Local Space Structure by Geometric Algebra of the Hurwitz Unit Quaternions

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Abstract
The aims to of this brief letter are to give a short introduction to the study of spatial locality structure using the even part of the Geometric Algebra $\mathcal{G}_3(\mathbb{R})$ which is called 2-spinor quaternions $\mathbb{H} = \mathcal{G}_{0,2}^{\perp}(\mathbb{R}) \sim \mathcal{G}_3^{+}(\mathbb{R})$. This is supplemented by the work of Adolf Hurwitz's (1859-1919) on Number Theory of Quaternions[6] to find a normal invariant subgroup of sixteen unit-½-quaternions by superposition of the orthonormal bivector basis which is commutator relations *interconnected*. This performs a *regular tetrahedron* space structure of four non-orthogonal bivector *directions* in local physical space which enables rotation invariant fluctuations.

Keywords: Bivector, Quaternion, Spinor, Spin-half, Angular-Momentum

Preface
This brief letter starts directly with the quaternion structure from an orthonormal bivector basis. To get an insight in the underlaying mother algebra $\mathcal{G}_3(\mathbb{R})$ generated from a Euclidian 1-vector space for the traditional quantum mechanics interpretation of angular momentum you can consult the preamble letter [1] :

Angular Space Structure of Geometric Algebra

To get an idea of the applications of this present letter consult the extra letter [2]

Post-processing the Geometric Algebra of the Hurwitz Unit Quaternions

[5] The book: Research on the a priori of Physic [3]. Book manuscript under preparation [6].
[6] Über die Zahlentheorie der Quaternionen. 1896 [4], 1919 [5]. (Not translated to English)

3. The Hurwitz Unit Quaternion Subgroup Structure for 2-spinors

3.1. The Unit Quaternion Group and the Linear Geometric Algebra $\mathbb{H} \sim \mathcal{G}_{0,2}(\mathbb{R})$

We define the fundamental units of the multiplicative quaternion group $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$.[7] We recall the multiplication *interconnected* structure of the quaternions from [3], [1]

$$\boxed{\boldsymbol{i}_1^{\,2} = \boldsymbol{i}_2^{\,2} = \boldsymbol{i}_3^{\,2} = -1, \qquad \begin{array}{ccccc} \boldsymbol{i}_1 & = & -\boldsymbol{i}_2\boldsymbol{i}_3 & = & \boldsymbol{i}_3\boldsymbol{i}_2\,, \\ \boldsymbol{i}_2 & = & -\boldsymbol{i}_3\boldsymbol{i}_1 & = & \boldsymbol{i}_1\boldsymbol{i}_3\,, \\ \boldsymbol{i}_3 & = & -\boldsymbol{i}_1\boldsymbol{i}_2 & = & \boldsymbol{i}_2\boldsymbol{i}_1\,, \end{array}}^{8} \quad \begin{array}{l} \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = +1, \\ \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1 = -1, \end{array} \qquad (3.$$

this presumes geometric perpendicular basis units which imply strong *interconnectivity*

$$\underbrace{\boldsymbol{i}_3 \perp \boldsymbol{i}_1 \perp \boldsymbol{i}_2 \perp \boldsymbol{i}_3}_{\text{Geometric perpendicular}} \implies \underbrace{\boldsymbol{i}_1\cdot\boldsymbol{i}_2 = \boldsymbol{i}_2\cdot\boldsymbol{i}_3 = \boldsymbol{i}_3\cdot\boldsymbol{i}_1 = 0}_{\text{Algebraic orthogonal}} \implies \underbrace{\boldsymbol{i}_3 \coloneqq \boldsymbol{i}_2\boldsymbol{i}_1}_{\textit{Perturbated}} \qquad (3.2)$$

The geometric perpendicular unit quaternion multiplication group have eight elements

$$\mathbb{U}_\perp(\mathbb{H})_8 = \left\{ \begin{array}{cccc} 1 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3, & \boldsymbol{i}_1 = \boldsymbol{i}_3\boldsymbol{i}_2, & \boldsymbol{i}_2 = \boldsymbol{i}_1\boldsymbol{i}_3, & \boldsymbol{i}_3 = \boldsymbol{i}_2\boldsymbol{i}_1, \\ -1 = \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1, & -\boldsymbol{i}_1 = \boldsymbol{i}_2\boldsymbol{i}_3, & -\boldsymbol{i}_2 = \boldsymbol{i}_3\boldsymbol{i}_1, & -\boldsymbol{i}_3 = \boldsymbol{i}_1\boldsymbol{i}_2 \end{array} \right\}_8, \qquad (3.3)$$

that stays closed inside, performing an orthonormal *interconnectivity* structure of its three perpendicular plane directions and an independent unit scalar. All these four units $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ have two opposite orientations in a full symmetry. To know the physical magnitude, we define a *measure unit* by division as the multiplicative inverse reciprocal

$$\boxed{\boldsymbol{i}_k^{-1} = \frac{\boldsymbol{i}_k}{\boldsymbol{i}_k^2} = \frac{\boldsymbol{i}_k}{-1} = -\boldsymbol{i}_k = \boldsymbol{i}_k^\dagger \iff \boldsymbol{i}_k^{-1}\boldsymbol{i}_k = \frac{\boldsymbol{i}_k}{\boldsymbol{i}_k} = 1} \implies \boldsymbol{i}_k^\dagger\boldsymbol{i}_k = \boldsymbol{i}_k\boldsymbol{i}_k^\dagger = 1. \qquad (3.4)$$

We use the conjugate notation ($\psi^\dagger$ to ψ) which is bivector inverse. – Performing the *isometric measure* of one plane unit ***direction*** $\boldsymbol{i}_k$ on the others $\boldsymbol{i}_j$ in $\mathbb{U}_\perp(\mathbb{H})_8$: $\boldsymbol{i}_k\boldsymbol{i}_j\boldsymbol{i}_k^\dagger = \boldsymbol{i}_j^\dagger = \boldsymbol{i}_j^{-1} = -\boldsymbol{i}_j$, for $j \neq k$, simply confirms that (3.3) is a *normal invariant* orthogonal unit subgroup which generates ***full linear quaternion algebra*** from the units

$$\mathbb{H} = \mathcal{G}_{0,2}^{\perp}(\mathbb{R}) \longleftarrow span_{\mathbb{R}}\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} \quad \longleftarrow \mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{i}_1, \boldsymbol{i}_2\}, \ \text{(etc. perturbated.)} \qquad (3.5)$$

Every 2-spinor of the quaternion algebra $\mathbb{H} = \mathcal{G}_{0,2}^{\perp}(\mathbb{R}) \sim \mathcal{G}_{3,0}^{+}(\mathbb{R}) \subset \mathcal{G}_{3,0}(\mathbb{R})$ is written

$$S \coloneqq \lambda_0 + \textstyle\sum_{k=1}^{3} \lambda_k\boldsymbol{i}_k \ = \ \lambda_0 1 + \lambda_1\boldsymbol{i}_1 + \lambda_2\boldsymbol{i}_2 + \lambda_3\boldsymbol{i}_3, \qquad \text{for } \forall\lambda_\kappa \in \mathbb{R}. \qquad (3.6)$$

3.2. The Hurwitz Unit Quaternion Subgroup

Adolf Hurwitz (1859-1919) introduced in 1896 [4], and 1919 [5] in his work on the number theory of quaternions a closed subgroup set of 24 unit quaternion elements

$$\mathbb{U}(\mathbb{H}) \coloneqq \left\{ \pm 1, \ \pm\boldsymbol{i}_1, \ \pm\boldsymbol{i}_2, \ \pm\boldsymbol{i}_3, \ \left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16} \right\}_{24}, \qquad (3.7)$$

which in a geometric context perform a *spatial structure* group from its subgroup (3.3)

$$\mathbb{U}_\perp(\mathbb{H})_8 \overset{\rightarrow}{\subset} \mathbb{U}(\mathbb{H})_{24} \overset{\rightarrow}{\subset} \mathbb{H} \ \overset{\leftarrow}{\sim} \mathcal{G}_3^{+}(\mathbb{R}) \overset{\leftarrow}{\subset} \mathcal{G}_{3,0}(\mathbb{R}) \leftarrow \mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}, \qquad (3.8)$$

3.3. The Sixteen Hurwitz unit-½-quaternions (½-versors)

This (3.3) orthonormal subgroup $\mathbb{U}_\perp(\mathbb{H})_8$ with eight 2^3 elements make by linear superposition the new units to the *Hurwitz Unit Quaternion Group* $\mathbb{U}(\mathbb{H}) \subset \mathbb{H}$, (3.7).

$$\varrho_\epsilon \in \mathbb{U}_{½}(\mathbb{H}) = \left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16}. \qquad (3.9)$$

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[7] Hamilton named the *quaternion* basis $\boldsymbol{i} \equiv \boldsymbol{i}_3$, $\boldsymbol{j} \equiv \boldsymbol{i}_2$, $\boldsymbol{k} \equiv \boldsymbol{i}_1$, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$.

[8] Remark opposite orientation noted by Hurwitz [4] §1.(1). – A product $\wedge$ algebra $\mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{i}_j, \boldsymbol{i}_k\}$.

These extra sixteen units: $\mathbb{U}_{½}(\mathbb{H})_{16} = \mathbb{U}(\mathbb{H})_{24}/\mathbb{U}_{\perp}(\mathbb{H})_8$, are not mutually orthogonal. The index $\boldsymbol{\epsilon}$ of the unit element $\varrho_{\boldsymbol{\epsilon}}$ indicate sixteen 2^4 combination of orientation signs

$$\boxed{\epsilon_\mu = \pm 1} \text{ for } \mu = 0,1,2,3\,; \quad \text{giving the index} \quad \boldsymbol{\epsilon} \leftarrow (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \tag{3.10}$$

From this, we write each unit

$$\varrho_{\boldsymbol{\epsilon}} = \tfrac{1}{2}\big(\epsilon_0 1 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3\big) \;=\; \varrho_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)}. \text{ (sixteen different)} \tag{3.11}$$

these sixteen specific 2-rotors (unit 2-spinors) we call *unit-½-quaternions* or ½-versors, all $\varrho_{\boldsymbol{\epsilon}} \in \mathbb{H}$. They have conjugation that just is a reversed 2-rotor (multiplication inverse)

$$\varrho_{\boldsymbol{\epsilon}}^{\dagger} = \varrho_{\boldsymbol{\epsilon}}^{-1} = \tfrac{1}{2}\big(\epsilon_0 1 - \epsilon_1 \boldsymbol{i}_1 - \epsilon_2 \boldsymbol{i}_2 - \epsilon_3 \boldsymbol{i}_3\big)\,, \tag{3.12}$$

The *unit* magnitude of these is confirmed by $|\varrho_{\boldsymbol{\epsilon}}| = 1$, defining a unit we have

$$\varrho_{\boldsymbol{\epsilon}}\varrho_{\boldsymbol{\epsilon}}^{\dagger} = \varrho_{\boldsymbol{\epsilon}}^{\dagger}\varrho_{\boldsymbol{\epsilon}} = \tfrac{1}{4}(\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2) = 1. \tag{3.13}$$

All the 24 unit elements of the *Hurwitz Unit Quaternion Subgroup* are now introduced

$$\mathbb{U}(\mathbb{H}) = \Big\{ \pm 1,\ \pm \boldsymbol{i}_1,\ \pm \boldsymbol{i}_2,\ \pm \boldsymbol{i}_3,\ \big\{\tfrac{1}{2}(\epsilon_0 1 + \epsilon_1 \boldsymbol{i}_1 + \epsilon_2 \boldsymbol{i}_2 + \epsilon_3 \boldsymbol{i}_3)\big\}_{16} \Big\}_{24} \tag{3.14}$$

To confirm that this is a *closed multiplication group* we make test products:
A unit basis bivector $\boldsymbol{i}_k$ operating on one of its ½-versors give another ½-versor

$$\varrho_{\boldsymbol{\epsilon}'} = \boldsymbol{i}_3 \varrho_{\boldsymbol{\epsilon}} = \tfrac{1}{2}(+\epsilon_0 \boldsymbol{i}_3 + \epsilon_2 \boldsymbol{i}_1 - \epsilon_1 \boldsymbol{i}_2 - \epsilon_3), \qquad \text{etc. for } \boldsymbol{i}_2,\ \boldsymbol{i}_3 \tag{3.15}$$

Having two ½-versors (unit-½-quaternions)

$$\begin{aligned} \varrho_a &= \tfrac{1}{2}(\epsilon_{a0} + \epsilon_{a1}\boldsymbol{i}_1 + \epsilon_{a2}\boldsymbol{i}_2 + \epsilon_{a3}\boldsymbol{i}_3), \quad \text{and} \\ \varrho_b &= \tfrac{1}{2}(\epsilon_{b0} + \epsilon_{b1}\boldsymbol{i}_1 + \epsilon_{b2}\boldsymbol{i}_2 + \epsilon_{b3}\boldsymbol{i}_3), \end{aligned} \tag{3.16}$$

the simple product of these two gives a unit member of $\mathbb{U}(\mathbb{H})$

$$\begin{aligned} \varrho_a\varrho_b = &+¼(\epsilon_{a0}\epsilon_{b0} - \epsilon_{a1}\epsilon_{b1} - \epsilon_{a2}\epsilon_{b2} - \epsilon_{a3}\epsilon_{b3}) \\ &+¼(\epsilon_{a1}\epsilon_{b0} + \epsilon_{a0}\epsilon_{b1} + \epsilon_{a3}\epsilon_{b2} - \epsilon_{a2}\epsilon_{b3})\,\boldsymbol{i}_1 \\ &+¼(\epsilon_{a2}\epsilon_{b0} - \epsilon_{a3}\epsilon_{b1} + \epsilon_{a0}\epsilon_{b2} + \epsilon_{a1}\epsilon_{b3})\,\boldsymbol{i}_2 \\ &+¼(\epsilon_{a3}\epsilon_{b0} + \epsilon_{a2}\epsilon_{b1} - \epsilon_{a1}\epsilon_{b2} + \epsilon_{a0}\epsilon_{b3})\,\boldsymbol{i}_3 \;= \\ = &\begin{cases} \pm\boldsymbol{i}_k, \text{ if } \epsilon_{ak} = \mp\epsilon_{bk} \text{ for one } k, \text{ and } \epsilon_{aj} = \pm\epsilon_{bj} \text{ for } j \neq k, & \text{: bivector,} \\ \pm 1\,, \text{ if } \epsilon_{a0} = \pm\epsilon_{b0} \text{ and } \epsilon_{ak} = \mp\epsilon_{bk}\,, & \text{: scalar,} \\ \text{else, } \tfrac{1}{2}(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3)\,, \text{ in all other cases} & \text{: ½-versor.} \end{cases} \end{aligned} \tag{3.17}$$

The simple multiplication by -1 gives $\varrho_{-\boldsymbol{\epsilon}} = -\varrho_{\boldsymbol{\epsilon}}$, again a ½-versor.
Squaring a ½-versor (3.11) turns the scalar part negative, but still a ½-versor

$$\varrho_{\boldsymbol{\epsilon}}^2 = \epsilon_0\varrho_{\boldsymbol{\epsilon}} - 1 = -\tfrac{1}{2} + \epsilon_0\tfrac{1}{2}\big(+\epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3\big), \tag{3.18}$$

a positive scalar $\epsilon_0 = 1$ prevents the reversing of the bivector part, then $\varrho^2 = \varrho - 1$, as in [5], simply $\Big(\tfrac{1}{2}(1 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3)\Big)^2 = \tfrac{1}{2}\big(-1 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3\big)$.

3.4. Stability of the Sixteen Hurwitz unit-½-quaternions

From the quaternion basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$, we have sixteen ½-versors (3.9)-(3.11). To simplify the possible perturbation structure between these sixteen elements we introduce extra as in [4] the reduced quaternion 1-spinors operators of the simplest form

$$\xi_k = \alpha(1 \pm \boldsymbol{i}_k) \quad \in \mathbb{H}, \text{ but } \notin \mathbb{U}(\mathbb{H}), \quad \text{and } \alpha \in \mathbb{R}, \quad k = 1,2,3 \tag{3.19}$$

in three orthogonal ⊥-plane-*directions* of the quaternion basis, with the reciprocal

$$\xi_k^{-1} = \alpha^{-1}(1 \pm \boldsymbol{i}_k)^{-1} = \alpha^{-1}\tfrac{1}{2}(1 \mp \boldsymbol{i}_k), \quad \text{where} \quad \xi_k \xi_k^{-1} = 1, \tag{3.20}$$

and freedom to choose $\alpha = \pm\sqrt{½}$ and we get unit 1-rotor operators of the form

$$U_{\boldsymbol{i}_k} = \sqrt{½}(1 + \boldsymbol{i}_k), \quad \text{and} \quad U_{\boldsymbol{i}_k}^\dagger = \sqrt{½}(1 - \boldsymbol{i}_k), \text{ then } U_{\boldsymbol{i}_k} U_{\boldsymbol{i}_k}^\dagger = U_{\boldsymbol{i}_k}^\dagger U_{\boldsymbol{i}_k} = 1. \tag{3.21}$$

The trick is to take the product of two different 1-rotors to get a 2-rotor ½-versor, e.g.,

$$\begin{aligned} \varrho_{\boldsymbol{1}} &= \sqrt{½}(1+\boldsymbol{i}_2)\sqrt{½}(1+\boldsymbol{i}_1) = (1+\boldsymbol{i}_2)\tfrac{1}{2}(1+\boldsymbol{i}_1) = \tfrac{1}{2}(1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3) \\ \text{or } = \varrho_{\boldsymbol{1}} &= (1+\boldsymbol{i}_3)\tfrac{1}{2}(1+\boldsymbol{i}_2) = (1+\boldsymbol{i}_1)\tfrac{1}{2}(1+\boldsymbol{i}_3) = \tfrac{1}{2}(1 + \boldsymbol{i}_1 + \boldsymbol{i}_2 + \boldsymbol{i}_3) \end{aligned} \tag{3.22}$$

We now make a rotation test example; first by sandwich operating with $U_{\boldsymbol{i}_1}$ on (3.11)

$$\begin{aligned} U_{\boldsymbol{i}_1}\varrho_{\boldsymbol{\epsilon}}U_{\boldsymbol{i}_1}^\dagger = (1 + \boldsymbol{i}_1)\left[\tfrac{1}{2}(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3)\right]\tfrac{1}{2}(1 - \boldsymbol{i}_1) = \\ \tfrac{1}{2}(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_3\boldsymbol{i}_2 - \epsilon_2\boldsymbol{i}_3)\,, \end{aligned} \tag{3.23}$$

and further operating with $U_{\boldsymbol{i}_2}$ to get a perturbation of signs $\epsilon_k = \pm 1$ in this way

$$\begin{aligned} f(\varrho_{\boldsymbol{\epsilon}}) = \varrho_{\boldsymbol{1}}\varrho_{\boldsymbol{\epsilon}}\varrho_{\boldsymbol{1}}^\dagger \; = \; U_{\boldsymbol{i}_2}U_{\boldsymbol{i}_1}\varrho_{\boldsymbol{\epsilon}}U_{\boldsymbol{i}_1}^\dagger U_{\boldsymbol{i}_2}^\dagger = \\ (1 + \boldsymbol{i}_2)(1 + \boldsymbol{i}_1)\left[\tfrac{1}{2}(\epsilon_0 + \epsilon_1\boldsymbol{i}_1 + \epsilon_2\boldsymbol{i}_2 + \epsilon_3\boldsymbol{i}_3)\right]\tfrac{1}{2}(1 - \boldsymbol{i}_1)\tfrac{1}{2}(1 - \boldsymbol{i}_2) \\ = \tfrac{1}{2}(\epsilon_0 + \epsilon_2\boldsymbol{i}_1 + \epsilon_3\boldsymbol{i}_2 + \epsilon_1\boldsymbol{i}_3). \end{aligned} \tag{3.24}$$

The scalar sign $\epsilon_0 = \pm 1$ is not altered by the permutation rotation. A dextral 2-rotor ½-versor permutating operator $\varrho_{\boldsymbol{1}}$ (3.22) building on the dextral basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ does not change the chirality by the permutation $\varrho_{\boldsymbol{\epsilon}} \longrightarrow f(\varrho_{\boldsymbol{\epsilon}})$, just the same as permutate changing the reference basis sequence of names.

$$\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\} \xrightarrow{f} \{\boldsymbol{i}_3, \boldsymbol{i}_1, \boldsymbol{i}_2\} \xrightarrow{f} \{\boldsymbol{i}_2, \boldsymbol{i}_3, \boldsymbol{i}_1\} \tag{3.25}$$

3.5. The Normal Invariant Unit ½-versor Part-group of Quaternions

In general, a sandwich operating on a *Unit-½-quaternion* (½-versor) ϱ_b (3.16) with left acting part ϱ_a and its right operating reversed part $\varrho_a^\dagger$ always gets one of the sixteen *Unit-½-quaternions*, normal invariant ½-versor units inside this subgroup $\mathbb{U}_{½}(\mathbb{H}) \subset \mathbb{H}$

$$\varrho_c \;=\; \varrho_a\varrho_b\varrho_a^\dagger \;=\; \varrho_a\varrho_b\varrho_a^{-1} \;=\; \tfrac{1}{2}(\epsilon_{c0} + \epsilon_{c1}\boldsymbol{i}_1 + \epsilon_{c2}\boldsymbol{i}_2 + \epsilon_{c3}\boldsymbol{i}_3), \tag{3.26}$$

were the sign signature components $\epsilon_{c\mu} = \pm 1$ fulfil

$$\begin{aligned} \epsilon_{c0} &= \epsilon_{b0}\,, \\ \epsilon_{c1} &= ½(\epsilon_{a1}\epsilon_{a2} + \epsilon_{a0}\epsilon_{a3})\epsilon_{b2} + ½(\epsilon_{a1}\epsilon_{a3} - \epsilon_{a0}\epsilon_{a2})\epsilon_{b3}\,, \\ \epsilon_{c2} &= ½(\epsilon_{a2}\epsilon_{a3} + \epsilon_{a0}\epsilon_{a1})\epsilon_{b3} + ½(\epsilon_{a2}\epsilon_{a1} - \epsilon_{a0}\epsilon_{a3})\epsilon_{b1}\,, \\ \epsilon_{c3} &= ½(\epsilon_{a3}\epsilon_{a1} + \epsilon_{a0}\epsilon_{a2})\epsilon_{b1} + ½(\epsilon_{a3}\epsilon_{a2} - \epsilon_{a0}\epsilon_{a1})\epsilon_{b2}\,. \end{aligned} \tag{3.27}$$

Then the sixteen ½-versor units (3.9) are mutually *interconnected* in a stable structure

$$\varrho_{\boldsymbol{\epsilon}} \in \mathbb{U}_{½}(\mathbb{H}) = \left\{\tfrac{1}{2}(\pm 1 \pm \boldsymbol{i}_1 \pm \boldsymbol{i}_2 \pm \boldsymbol{i}_3)\right\}_{16},$$

by their mutual interaction of its own members as unit rotations operators. All these sixteen possibilities of unit 2-rotor ½-versors perform a normal invariant subgroup, which comparts in a tetrahedron structure of ***directions***. This we designate it *the normal regular tetrahedron subgroup of quaternions* $\mathbb{U}_{½}(\mathbb{H}) \subset \mathbb{U}(\mathbb{H})_{24} \subset \mathbb{H}$. (the half-versors)

3.6. The Sixteen *Unit ½-versor Quaternion* Basis with Tetrahedron Directions

The sixteen distinct ½-versors of $\mathbb{U}_{½}(\mathbb{H})$ (3.9) we written out as

$$\begin{aligned}
\varrho_0 &= \tfrac{1}{2}(+1-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3), & \varrho_0^\dagger &= \tfrac{1}{2}(+1+\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3),\\
\varrho_1 &= \tfrac{1}{2}(+1-\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3), & \varrho_1^\dagger &= \tfrac{1}{2}(+1+\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3),\\
\varrho_2 &= \tfrac{1}{2}(+1+\boldsymbol{i}_1-\boldsymbol{i}_2+\boldsymbol{i}_3), & \varrho_2^\dagger &= \tfrac{1}{2}(+1-\boldsymbol{i}_1+\boldsymbol{i}_2-\boldsymbol{i}_3),\\
\varrho_3 &= \tfrac{1}{2}(+1+\boldsymbol{i}_1+\boldsymbol{i}_2-\boldsymbol{i}_3), & \varrho_3^\dagger &= \tfrac{1}{2}(+1-\boldsymbol{i}_1-\boldsymbol{i}_2+\boldsymbol{i}_3),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
\varrho_0^2 = -\varrho_0^\dagger &= \tfrac{1}{2}(-1-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3), & -\varrho_0 &= \tfrac{1}{2}(-1+\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3) = (\varrho_0^\dagger)^2,\\
\varrho_1^2 = -\varrho_1^\dagger &= \tfrac{1}{2}(-1-\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3), & -\varrho_1 &= \tfrac{1}{2}(-1+\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3) = (\varrho_1^\dagger)^2,\\
\varrho_2^2 = -\varrho_2^\dagger &= \tfrac{1}{2}(-1+\boldsymbol{i}_1-\boldsymbol{i}_2+\boldsymbol{i}_3), & -\varrho_2 &= \tfrac{1}{2}(-1-\boldsymbol{i}_1+\boldsymbol{i}_2-\boldsymbol{i}_3) = (\varrho_2^\dagger)^2,\\
\varrho_3^2 = -\varrho_3^\dagger &= \tfrac{1}{2}(-1+\boldsymbol{i}_1+\boldsymbol{i}_2-\boldsymbol{i}_3), & -\varrho_3 &= \tfrac{1}{2}(-1-\boldsymbol{i}_1-\boldsymbol{i}_2+\boldsymbol{i}_3) = (\varrho_3^\dagger)^2.
\end{aligned} \tag{3.29}$$

These unit ½-versors consist of a scalar ±½, plus a spinning[9] plane area bivector $\pm\mathbf{A}_\kappa$

$$\varrho_\kappa = +\tfrac{1}{2}+\mathbf{A}_\kappa, \quad \text{and} \quad \varrho_\kappa^\dagger = +\tfrac{1}{2}+\mathbf{A}_\kappa^\dagger, \quad \text{with} \quad \varrho_\kappa + \varrho_\kappa^\dagger = 1 \tag{3.30}$$

$\kappa = 0,1,2,3$. Recall (3.13) $\not{\Sigma}_{\kappa}$: $\varrho_\kappa\varrho_\kappa^\dagger = \frac{1}{4}(\epsilon_0^2+\epsilon_1^2+\epsilon_1^2+\epsilon_1^2)_\kappa = \frac{1}{4}+\frac{3}{4} = 1$, and spin½ state $j=\frac{1}{2}$, gives $\lambda = j(j+1) = \frac{3}{4}$. The plane area bivectors in (3.30) are achieved by

$$\mathbf{A}_\kappa = \tfrac{1}{2}(\varrho_\kappa - \varrho_\kappa^\dagger), \quad \text{and} \quad \mathbf{A}_\kappa^\dagger = \tfrac{1}{2}(\varrho_\kappa^\dagger - \varrho_\kappa), \qquad \mathbf{A}_\kappa\mathbf{A}_\kappa^\dagger = \tfrac{3}{4}, \ \not{\Sigma}_{\kappa}. \tag{3.31}$$

These eight bivectors in four pairs of reversed orientations $\mathbf{A}_\kappa^\dagger = -\mathbf{A}_\kappa$ are just written

$$\begin{aligned}
\mathbf{A}_0 &= \tfrac{1}{2}(-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3), & \mathbf{A}_0^\dagger &= \tfrac{1}{2}(+\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3),\\
\mathbf{A}_1 &= \tfrac{1}{2}(-\boldsymbol{i}_1+\boldsymbol{i}_2+\boldsymbol{i}_3), & \mathbf{A}_1^\dagger &= \tfrac{1}{2}(+\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3),\\
\mathbf{A}_2 &= \tfrac{1}{2}(+\boldsymbol{i}_1-\boldsymbol{i}_2+\boldsymbol{i}_3), & \mathbf{A}_2^\dagger &= \tfrac{1}{2}(-\boldsymbol{i}_1+\boldsymbol{i}_2-\boldsymbol{i}_3),\\
\mathbf{A}_3 &= \tfrac{1}{2}(+\boldsymbol{i}_1+\boldsymbol{i}_2-\boldsymbol{i}_3), & \mathbf{A}_3^\dagger &= \tfrac{1}{2}(-\boldsymbol{i}_1-\boldsymbol{i}_2+\boldsymbol{i}_3),
\end{aligned} \tag{3.32}$$

representing four plane ***direction*** of face areas of a regular tetrahedron internal in a ³space locality sphere **Fig. 1**

In the geometric mother algebra $\mathcal{G}_{3,0}(\mathbb{R})$ with the pseudoscalar $\boldsymbol{i} := \boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_2\wedge\boldsymbol{\sigma}_3$ for the quaternions $\mathbb{H} = \mathcal{G}_{0,2}^{\perp}(\mathbb{R}) \sim \mathcal{G}_{3,0}^{+}(\mathbb{R}) \subset \mathcal{G}_{3,0}(\mathbb{R}) \leftarrow \mathcal{G}en_{\mathbb{R}}^{\wedge}\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ from which the bivectors coms, gives: $\boldsymbol{i} \Longrightarrow$

$$\boldsymbol{i}_1 = \boldsymbol{i}\boldsymbol{\sigma}_1, \quad \boldsymbol{i}_2 = \boldsymbol{i}\boldsymbol{\sigma}_2, \quad \boldsymbol{i}_3 = \boldsymbol{i}\boldsymbol{\sigma}_3, \tag{3.33}$$

and $\mathbf{A}_\kappa = \boldsymbol{i}\mathbf{k}_\kappa$, where the ***direction*** 1-vectors dual to (3.31) are four and its parity inversions, in all eight

$$\begin{aligned}
\mathbf{k}_\kappa &= \boldsymbol{i}\tfrac{1}{2}(\varrho_\kappa^\dagger - \varrho_\kappa), \qquad \mathbf{k}_\kappa^2 = \tfrac{3}{4}, \ \text{and}\\
\overline{\mathbf{k}_\kappa} &= -\mathbf{k}_\kappa = \boldsymbol{i}\tfrac{1}{2}(\varrho_\kappa - \varrho_\kappa^\dagger)
\end{aligned} \tag{3.34}$$

These 1-vectors are ***directed*** outward from a locality center perpendicular normal to the faces (3.32) of a regular tetrahedron. The set $\mathbf{k}_\kappa$ we call a *tetraon*

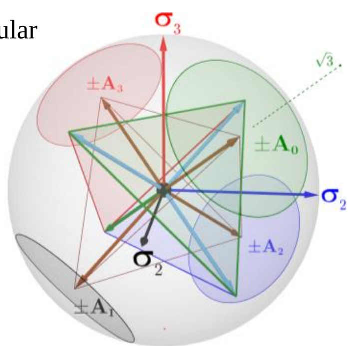


Fig. 1 The tetrahedron structure. Four bivector ***directions*** of spinning plane angular momenta, illustrated circular rotating areas $|\mathbf{A}_\kappa| = \sqrt{¾}$. Eight *invariant* possibilities $\pm\mathbf{A}_\kappa$.

[9] Remember that a bivector is rotating *invariant* in its own plane, so its angular spin is free.

$$\left\{\begin{array}{lcl} \mathbf{k}_0 & = & \frac{1}{2}(-\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3), \\ \mathbf{k}_1 & = & \frac{1}{2}(-\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3), \\ \mathbf{k}_2 & = & \frac{1}{2}(+\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3), \\ \mathbf{k}_3 & = & \frac{1}{2}(+\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_3) \end{array}\right\}_4, \qquad (3.35)$$

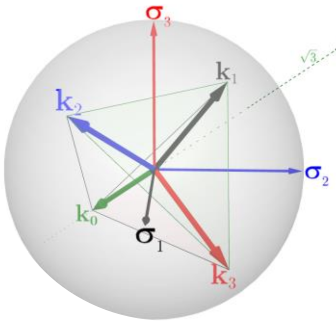


Fig. 2 A *tetraon* basis set $\{\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\}$, where the dextral structure $\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3$ is preserved from Cartesian basis $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$, armed from $\mathbf{k}_0$. Note $|\mathbf{k}_\kappa| = \frac{1}{2}\sqrt{3}$.

are displayed outwards in **Fig. 2** and their parity inverted is then inwards $\overline{\mathbf{k}_\kappa} = -\mathbf{k}_\kappa$. These ***directions*** are dual to the plane area invariant spin rotating bivectors $\pm\mathbf{A}_\kappa = \pm\boldsymbol{i}\mathbf{k}_\kappa$, displayed **Fig. 1**

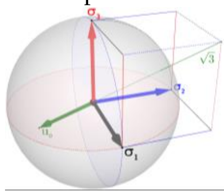
The disadvantage with this basis set $\{\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\}$ is the magnitudes $\sqrt{3/4}$, for each element $\mathbf{k}_\kappa^2 = \frac{3}{4}$. The advantage is the balance $\mathbf{k}_0+\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 = 0$ of a regular *tetraon* symmetry which entails

$$\mathbf{k}_0 = -(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3), \qquad (3.36)$$

that is the antagonist 1-vector ***direction*** to the dextral structure displayed in **Fig. 2**

It is urgent to note that a four permutation {0,1,2,3} → {1,2,3,0} change the chirality, while {0, 1,2,3} → {0, 2,3,1} preserve **Fig. 2** as dextral. Conversely from a presumed set $\{\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\} \longrightarrow \{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$, we find the orthonormal set

$$\boldsymbol{\sigma}_1 = \mathbf{k}_2+\mathbf{k}_3, \quad \boldsymbol{\sigma}_2 = \mathbf{k}_3+\mathbf{k}_1, \quad \boldsymbol{\sigma}_3 = \mathbf{k}_1+\mathbf{k}_2, \qquad (3.37)$$



From this orthonormal set, we need a fourth 1-vector to support a local unit sphere $\mathbf{u}_0 = -\frac{1}{\sqrt{3}}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)$. This locality requirement is that four points support a sphere (surface);
three points support a circle (circumference) like a three-1-vectors Mercedes star ☮;
two points support a line segment as one 1-vector ⟼ ***direction***; and
one single point supports nothing nor any locality *direction* structure of physics.
This set $\{\mathbf{u}_0,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ is not symmetric in its support of locality sphere in ^{3}space.
In the orthogonal case the spin½ amplitude in the perpendicular ***directions*** is ½$\boldsymbol{i}_\kappa$, giving the invariant spin oscillation ***directions*** in the tetrahedron stats (3.31) $|\mathbf{A}_\kappa| = \sqrt{3/4}$.

To make a normalised the unit sphere symmetric support *tetraon* structure basis is $\{\mathbf{u}_0,\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ defined by $\mathbf{u}_\kappa = \frac{2}{\sqrt{3}}\mathbf{k}_\kappa$, $|\mathbf{u}_\kappa| = 1$, fulfilling the algebraic rules:

$$\mathbf{u}_0^2 = \mathbf{u}_1^2 = \mathbf{u}_2^2 = \mathbf{u}_3^2 = 1, \quad \text{and} \quad \mathbf{u}_0+\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3 = 0 \qquad (3.38)$$

Note $\mathbf{u}_\kappa\mathbf{u}_\kappa = 4$, and further the linear dependency of one of the three others, e.g., $\mathbf{u}_0 = -(\mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3)$. This regular tetrahedron symmetry obeys six 1-rotor relations with a scalar for covariant projection

$$\begin{array}{lcl} \mathbf{u}_1\mathbf{u}_0 = \mathbf{u}_1\wedge\mathbf{u}_0 - \frac{1}{3}, & & \mathbf{u}_3\mathbf{u}_0 = \mathbf{u}_3\wedge\mathbf{u}_0 - \frac{1}{3}, \\ \mathbf{u}_2\mathbf{u}_1 = \mathbf{u}_2\wedge\mathbf{u}_1 - \frac{1}{3}, & & \mathbf{u}_3\mathbf{u}_1 = \mathbf{u}_3\wedge\mathbf{u}_1 - \frac{1}{3}, \\ \mathbf{u}_0\mathbf{u}_2 = \mathbf{u}_0\wedge\mathbf{u}_2 - \frac{1}{3}, & & \mathbf{u}_3\mathbf{u}_2 = \mathbf{u}_3\wedge\mathbf{u}_2 - \frac{1}{3}, \end{array} \qquad (3.39)$$

where $\mathbf{u}_\kappa\cdot\mathbf{u}_\kappa = 1$, $\mathbf{u}_\kappa\cdot\mathbf{u}_\nu = -\frac{1}{3}$, for $\kappa \neq \nu$, and $\cos\beta = -\frac{1}{3} \Rightarrow \beta \sim 109.47^\circ$.

In our artefact humanistic mathematical approach to physics, we have the freedom to model in myriad ways. The known example is the Cartesian $\{\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ and now the transformation $\{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\} \longrightarrow \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to a non-orthogonal unit basis of a regular tetraon inside a symmetric unit sphere. Taking spin½ bivectors in these ***directions*** $\mathbf{S}_\kappa = +\frac{1}{2}\boldsymbol{i}\mathbf{u}_\kappa = \frac{1}{\sqrt{3}}\mathbf{A}_\kappa$, forming four 1-spinor cyclic oscillators

$$\psi^{½}_{\kappa\pm} \sim \rho_\kappa U_{\phi_\kappa} = \rho_\kappa e^{\pm ½\boldsymbol{i}\mathbf{u}_\kappa\phi_\kappa} = \rho_\kappa(\cos ½\phi_\kappa \pm \boldsymbol{i}\mathbf{u}_\kappa \sin ½\phi_\kappa) \tag{3.40}$$

in the relative invariant $\mathbf{A}_\kappa$ plane ***directions*** in a locality sphere displayed in **Fig. 3**

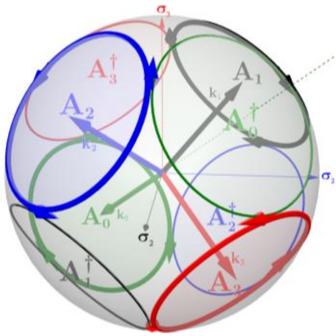


Fig. 3 Display idea of spin½ Fermion cyclic 1-spinor oscillations of the tetrahedron ***direction*** symmetry in a local sphere. The four circle angular momenta bivectors $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ outwards dextral orientated, and their reversed outwards sinistral $\mathbf{A}_\kappa^\dagger = -\mathbf{A}_\kappa$ for the fluctuating oscillations that on the spherical surface does not exceed the *retarding* speed of information. (Fit of display $\mathbf{A}_\kappa \sim 1.16\cdot$ (3.32) Besides the eight circular oscillating bivector areas, there are six shapes ✧ in the three perpendicular *directions* of the orthogonal basis idea, they are not cyclic closed, hence no angular momentum.

The bivector directions $\mathbf{S}_\kappa = \sqrt{⅓}\mathbf{A}_\kappa$ is the same as the spatial *directions* of the ½-versors ϱ_κ (3.28)-(3.30) that is mutual interconnected by measure operation (3.26).

3.7. The Hurwitz Integer Quaternions

Adolf Hurwitz introduced integer quaternions [4],§3,eq.(3) and [5],Vor.4,eq.(14),

$$g = \frac{1}{2}(g_0 + g_1\boldsymbol{i}_1 + g_2\boldsymbol{i}_2 + g_3\boldsymbol{i}_3), \tag{3.41}$$

where all the $g_\mu \in \mathbb{Z}$ are exclusive *odd* or *even* integers for different excitation combinations of physical localised quaternions, which should be studied more.

We have above studied the indivisible case with $g_\mu = \epsilon_\mu = \pm 1$, for $\mu = 0{,}1{,}2{,}3$; giving the geometric tetrahedron structure of a physical space locality.

Alternative choosing some $g_\nu = 0$, force the others even, $g_\mu \geq 2$ or 0, for $\mu \neq \nu$. The simple case is counting in the ***directions*** of the quaternion basis $\{1, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$, e.g., one direction $\boldsymbol{i}_3$ gives the 1-rotor oscillation $e^{\boldsymbol{i}_3\phi}$ transversal wavefront propagating external with the light speed of information. The next often used case is $g_0 = 2$ and one $g_k = \pm 2$ and two $g_j = 0$ for $j \neq k$, $j, k = 1{,}2{,}3$, as $(1 \pm \boldsymbol{i}_k)$ with the modulo 2 norm

$$(1 + \boldsymbol{i}_k)(1 - \boldsymbol{i}_k) = 2 \quad \leftrightarrow \quad (1 + \boldsymbol{i}_k)(1 + \boldsymbol{i}_k)^{-1} = 1, \tag{3.42}$$

that also preserve as (3.23) the *interconnectivity* tetrahedron structure of (3.28).

3.8. Concluding Idea of Unit Quaternions in Physics

This brief introduction the Unit Quaternion Group based on the over hundred years old work by Adolf Hurwitz on integer quaternions inspired an algebraic view of the geometric tetrahedron structure of local Space in physics, where the *non-directional* scalar parts of quaternions mix their impact into normal invariant stable symmetry structure of *interconnected* bivector *directions* of plane rotation invariant angular retarded activity inside a spherical *locality structure* considered as one indivisible.

– This letter is continued in a post-processing letter [2].

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[1] J. E. Andresen, *Angular Space Structure of Geometric Algebra,* Køge, Denmark: Independent Scientist, 2024.

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