Bridge between hyperbolic and circular symmetry illuminates spacetime spinors

Peter J. Brands
Lumina Innovation, Zoutelande, the Netherlands

Summary of the Abstract

In this paper it will be shown that there is a bridge between hyperbolic and circular symmetry. This bridge, based on a hyperbolic rotation with Euclidean rotation parameter, reveals a hidden spacetime property - connection between future and past mass-shell - that is obscured when using a hyperbolic rotation parameter. The hyperbolic and circular symmetry are connected by a single Euclidean rotation parameter. A full circular rotation in the circular symmetry is one to one connected to a full hyperbolic rotation in the hyperbolic symmetry, connecting the future and past mass-shell (future and past part of the hyperbola). So, the bridge includes passing infinity with a single Euclidean rotation parameter. The spacetime spinor derived from this bridge is a solution of the Dirac equation.

1. Introduction

The six independent generators of the Lorentz group [1-4] divide in two parts: (a) three generators related to temporal (hyperbolic) rotations in the three temporal planes (xt, yt and zt), and (b) three generators related to spatial (circular) rotations in the three spatial planes (xy, yz and zx). Using three from the six generators is sufficient to cover all possible spacetime rotations. These three generators can be chosen as: one temporal generator (zt plane) with a hyperbolic rotation angle φ, and two spatial generators (zx, xy planes) with two Euclidean rotation angles $\{\theta, \phi\}$. So, using a mixed set of generators (temporal and spatial) with a mixed set of rotation parameters (hyperbolic and Euclidean) $\{\varphi, \theta, \phi\}$. In section 2 it will be shown that the division in hyperbolic and Euclidean rotation parameters can be broken by the introduction of a temporal (hyperbolic) rotor with Euclidean rotation parameter β, utilizing an all-Euclidean set of rotation parameters $\{\beta, \theta, \phi\}$ [5].

A physical connection to the Euclidean set of rotation parameters can be made by mapping Euclidean angle β to relative speed v/c, in the same way has done with hyperbolic angle φ (known as rapidity). As will be shown in section 3 the mapping of $\tanh(\varphi) = \sin(\beta) = v/c$ reveals a bridge between hyperbolic and circular symmetry $\{\sec^2(\beta) - \tan^2(\beta) = \cos^2(\beta) + \sin^2(\beta) = 1\}$ that is using a single Euclidean rotation parameter $\beta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

The all-Euclidean set of rotation parameters $\{\beta, \theta, \phi\}$ form the polar coordinates of a causal three-sphere $\mathbb{S}_C^3$ (the light cone of a past event) with the squared spacetime proper distance $s^2 = c^2t^2 - x^2 - y^2 - z^2$ as symmetrical condition. In section 4 it will be shown that all possible rotations in causal three-sphere $\mathbb{S}_C^3$ can be obtained by a spacetime spinor with an all-Euclidean set of rotation parameters $\{\beta, \theta, \phi\}$. This spacetime spinor is a solution of the Dirac equation and is composed of three irreducible rotors represented by the orthogonal spacetime bivectors $\{\sigma_3, \mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$ (one temporal zt plane orthogonal to two orthogonal spatial zx, xy planes).

The mathematical formalism used in this paper is based on the geometric algebra (*GA*) of spacetime (*STA*) as developed by David Hestenes [6-8]. Foundations of geometric algebra where jointly developed by Grassmann [9] and Clifford [10] in the late 19th century. There are many positive arguments for using *GA*, especially in physics [11-22]. However, the most decisive argument is the generalization of rotation which can be applied in any dimension, and which can act on any multi-vector by means of the so-called rotors [6, 7, 11-20, 23, 24]. Rotors are directly related to spinors and automatically integrate Lie algebra [25-27] by the *GA* bivectors.

In spacetime algebra (*STA*) [6] a spacetime inertial frame (reference frame) $\{t, x, y, z\}$ is represented by a set of four orthogonal basis vectors $\gamma_\mu = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$. The temporal basis vector γ_0 squares to one, while the spatial basis vectors γ_k square to minus one, i.e., a Minkowski space with a $\mathbb{R}^{1,3}$ metric. The *STA* orthogonal basis vectors γ_μ satisfy the algebra of the Dirac gamma matrices.

2. Hyperbolic rotation

For clarity in this section on hyperbolic rotation, we will focus only on the tz-plane as spanned by the temporal and spatial basis vectors $\{\gamma_0, \gamma_3\}$. A generalization follows from section 3 on. A hyperbolic unit-vector $w(\varphi)$ in the tz-plane as function of a hyperbolic angel φ is given by:

$$w(\varphi) = \cosh(\varphi)\gamma_0 + \sinh(\varphi)\gamma_3 \qquad \varphi \in [-\infty, \infty] \qquad w^2 = \cosh^2(\varphi) - \sinh^2(\varphi) = 1 \tag{2.1}$$

where $w(\varphi)$ describes only the future side of an implicit hyperbolic symmetry: $\cosh^2(\varphi) - \sinh^2(\varphi) = 1$. The past side of the hyperbola is missing (Fig. 2.1a). However, a hyperbolic unit vector can also be described as function of a Euclidean angle β:

$$p(\beta) = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3 \qquad \beta \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \qquad p^2 = \sec^2(\beta) - \tan^2(\beta) = 1 \tag{2.2}$$

where $p(\beta)$ covers the full hyperbolic symmetry - both the future and the past side - of the hyperbolic symmetry: $\sec^2(\beta) - \tan^2(\beta) = 1$. These two hyperbolic unit-vectors $\{w(\varphi), p(\beta)\}$ are equal $\pm w(\varphi) = p(\beta)$ if the two different angle types $\{\varphi, \beta\}$ have the following implicit relationship:

$$\tanh(\varphi) = \sin(\beta) \qquad \varphi \in [-\infty, \infty \rightarrow \infty, -\infty] \leftrightarrow \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \rightarrow \frac{\pi}{2}, \frac{3\pi}{2}\right] \tag{2.3}$$

The hyperbolic angle interval $\varphi \in [-\infty, \infty \rightarrow \infty, -\infty]$ is bound by infinities, while angle $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ has an interval that is periodic and that provides a full hyperbolic symmetry (Fig. 2.1b). By substitution of the Gudermannian function [28] $\varphi = tanh^{-1}(sin(\beta))$ (2.3) in $w(\varphi)$, the two-hyperbola will become equal:

$$\left.\begin{aligned} +w\left(\tanh^{-1}(\sin(\beta))\right) &= p(\beta) = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3 \quad \beta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \\ -w\left(\tanh^{-1}(sin(\beta))\right) &= p(\beta) = \sec(\beta)\gamma_0 + \tan(\beta)\gamma_3 \quad \beta \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \end{aligned}\right\} \beta \in [-\tfrac{\pi}{2}, \tfrac{3\pi}{2}] \tag{2.4}$$

Therefore, $p(\beta)$ is under the implicit relationship $\tanh(\varphi) = \sin(\beta)$ (2.3) a hyperbolic unit-vector $\pm w(\varphi)$ as function of Euclidean angle $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ (Fig. 2.1b). The next step involves formulating a mixed type of rotor that combines hyperbolic rotation with a Euclidean rotation angle β.

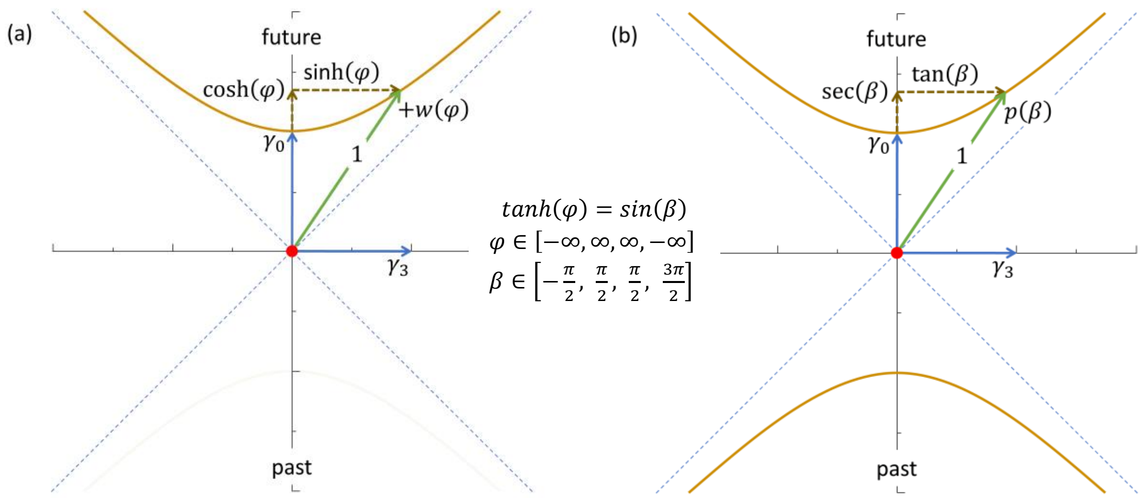


Fig. 2.1: (a) The hyperbolic unit-vector $+w(\varphi); \varphi \in [-\infty, \infty]$ covers only half of the hyperbolic symmetry: $\cosh^2(\varphi) - \sinh^2(\varphi) = 1$. (b) The hyperbolic unit-vector $p(\beta)$; $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ covers the full hyperbolic symmetry: $\sec^2(\beta) - \tan^2(\beta) = 1$. In (a) the future and past hyperbola are disconnected (missing part of the symmetry), whereas in (b) there is a full hyperbolic symmetry $\sec^2(\beta) - \tan^2(\beta) = 1 \mapsto \beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ with a connection between the future and past hyperbola.

A *STA* rotor $R = \rho S_R \in (\langle M\rangle_0 + \langle M\rangle_2 + \langle M\rangle_4)$ is unitary $R\tilde{R} = 1$ and consists of a scalar density ρ times a spinor S_R [11]. Where spinor S_R - part of the STA even subalgebra - provides a rotation that includes scaling. The scalar density ρ is determined by taking the inverse magnitude of the spinor part, ensuring the unitarity of the rotor R:

$$R = \rho S_R \quad R\tilde{R} = 1 \;\mapsto\; \rho = \left(S_R \tilde{S}_R\right)^{-1/2} \qquad \text{Unitarity of } R\tilde{R} = 1 \text{ defines the scalar density } \rho \tag{2.5}$$

An irreducible rotor $R \in (\langle M\rangle_0 + \langle M\rangle_2)$ is unitary $R\tilde{R} = 1$ and consists of a scalar $\langle M\rangle_0$ plus a bivector $\langle M\rangle_2$. Where $\langle M\rangle_2$ can be either a temporal σ_j or spatial $\mathbb{i}\sigma_j$ bivector. The calculation of a irreducible rotor can be realized by taking the square root of the geometric product (*GP*) of two unit-vectors that span a bivector plane [29]. Hence, an irreducible hyperbolic rotor for the $\{\gamma_3\gamma_0 = \sigma_3 \to zt\text{-plane}\}$ can be calculated from the square root of the *GP* of hyperbolic unit vector $w(\varphi)$ (2.1) with temporal basis vector γ_0:

$$\begin{aligned} R(\varphi) &= \sqrt{w\gamma_0} = \sqrt{\cosh(\varphi) + \sinh(\varphi)\sigma_3} = \sqrt{e^{\sigma_3\,\varphi}} = e^{\sigma_3\,\varphi/2} \quad \varphi \in [-\infty, \infty] \quad (\sigma_3)^2 = 1 \\ R(\varphi) &= e^{\sigma_3\,\varphi/2} = \cosh(\varphi/2) + \sinh(\varphi/2)\sigma_3 \quad M' = RM\tilde{R} \quad R\tilde{R} = 1 \quad \tilde{R} = e^{-\sigma_3\,\varphi/2} \end{aligned} \tag{2.6}$$

where $R(\varphi)$ is an irreducible hyperbolic rotor with temporal bivector (generator) σ_3. The type of a rotor is defined by the bivector. A hyperbolic rotor has a positive squared bivector $\left(\sigma_j\right)^2 = +1$ whereas a Euclidean rotor has a negative squared bivector $\left(\mathbb{i}\sigma_j\right)^2 = -1$. Substitution of the Gudermannian function [28] $\varphi = \tanh^{-1}\left(\sin(\beta)\right)$ (2.3) in the irreducible hyperbolic rotor $R(\varphi)$ (2.6) yields a mixed type of rotor:

$$\begin{aligned} &R\left(\tanh^{-1}\left(\sin(\beta)\right)\right) = L_z(\beta) = \rho L_{u1}(\beta) \qquad L_z\tilde{L}_z = 1 \;\mapsto\; \rho = \left(L_{u1}\tilde{L}_{u1}\right)^{-1/2} = \sqrt{\sec(\beta)} \\ &L_z(\beta) = \sqrt{\sec(\beta)}\left(\cos(\beta/2) + \sin(\beta/2)\sigma_3\right) \qquad L_{u1}(\beta) = \cos(\beta/2) + \sin(\beta/2)\sigma_3 \end{aligned} \tag{2.7}$$

where $L_z(\beta) = \rho L_{u1}(\beta)$ is an irreducible hyperbolic rotor ($(\sigma_3)^2 = +1$) with a Euclidean rotation parameter β and $L_{u1}(\beta)$ is a temporal spinor with typical spinor characteristics: picking up a minus sign at $L_{u1}(\beta + 2\pi) = -L_{u1}(\beta)$ and remaining unchanged at $L_{u1}(\beta + 4\pi) = L_{u1}(\beta)$. Note that temporal spinor $L_{u1}(\beta)$ with a squared bivector $\left(\sigma_j\right)^2 = +1$ is not a Euler relationship $L_{u1}(\beta) \neq \exp(\sigma_3\,\beta/2)$.

3. Spacetime symmetries

A connection to spacetime symmetries can be made by mapping relative speed v/c to Euclidean rotation angle β, similar to how relative speed v/c is mapped to hyperbolic angle φ (known as rapidity):

$$\begin{aligned} &\tanh(\varphi) = \sin(\beta) && \mapsto\; \tanh(\varphi) = \sin(\beta) = \pm\, v/c && \text{Mapping to relative speed} \\ &\tfrac{v}{c} \in [-1, +1] && \leftrightarrow\; \varphi \in [-\infty, \infty] && \text{One to one and part of symmetry} \\ &\tfrac{v}{c} \in [-1, +1 \to +1, -1] && \leftrightarrow\; \beta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2} \to \tfrac{\pi}{2}, \tfrac{3\pi}{2}\right] && \text{Periodic and full symmetry} \end{aligned} \tag{3.1}$$

Hyperbolic angle φ is non-periodic and covers only a part of the hyperbolic symmetry, whereas Euclidean angle β is periodic and covers the full hyperbolic symmetry (Fig. 2.1b). By mapping $\tanh(\varphi) = \sin(\beta) = \pm v/c$, four trigonometric relativistic proportionality factors emerge, revealing a bridge between hyperbolic and circular symmetry:

$$\begin{aligned} &\sin(\beta) = \pm\, v/c && \cos(\beta) = \pm\sqrt{1 - (v/c)^2} \;\mapsto\; \sin^2(\beta) + \cos^2(\beta) = 1 \;\; \mathit{Circular} \\ &\sec(\beta) = \frac{1}{\pm\sqrt{1 - (v/c)^2}} && \tan(\beta) = \frac{\pm\, v/c}{\pm\sqrt{1 - (v/c)^2}} \;\mapsto\; \sec^2(\beta) - \tan^2(\beta) = 1 \;\; \mathit{Hyperbolic} \end{aligned} \tag{3.2}$$

So, this mapping $\tanh(\varphi) = \sin(\beta) = \pm v/c$ creates a bridge between hyperbolic and circular symmetry $\{\sec^2(\beta) - \tan^2(\beta) = \cos^2(\beta) + \sin^2(\beta) = 1\}$ via a single Euclidean rotation parameter $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$. This bridge reveals a connection between the future and past mass-shell that is obscured when using a hyperbolic angle. So, a rotation specified by rotation parameter β in the circular symmetry $\cos^2(\beta) + \sin^2(\beta) = 1$ has a one to one connection to a rotation in the hyperbolic symmetry $\sec^2(\beta) - \tan^2(\beta) = 1$. Where, the circular symmetry is related to the causality relationship $s^2 = c^2t^2 - \vec{x}^2$, while the hyperbolic symmetry is related to the energy-momentum relationship $m_0^2c^4 = E^2 - \vec{p}^2c^2$ (mass-shell). So, the future and past mass-shell (future and past hyperbola) are connected in terms of their full hyperbolic symmetry, revealing a hidden spacetime property.

4. Spacetime rotor and spinor

Each position q in *3D* space can be represented by three cartesian coordinates $\{x, y, z\}$ or by two polar coordinates $\{\theta, \phi\}$ connected to a symmetrical *3D* condition $r^2 = x^2 + y^2 + z^2$, an invariant squared spatial distance. For spacetime each event q can be represented by four cartesian coordinates $\{ct, x, y, z\}$. However, by using the Euclidean Lorentz group rotation parameters $\{\beta, \theta, \phi\}$ (section 2), each event q can also be represented by three polar coordinates $\{\beta, \theta, \phi\}$ (Fig. 4.1) and the condition of an invariant squared spacetime proper distance ($s^2 = c^2t^2 - x^2 - y^2 - z^2$).

Spacetime rotor R_1:

$R_1(\beta, \theta, \phi) = \eta_1 U_1(\beta, \theta, \phi)$

$R_1\tilde{R}_1 = 1$

Spacetime spinor U_1:

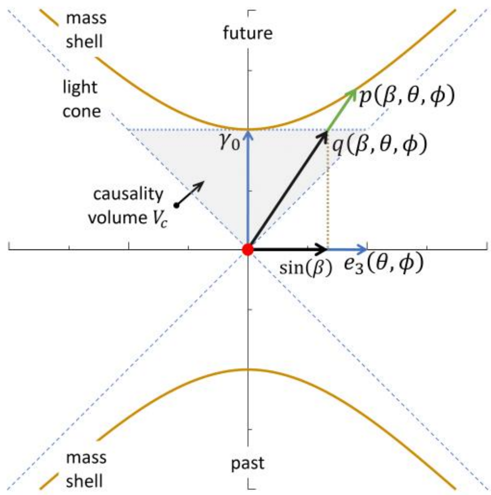
$U_1(\beta, \theta, \phi) = S_1(\theta, \phi) L_{u1}(\beta)$

$U_1\tilde{U}_1 = \cos(\beta) = \pm\sqrt{1 - (v/c)^2}$

Scalar density η_1:

$\eta_1 = \left(U_1\tilde{U}_1\right)^{-1/2} = \sqrt{\sec(\beta)}$

$R_1 = \sqrt{\sec(\beta)}\, U_1$



Momentum vector p:

$p = R_1\gamma_0\tilde{R}_1$ on mass-shell

$p = \sec(\beta)\gamma_0 + \tan(\beta)\, e_3(\theta, \phi)$

$p^2 = \sec^2(\beta) - \tan^2(\beta) = 1$

Event vector q:

$q = U_1\gamma_0\tilde{U}_1$ on area of causality V_c

$q = \gamma_0 + \sin(\beta)\, e_3(\theta, \phi)$

$q^2 = s^2 = \cos^2(\beta) = 1 - (v/c)^2$

Fig. 4.1: All possible spacetime event vectors $q(\beta, \theta, \phi)$ pointing to the surface of causality volume V_c together with all possible spacetime momentum vectors $p(\beta, \theta, \phi)$ pointing to the future and past mass-shell in a *2D* representation. This *4D* object in $\mathbb{R}^{1,3}$ can be depicted in a *2D* plane because all spatial unit vector $e_3(\theta, \phi)$ (spanning two-sphere $\mathbb{S}^{0,2}$) are orthogonal to temporal basis vector γ_0.

The Lorentz group consist of six independent generators, represented by the six spacetime bivectors $\{\sigma_k, \mathbb{i}\sigma_k\}$. To perform all possible spacetime rotations three of the six bivectors are needed. They can be chosen as: (a) temporal bivector σ_3 (zt plane) and (b) two spatial bivectors $\{\mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$ (zx, xy plane) [18, 24]. From this set of three orthogonal planes represented by the spacetime bivectors $\{\sigma_3, \mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$, three unitary irreducible rotors $\{L_z(\beta), \mathcal{S}_\theta(\theta), \mathcal{S}_\phi(\phi)\}$ can be calculated [29] (2.6) (2.7):

$$L_z(\beta) = \sqrt{\sec(\beta)}\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)\sigma_3\right) \quad \text{Temporal rotor} \quad : \sigma_3 \mapsto zt \text{ plane} \quad (\sigma_3)^2 = +1$$

$$\mathcal{S}_\theta(\theta) = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2 \quad \text{Spatial rotor} \quad : \mathbb{i}\sigma_2 \mapsto zx \text{ plane} \quad (\mathbb{i}\sigma_2)^2 = -1 \qquad (4.1)$$

$$\mathcal{S}_\phi(\phi) = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3 \quad \text{Spatial rotor} \quad : \mathbb{i}\sigma_3 \mapsto xy \text{ plane} \quad (\mathbb{i}\sigma_3)^2 = -1$$

where $L_z(\beta) = \sqrt{\sec(\beta)} L_{u1}(\beta)$ is an unitary irreducible temporal rotor (boost in the z-direction) with scalar density $\sqrt{\sec(\beta)}$ times temporal spinor $L_{u1}(\beta)$ (2.7) and $\{\mathcal{S}_\theta(\theta), \mathcal{S}_\phi(\phi)\}$ are two unitary irreducible spatial rotors.

The *GP* of the two irreducible spatial rotors $\{\mathcal{S}_\phi, \mathcal{S}_\theta\}$ connected to the orthogonal spatial bivectors $\{\mathbb{i}\sigma_3, \mathbb{i}\sigma_2\}$, covers all possible spatial rotations in a *3D* two-sphere $\mathbb{S}^{0,2}$:

$$S_1(\theta,\phi) = \mathcal{S}_\phi(\phi)\mathcal{S}_\theta(\theta) = \lambda_1 W_1 = \left(cos\left(\frac{\phi}{2}\right) - sin\left(\frac{\phi}{2}\right)\mathbb{i}\sigma_3\right)\left(cos\left(\frac{\theta}{2}\right) - sin\left(\frac{\theta}{2}\right)\mathbb{i}\sigma_2\right)$$
$$\lambda_1 = \left(W_1\widetilde{W}_1\right)^{-1/2} = 1 \qquad \mathbb{S}^{0,2} = \left\{e_k = S_1\gamma_k\tilde{S}_1 \in R^{0,3}: e_k^2 = -1\right\} \qquad e_k.e_j = diag(-1,-1,-1) \tag{4.2}$$

Spatial rotor/spinor $S_1(\theta,\phi)$ is equal to a complex Pauli spinor [11, 16, 30, 31].

The *GP* of spatial spinor $S_1(\theta,\phi)$ with irreducible temporal rotor $L_z(\beta)$ (4.1), yields a spacetime rotor $R_1(\beta,\theta,\phi) = S_1(\theta,\phi)L_z(\beta)$:

$$R_1 = \sqrt{\sec(\beta)}S_1(\theta,\phi)\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)\sigma_3\right) \qquad R_1\tilde{R}_1 = +1 \tag{4.3}$$

Spacetime rotor $R_1(\beta,\theta,\phi)$ is characterized by a hyperbolic symmetry and covers all possible momentum vector $p = R_1\gamma_0\tilde{R}_1$ rotations. These rotations are represented by the light cone of a future and past event and are bound by the future and past mass-shell ($p^2 = +1$) (Fig. 4.1), i.e., a hyperbolic three-sphere $\mathbb{S}_H^3$. Spacetime rotor $R_1 = \eta_1 U_1$ consist of a scalar density $\eta_1 = \sqrt{sec(\beta)}$ times a spacetime spinor $U_1 = S_1 L_{u1}$, which is the *GP* of a spatial spinor S_1 times temporal spinor L_{u1} (2.7):

$$U_1(\beta,\theta,\phi) = S_1(\theta,\phi)\left(\cos\left(\frac{\beta}{2}\right) + \sin\left(\frac{\beta}{2}\right)\sigma_3\right) \qquad U_1\widetilde{U}_1 = \cos(\beta) = \pm\sqrt{1-(v/c)^2} \tag{4.4}$$

Spacetime spinor $U_1(\beta,\theta,\phi)$ is characterized by a circular symmetry and covers all possible event vector rotations $q = U_1\gamma_0\widetilde{U}_1$. These rotations are represented by the light cone of a past event and are bound by the proper length of event vector q ($q^2 = \cos^2(\beta)$) (Fig. 4.1), i.e., a causal three-sphere $\mathbb{S}_C^3$ covering a causality volume. Although $U_1 = S_1 L_{u1}$ is *STA* even ($\langle M\rangle_0 + \langle M\rangle_2 + \langle M\rangle_4$), the intensity $U_1\widetilde{U}_1 = \cos(\beta) = \pm\sqrt{1-(v/c)^2}$ is a scalar value. The pseudoscalar part $\langle M\rangle_4$ is zero because the spatial spinor is wrapped inside the temporal spinor $U_1 = \cos\left(\frac{\beta}{2}\right)S_1 + \sin\left(\frac{\beta}{2}\right)S_1\sigma_3$. So, the demand for unitarity of $R_1\tilde{R}_1$ gives a scalar density factor $\eta_1 = \sqrt{\sec(\beta)}$.

Spacetime spinor $U_1 = S_1 L_{u1}$ is a solution of the Dirac equation. Hence, all rotations $q = U_1\gamma_0\widetilde{U}_1$ spanning causal three-sphere $\mathbb{S}_C^3$ yield the same result as solving the Dirac equation with complex quantum mechanical eigenvalue eigenvector matrix equations. Causal three-sphere $\mathbb{S}_C^3$ is a circular symmetry related geometrical object that is bound by the light cone of a past event and the proper length of event vectors q, i.e., the causality volume of the light cone of a past event (Fig. 4.1). Whereas spacetime rotor $R_1 \mapsto p = R_1\gamma_0\tilde{R}_1$ is spanning a hyperbolic three-sphere $\mathbb{S}_H^3$, which is a hyperbolic symmetry related geometrical object that is bound by the light cone of a future and past event and the future and past mass-shell (Fig. 4.1).

5. Discussion

Introducing a irreducible hyperbolic rotor $L_z(\beta)$ (2.7) (4.1) with Euclidean rotation parameter β eliminates the division between hyperbolic and Euclidean rotation parameters in the Lorentz group, and this allows the utilization of an all Euclidean set of rotation parameters $\{\beta,\theta,\phi\}$. Hyperbolic rotor $L_z(\beta) = \rho L_{u1}(\beta)$ (4.1) (boost in the z-direction) combines a scalar density ρ and a temporal spinor $L_{u1}(\beta)$ (2.7).

A connection to spacetime symmetries can be made by mapping relative speed v/c to Euclidean rotation angle β. This mapping $\tanh(\varphi) = \sin(\beta) = \pm v/c$ (3.1) reveals a bridge between hyperbolic and circular symmetry $\{\sec^2(\beta) - \tan^2(\beta) = \cos^2(\beta) + \sin^2(\beta) = 1\}$ with a single Euclidean rotation parameter $\beta \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$. Hence, a full 2π Euclidean rotation in the circular symmetry $\cos^2(\beta) + \sin^2(\beta) = 1$ has a direct connection to a full hyperbolic rotation in the hyperbolic symmetry $\sec^2(\beta) - \tan^2(\beta) = 1$. The hyperbolic $\sec^2(\beta) - \tan^2(\beta) = 1$ and circular symmetry $\cos^2(\beta) + \sin^2(\beta) = 1$ cannot exist in the same bivector plane. So, to perform all possible spacetime rotations the dimensionality of the bridge must increase to $\mathbb{R}^{1,3}$ (*4D* Minkowski space). Causal three-sphere $\mathbb{S}_C^3$ in $\mathbb{R}^{1,3}$ and hyperbolic three-sphere $\mathbb{S}_H^3$ in $\mathbb{R}^{1,3}$ are depicted in a *2D* plane because all spatial unit vectors $e_3(\theta,\phi) \in \mathbb{S}^{0,2} \mapsto (e_3)^2 = -1$ are orthogonal to temporal basis vector γ_0 (Fig. 4.1).

To perform all possible Lorentz group spacetime rotations, a selection of three bivectors is necessary. These can be chosen as: (a) temporal bivector σ_3 (zt plane) and (b) two spatial bivectors $\{\mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$ (zx, xy plane) [18, 24]. Using these three orthogonal planes represented by the spacetime bivectors $\{\sigma_3, \mathbb{i}\sigma_2, \mathbb{i}\sigma_3\}$, three unitary irreducible rotors $\{L_z(\beta), \mathcal{S}_\theta(\theta), \mathcal{S}_\phi(\phi)\}$ can be calculated [29] (4.1). This set of irreducible rotors enables the composition of a spatial rotor $S_1(\theta,\phi)$ (4.2), a spacetime rotor $R_1(\beta,\theta,\phi)$ (4.3) and a spacetime spinor $U_1(\beta,\theta,\phi)$ (4.4).

Spatial rotor $S_1(\theta,\phi)$ (4.2) allows to obtain all possible spatial rotations in two-sphere $\mathbb{S}^{0,2}$ and is equal to a complex Pauli spinor [11, 16, 30, 31]. Spacetime rotor $R_1(\beta,\theta,\phi)$ (4.3) allows to obtain all possible momentum vector $p = R_1\gamma_0\tilde{R}_1$ rotations, which are characterized by a hyperbolic symmetry and point to the future and past mass-shell (Fig. 4.1). Whereas spacetime spinor $U_1(\beta,\theta,\phi)$ (4.4) allows to obtain all possible causal event vector $q = U_1\gamma_0\tilde{U}_1$ rotations, which are characterized by circular symmetry and point at the surface of causality volume V_c (Fig. 4.1). The shape of causality-volume V_c is a causal three-sphere $\mathbb{S}_C^3$ (Fig. 4.1). Spacetime spinor $U_1(\beta,\theta,\phi)$ (4.4) - related to all possible causal rotations in three-sphere $\mathbb{S}_C^3$ - is a solution of the Dirac equation.

References:

1. Gray, J., *Henri Poincaré*, in *Henri Poincaré*. 2012, Princeton University Press.
2. Schwichtenberg, J., *Physics from symmetry*. 2018: Springer.
3. Bonolis, L., *From the rise of the group concept to the stormy onset of group theory in the new quantum mechanics: A saga of the invariant characterization of physical objects, events and theories.* La Rivista del Nuovo Cimento, 2004. **27**(4-5): p. 1-110.
4. Thomson, M., *Modern particle physics*. 2013: Cambridge University Press.
5. Brands, P.J., *Hyperbolic Rotation with Euclidean Angle Illuminates Spacetime Spinors.* J Math Tech-niques Comput Math, 2023. **2**(11): p. 456-478.
6. Hestenes, D., *Space-time algebra*. 2015: Springer.
7. Hestenes, D., *Spacetime physics with geometric algebra.* American Journal of Physics, 2003. **71**(7): p. 691-714.
8. Hestenes, D., *The genesis of geometric algebra: A personal retrospective.* Advances in Applied Clifford Algebras, 2017. **27**: p. 351-379.
9. Grassmann, H., *A New Branch of Mathematics: The "Ausdehnungslehre" of 1844 and Other Works*. 1995: Open Court.
10. Chisholm, M., *Such silver currents: the story of William and Lucy Clifford, 1845-1929*. 2021: The Lutterworth Press. 1-100.
11. Doran, C., et al., *Spacetime algebra and electron physics.* Advances in imaging and electron physics, 1996. **95**: p. 271-386.
12. Dressel, J., K.Y. Bliokh, and F. Nori, *Spacetime algebra as a powerful tool for electromagnetism.* Physics Reports, 2015. **589**: p. 1-71.
13. Gull, S., A. Lasenby, and C. Doran, *Imaginary numbers are not real—The geometric algebra of spacetime.* Foundations of Physics, 1993. **23**(9): p. 1175-1201.
14. Hestenes, D., *Vectors, spinors, and complex numbers in classical and quantum physics.* American Journal of Physics, 1971. **39**(9): p. 1013-1027.
15. Hestenes, D., *Clifford algebra and the interpretation of quantum mechanics.* Clifford Algebras and their Applications in Mathematical Physics, 1986: p. 321-346.
16. Lasenby, A.N., *Geometric algebra as a unifying language for physics and engineering and its use in the study of gravity.* Advances in Applied Clifford Algebras, 2017. **27**(1): p. 733-759.
17. De Sabbata, V. and B.K. Datta, *Geometric algebra and applications to physics*. 2006: CRC Press.
18. Doran, C., A. Lasenby, and J. Lasenby, *Geometric algebra for physicists*. 2003: Cambridge University Press.
19. Josipović, M., *Geometric Multiplication of Vectors An Introduction to Geometric Algebra in Physics*. Springer.
20. Macdonald, A., *Linear and geometric algebra*. 2010: Alan Macdonald.
21. Macdonald, A., *A survey of geometric algebra and geometric calculus.* Advances in Applied Clifford Algebras, 2017. **27**: p. 853-891.
22. Hestenes, D. and G. Sobczyk, *Clifford algebra to geometric calculus: a unified language for mathematics and physics*. Vol. 5. 2012: Springer Science & Business Media.
23. Hestenes, D., *Real Dirac theory.* Advances in Applied Clifford Algebras, 1997. **7**: p. 97-144.
24. Doran, C. and A. Lasenby, *Physical applications of geometric algebra.* Cambridge University Lecture Course., 1999.
25. Stubhaug, A., *The Mathematician Sophus Lie: It was the audacity of my thinking*. 2002: Springer.
26. Hall, B.C. and B.C. Hall, *Lie groups, Lie algebras, and representations*. 2013: Springer.
27. Georgi, H., *Lie algebras in particle physics: from isospin to unified theories*. 2000: Taylor & Francis.
28. Ernst, T., *On Elliptic and Hyperbolic Modular Functions and the Corresponding Gudermann Peeta Functions.* axioms, 2015. **4**(3): p. 235-253.
29. Roelfs, M. and S. De Keninck, *Graded symmetry groups: plane and simple.* Advances in Applied Clifford Algebras, 2023. **33**(3): p. 30.
30. Doran, C., A. Lasenby, and S. Gull, *States and operators in the spacetime algebra.* Foundations of physics, 1993. **23**(9): p. 1239-1264.
31. Steane, A.M., *An introduction to spinors.* arXiv preprint arXiv:1312.3824, 2013.