_{67}$	e_{157}
2	$-e_{123} + e_{145} + e_{167} + e_{246} + e_{257} + e_{347} + e_{356}$	e_{257}	$(23)(45)$	$e_{23}e_{45}$	e_{12467}
3	$e_{123} - e_{145} + e_{167} + e_{246} + e_{257} + e_{347} + e_{356}$	e_{347}	$(45)(67)$	$e_{45}e_{67}$	e_{12346}
4	$e_{123} + e_{145} - e_{167} + e_{246} + e_{257} + e_{347} + e_{356}$	e_{356}	$(1463)(25)$	$e_{14}e_{16}e_{13}e_{25}$	e_{127}
6	$e_{123} + e_{145} + e_{167} + e_{246} - e_{257} + e_{347} + e_{356}$	e_{123}	$(2\ -4)(3\ -5)$	$e_{24}e_{35}$	1
7	$e_{123} + e_{145} + e_{167} + e_{246} + e_{257} - e_{347} + e_{356}$	e_{145}	$(-2)(-3)(46)(57)$	$e_{46}e_{57}$	e_{145}
8	$e_{123} + e_{145} + e_{167} + e_{246} + e_{257} + e_{347} - e_{356}$	e_{167}	$(-12)(4576)$	$e_{12}e_{45}e_{47}e_{46}$	e_{34}

Permutations $(i, j, k) \approx (i \to j, j \to k, k \to i) \approx 90^\circ$ rotations $e_{ij}e_{ik}$ and reflections e_j and e_k

Keeps $\rho^2 + 1$ the same and has been verified for all primaries

Classification Theorem proof 2

Simple Sign Single Basis Reflections

Reflection	$\mathbf{\Phi_{1,1}}$	$\mathbf{\Phi_{1,2}}$	$\mathbf{\Phi_{1,3}}$	$\mathbf{\Phi_{1,4}}$	$\mathbf{\Phi_{1,5}}$	$\mathbf{\Phi_{1,6}}$	$\mathbf{\Phi_{1,7}}$	$\mathbf{\Phi_{1,8}}$
e_1	$\Phi_{1,99}$	$\Phi_{1,114}$	$\Phi_{1,119}$	$\Phi_{1,120}$	$\Phi_{1,64}$	$\Phi_{1,63}$	$\Phi_{1,62}$	$\Phi_{1,61}$
e_2	$\Phi_{1,90}$	$\Phi_{1,105}$	$\Phi_{1,60}$	$\Phi_{1,54}$	$\Phi_{1,117}$	$\Phi_{1,118}$	$\Phi_{1,48}$	$\Phi_{1,47}$
e_3	$\Phi_{1,85}$	$\Phi_{1,100}$	$\Phi_{1,55}$	$\Phi_{1,49}$	$\Phi_{1,46}$	$\Phi_{1,45}$	$\Phi_{1,115}$	$\Phi_{1,116}$
e_4	$\Phi_{1,79}$	$\Phi_{1,59}$	$\Phi_{1,104}$	$\Phi_{1,43}$	$\Phi_{1,111}$	$\Phi_{1,38}$	$\Phi_{1,113}$	$\Phi_{1,36}$
e_5	$\Phi_{1,76}$	$\Phi_{1,56}$	$\Phi_{1,101}$	$\Phi_{1,40}$	$\Phi_{1,37}$	$\Phi_{1,110}$	$\Phi_{1,35}$	$\Phi_{1,112}$
e_6	$\Phi_{1,72}$	$\Phi_{1,52}$	$\Phi_{1,42}$	$\Phi_{1,103}$	$\Phi_{1,106}$	$\Phi_{1,33}$	$\Phi_{1,32}$	$\Phi_{1,109}$
e_7	$\Phi_{1,71}$	$\Phi_{1,51}$	$\Phi_{1,41}$	$\Phi_{1,102}$	$\Phi_{1,34}$	$\Phi_{1,107}$	$\Phi_{1,108}$	$\Phi_{1,31}$

Single basis reflections cover all sign combinations and keep the same remainder and associativity

Column 5 is 8 calibrations plus 8 negated representations giving $30 \times 16 = 480$ octonions representations

All primaries have distribution 5 $\mathbb{O}^+$ and 3 $\mathbb{O}^-$, $j \in \mathbb{N}_1^{64}$, except $\Phi_{11,1}$ (Baez) and $\Phi_{20,1}$ have 7 $\mathbb{O}^+$ and 1 $\mathbb{O}^-$. Negations are opposite

Correspondence with spinors

The reason why $\rho_{1,5}$ is invertible is because $\Phi^*_{1,5} = 1 - 2\sigma$ where $\sigma = \frac{1}{8}(1 - \mathrm{e}_{1247})(1 + \mathrm{e}_{1357})(1 - \mathrm{e}_{4567})$. These are 3 independent idempotents defining the projection operator $\sigma^2 = \sigma$ which means

$$\begin{aligned}(\rho^*_{1,5})^2 &= (\tfrac{1}{4}(3 + \Phi^*_{1,5}))^2 \\ &= (1 - \tfrac{1}{2}\sigma)^2 \\ &= 1\end{aligned}$$

This works for 28 of the 35 combinations of 3 terms from $\Phi^*_{i,O}$ for all 30 octonion representations

It does not work for the sub-octonion algebra representations nor for positions $(1,2,7)$, $(1,3,6)$, $(1,4,5)$, $(2,3,5)$, $(2,4,6)$, $(3,4,7)$ or $(5,6,7)$

E.g. $\Phi^*_{1,5} = 1 - \frac{1}{4}(1 - \mathrm{e}_{1247})(1 - \mathrm{e}_{1256})(1 - \mathrm{e}_{1346})$

Non-associative algebras

Octonions, $\mathbb{O}$, are alternate-associative meaning $(ab)c - a(bc)$ is alternate

Sedenions, $\mathbb{S}$, are the next algebra in the Cayley-Dickson construction and are power associative and have zero divisors

The non-associative triples of $\mathbb{S}_4$ are a subset of $\mathbb{O}$ hence sub-octonion

$$(\mathrm{p}_4, \mathrm{p}_5, \mathrm{p}_6), (\mathrm{p}_4, \mathrm{p}_5, \mathrm{p}_7), (\mathrm{p}_4, \mathrm{p}_6, \mathrm{p}_7), (\mathrm{p}_5, \mathrm{p}_6, \mathrm{p}_7)$$

$\mathbb{S}_4^2$	$\mathbf{p}_1$	$\mathbf{p}_2$	$\mathbf{p}_3$	$\mathbf{p}_4$	$\mathbf{p}_5$	$\mathbf{p}_6$	$\mathbf{p}_7$
$\mathbf{p}_1$	-1	p_3	$-\mathrm{p}_2$	p_5	$-\mathrm{p}_4$	p_7	$-\mathrm{p}_6$
$\mathbf{p}_2$	$-\mathrm{p}_3$	-1	p_1	p_6	$-\mathrm{p}_7$	$-\mathrm{p}_4$	p_5
$\mathbf{p}_3$	p_2	$-\mathrm{p}_1$	-1	p_7	p_6	$-\mathrm{p}_5$	$-\mathrm{p}_4$
$\mathbf{p}_4$	$-\mathrm{p}_5$	$-\mathrm{p}_6$	$-\mathrm{p}_7$	-1	p_1	p_2	p_3
$\mathbf{p}_5$	p_4	p_7	$-\mathrm{p}_6$	$-\mathrm{p}_1$	-1	p_3	$-\mathrm{p}_2$
$\mathbf{p}_6$	$-\mathrm{p}_7$	p_4	p_5	$-\mathrm{p}_2$	$-\mathrm{p}_3$	-1	p_1
$\mathbf{p}_7$	p_6	$-\mathrm{p}_5$	p_4	$-\mathrm{p}_3$	p_2	$-\mathrm{p}_1$	-1

Zero divisors

$$(a+b)(c+d)=0, \text{ for } a,b,c,d \in \mathbb{S}, \text{ all unique}$$

Split octonions, $\tilde{\mathbb{O}}$, have elements $u_i^2 = 1$, $(\frac{1}{2}(1 \pm u_i))^2 = \frac{1}{2}(1 \pm u_i), i \in \mathbb{N}_1^4$

Zero divisors $(1+u_i)(1-u_i) = 0$, $i \in \mathbb{N}_1^4$ and 12 zero divisors $a,b,c,d \neq 1$ or -1

$\mathbb{S}_k$ also have 12 zero divisors if squares are < 0, $k \in \{4, 8, 10, 12, 14, 16\}$

Zero divisors for $\mathbb{S}_4$

$$(\mathrm{p}_7+\mathrm{p}_1)(\mathrm{p}_2+\mathrm{p}_4), (\mathrm{p}_6+\mathrm{p}_1)(\mathrm{p}_2+\mathrm{p}_5), (-\mathrm{p}_5+\mathrm{p}_1)(\mathrm{p}_2+\mathrm{p}_6), (-\mathrm{p}_4+\mathrm{p}_1)(\mathrm{p}_2,\mathrm{p}_7),$$
$$(-\mathrm{p}_6+\mathrm{p}_1)(\mathrm{p}_3+\mathrm{p}_4), (\mathrm{p}_7+\mathrm{p}_1)(\mathrm{p}_3+\mathrm{p}_5), (\mathrm{p}_4+\mathrm{p}_1)(\mathrm{p}_3+\mathrm{p}_6), (-\mathrm{p}_5+\mathrm{p}_1)(\mathrm{p}_3+\mathrm{p}_7),$$
$$(\mathrm{p}_5+\mathrm{p}_2)(\mathrm{p}_3+\mathrm{p}_4), (-\mathrm{p}_4+\mathrm{p}_2)(\mathrm{p}_3+\mathrm{p}_5), (\mathrm{p}_7+\mathrm{p}_2)(\mathrm{p}_3+\mathrm{p}_6), (-\mathrm{p}_6+\mathrm{p}_2)(\mathrm{p}_3+\mathrm{p}_7)$$

Sedenion representation

15 dimensional so e_F and $e_{123456789ABCDEF}$ are in $\mathrm{GA}(15)$ a cross product is

$$\begin{aligned}\Phi = {} & e_{123} + e_{145} + e_{167} + e_{189} + e_{1AB} + e_{1CD} + e_{1EF} + e_{246} + e_{257} \\ & + e_{28A} + e_{29B} + e_{2CE} + e_{2DF} + e_{347} + e_{356} + e_{38B} + e_{39A} + e_{3CF} \\ & + e_{3DE} + e_{48C} + e_{49D} + e_{4AE} + e_{4BF} + e_{58D} + e_{59C} + e_{5AF} + e_{5BE} \\ & + e_{68E} + e_{69F} + e_{6AC} + e_{6BD} + e_{78F} + e_{79E} + e_{7AD} + e_{7BC}\end{aligned}$$

For each sign change count the number of algebra non-associative products

$$210, 210, 210, 210, 210, 210, 210, 210, 224, 200, 224, 200, 224, 200, 208, 208, 208, 208,$$
$$208, 208, 252, 228, 204, 180, 236, 236, 188, 188, 220, 196, 220, 196, 204, 204, 204, 204$$

This represents 12 unique algebras and $-e_{48C}$ gives 252 non-associative triples - isomorphic to sedenions, $\mathbb{S}$

The enabling algebra in Spin(15) is a commuting subalgebra of 15 8-forms with $(\frac{1}{8}P)^2 = 1$

$$\begin{aligned}P = {} & e_{12478BDE} + e_{12479ACF} + e_{12568BCF} + e_{12569ADE} + e_{13468ADF} + e_{13469BCE} \\ & + e_{13578ACE} + e_{13579BDF} + e_{234589EF} + e_{2345ABCD} + e_{236789CD} + e_{2367ABEF} \\ & + e_{456789AB} + e_{4567CDEF} + e_{89ABCDEF}\end{aligned}$$

Spin and Pin groups

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\ldots\mathbf{a}_n = \sum_{i=0}^{[\frac{n}{2}]}\sum_{\mu\in\mathcal{C}^n_{2i}}(-1)^k\,\mathrm{pf}(\mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2},\ldots,\mathbf{a}_{\mu_{2i-1}}\cdot\mathbf{a}_{\mu_{2i}})\mathbf{a}_{\mu_{2i+1}}\wedge\ldots\wedge\mathbf{a}_{\mu_n}$$

where $\mathrm{pf}(A)$ is the Pfaffian of A and $\mathcal{C}^n_{2i} = \binom{n}{2i}$ provides all combinations of $2i$ and $n-2i$ parts with parity k

Multivectors are invertible due to associativity from the Pfaffian co-factor expansion

$$\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_n = (\mathbf{a}_1\ldots\mathbf{a}_r)(\mathbf{a}_{r+1}\ldots\mathbf{a}_n)$$

Spin and Pin groups are even and odd multivectors of unit length vectors but Spin is also pair of rotations for all generalised Euler angles and Pin an extra vector with conjugation $A' = R(\theta)AR(-\theta)$

$$R(\theta) = \prod_{i,j\in\mathcal{C}^n_2} R_{i,j}(\theta_{i,j}) \text{ and } R_{ij}(\phi) = \cos\left(\frac{\phi}{2}\right) + \mathrm{e}_{ij}\sin\left(\frac{\phi}{2}\right)$$

E.g. $R_{ij}R_{kl} = \frac{1}{2}(1+\mathrm{e}_{ij}+\mathrm{e}_{kl}+\mathrm{e}_{ijkl})$ with $i<j<k<l\in\mathbb{N}^7_1$.

Define $R_{ijkl} = \{R_{ij}R_{kl}, R_{ik}R_{jl}, R_{il}R_{jk}\}$ all contain e_{ijkl}

Automorphism Theorem

Theorem

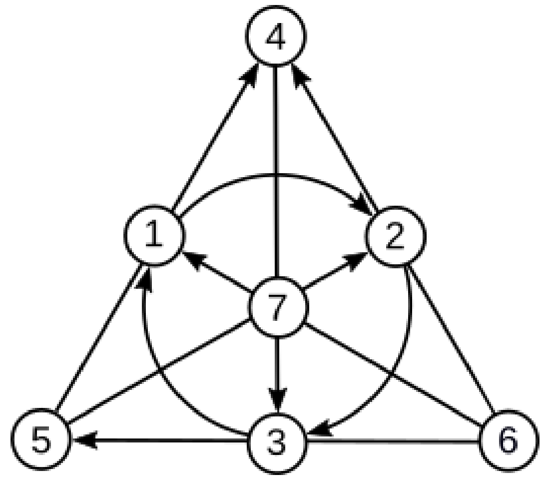
$$R_{jklm}\Phi_{i,O}R_{mlkj} = \Phi'_{i,O}, \quad \text{where } \mathrm{e}_{jklm} \text{ is a term of } \Phi_i^*, \text{ and}$$
$$\Phi'_{i,O} = \Phi_{i,O}, \quad \text{if } \mathrm{e}_{jklm} \text{ is a term of } \Phi_{i,O}^*$$

Definition $R_{ijkl} = \alpha + \beta$, with $\alpha = \frac{1}{2}(1 + \mathrm{e}_{ijkl})$ commutes with Φ_i and β acts on Φ_i

$$\alpha^2 = \alpha, \quad \beta^2 = \alpha - 1, \quad \alpha\beta = \beta\alpha = 0$$
$$\Phi_{i,O}\beta = \beta\Phi'_{i,O} \quad \text{and} \quad \alpha\Phi_{i,O} = \alpha\Phi'_{i,O}$$

E.g. R_{12} and R_{47} are symmetric in the Fano diagram (or they both have the same action)

$$\begin{aligned} R_{jk}R_{lm}\Phi_{i,O}R_{ml}R_{kj} &= (\alpha + \beta)\Phi_{i,O}(\alpha - \beta) \\ &= \alpha\Phi_{i,O} - \beta^2\Phi'_{i,O} \\ &= \alpha\Phi_{i,O} - (\alpha - 1)\Phi'_{i,O} \\ &= \Phi'_{i,O} \end{aligned}$$



Action of enabling algebra in $\mathrm{GA}(7)$ is governed by β

Derivation of G2

Using the Bryant the calibration $\Phi_{1,64}$ this gives the Bryant representation of G2

$\mathbf{\Phi^*_{1,64}}$ term	**Normal Rotations**	**Mixed Rotations**	**Outer Rotations**
e_{1247}	$C = (\mathrm{e}_{12} + \mathrm{e}_{47})/2$	$E = (\mathrm{e}_{14} - \mathrm{e}_{27})/2$	$F = (\mathrm{e}_{17} + \mathrm{e}_{24})/2$
e_{1256}	$C + J = (\mathrm{e}_{12} + \mathrm{e}_{56})/2$	$-D = (\mathrm{e}_{15} - \mathrm{e}_{26})/2$	$-G = (\mathrm{e}_{16} + \mathrm{e}_{25})/2$
e_{1346}	$-B = (\mathrm{e}_{13} + \mathrm{e}_{46})/2$	$E - L = (\mathrm{e}_{14} - \mathrm{e}_{36})/2$	$-G - N = (\mathrm{e}_{16} + \mathrm{e}_{34})/2$
$-\mathrm{e}_{1357}$	$-B - I = (\mathrm{e}_{13} - \mathrm{e}_{57})/2$	$-D - K = (\mathrm{e}_{15} + \mathrm{e}_{37})/2$	$-M = (\mathrm{e}_{17} - \mathrm{e}_{35})/2$
$-\mathrm{e}_{2345}$	$A = (\mathrm{e}_{23} - \mathrm{e}_{45})/2$	$F + M = (\mathrm{e}_{24} + \mathrm{e}_{35})/2$	$N = (\mathrm{e}_{25} - \mathrm{e}_{34})/2$
$-\mathrm{e}_{2367}$	$A + H = (\mathrm{e}_{23} - \mathrm{e}_{67})/2$	$-K = (\mathrm{e}_{26} + \mathrm{e}_{37})/2$	$-L = (\mathrm{e}_{27} - \mathrm{e}_{36})/2$
$-\mathrm{e}_{4567}$	$H = (\mathrm{e}_{45} - \mathrm{e}_{67})/2$	$I = (\mathrm{e}_{46} + \mathrm{e}_{57})/2$	$-J = (\mathrm{e}_{47} - \mathrm{e}_{56})/2$

Procedure to construct G2 from $\mathrm{GA}(7)$ apart from overall sign

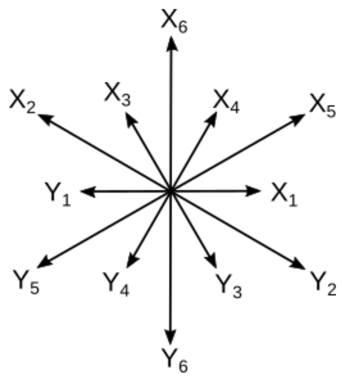
Cartan representation of G2

Cartan G2 is X_i and $Y_i, i \in \mathbb{N}_1^6$ and $H_j = [X_j, Y_j]$, $j \in \mathbb{N}_1^2$

Humphreys matrix representation can be mapped to orthogonal matrices then $\mathrm{GA}(7)$

$$X_1 = g_2^T, X_2 = g_{1,-2}, X_3 = -g_1^T, X_4 = g_3, X_5 = -g_{2,-3}, X_6 = g_{1,-3},$$
$$Y_1 = g_2,\ \ Y_2 = g_{1,-2}^T,\ Y_3 = -g_1,\ \ Y_4 = g_3^T, Y_5 = -g_{2,-3}^T,\ Y_6 = g_{1,-3}^T$$

$$E - L = X_1 - Y_1, \quad D + K = X_2 - Y_2, \quad H = Y_3 - X_3$$



Partial mapping but shows remainders correspond to G2 root diagram symmetries

Summary

The 3-form calibrations from differential geometry satify $\rho_O^2 = -1$ in $\mathrm{GA}(7)$ where

$$\rho_O = \frac{1}{4}(3\mathrm{e}_{1234567} - (-1)^\sigma \Phi_O),$$

Replacing Φ_O with any $\Phi_{i,j}$ generates 6 algebras related to split octonions

The complementary 4-forms extend Pauli and Dirac spinors to 3 projection operators and have terms that generate automorphisms that are invariant for $\Phi_{i,O}$ and partially invariant for $\Phi_{i,j}$

These terms belong to $\mathrm{Spin}(7)$, $R_{ijkl} = \alpha + \beta$, and generate G2 terms from β, with the same action

The concepts used extend to $\mathrm{GA}(15)$ to uncover sedenions and another 90+ power associative algebras

Rotations of 90° and reflections are trivial in geometric algebra and the notation is concise enough to provide all transformations for the 480 representations of octonions and G2

This work was verified with the use of Python calculators at https://github.com/GPWilmot/geoalg