Transformation in four phase power system using CGA.

J. Brdečková[a]

[a]Institute of Mathematics
Faculty of Mechanical Engineering, Brno University of Technology
Technická 2896/2, 616 69 Brno, Czech Republic
fme.vutbr.cz

Summary of the Abstract

We use four dimensional geometric algebra for transforming four dimensional circles that correspond to voltage signal of four phase power system. We also show how rotation in four dimensions can be done by quaternions.

Actual Abstract Sections

1 Introduction

We consider two voltage signals from 4-phase power systems. Our aim is to transform one to the other. First we deal with symmetric signals with the same amplitude with phase shift φ. That is to transform $\boldsymbol{v} = v_1e_1 + v_2e_2 + v_3e_3 + v_4e_4$, where

$$\begin{aligned} v_1(t) &= v\cos(t\omega), \qquad v_2(t) = v\sin(t\omega), \\ v_3(t) &= -v\cos(t\omega), \qquad v_4(t) = -v\sin(t\omega), \end{aligned} \tag{1.1}$$

to $\boldsymbol{v}' = v_1'e_1 + v_2'e_2 + v_3'e_3 + v_4'e_4$, where

$$\begin{aligned} v_1'(t) =& v\cos(t\omega+\varphi), \qquad v_2'(t) = v\sin(t\omega+\varphi), \\ v_3'(t) =& -v\cos(t\omega+\varphi), \quad v_4'(t) = -v\sin(t\omega+\varphi). \end{aligned} \tag{1.2}$$

for each t. Such a curve will be called 4D circle.

That transformation is a rotation and we will show how it can be represented in Conformal Geometric Algebra, we will also see how it can be done using quaternions. Before analyzing the situation in 4 dimensions, let's recall rotations in 3 dimensions. The main idea is, that in 3 dimensions we are used to rotations around axis, which cannot be widened to more dimensions. While rotation in plane can be. For rotation in a plane π, the rotated vector $\boldsymbol{v}$ can be decomposed

$$\boldsymbol{v} = \boldsymbol{v}_\perp + \boldsymbol{v}_\parallel,$$

where $\boldsymbol{v}_\perp \perp \pi$ and $\boldsymbol{v}_\parallel \parallel \pi$. During rotation, the part $\boldsymbol{v}_\perp$ remains unchanged. The resultant vector is in form of $\boldsymbol{v}' = \boldsymbol{v}_\parallel' + \boldsymbol{v}_\perp$, where $\boldsymbol{v}_\parallel'$ is rotated $\boldsymbol{v}_\parallel$:

$$\boldsymbol{v}_\parallel' = \cos(\varphi)\boldsymbol{v}_\parallel + \sin(\varphi)\boldsymbol{v}_\top,$$

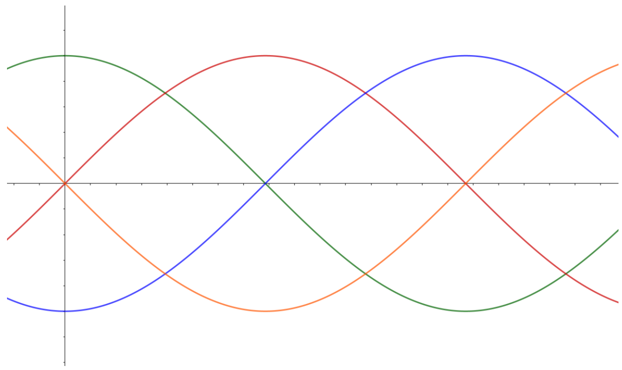

Figure 1: Four phase signal

where $\boldsymbol{v}_\top$ is vector in π perpendicular to $\boldsymbol{v}_{||}$ and $\|\boldsymbol{v}_\top\| = \|\boldsymbol{v}_{||}\|$. So the resultant vector is

$$\boldsymbol{v}' = \cos(\varphi)\boldsymbol{v}_{||} + \sin(\varphi)\boldsymbol{v}_\top + \boldsymbol{v}_\perp. \tag{1.3}$$

The composition of two such rotations in 3 dimensions is again a rotation in a plane.

2 Rotation in 4 dimensions

In four dimension the situation is a bit more complicated. In case (1.1), (1.2), the rotation is in two perpendicular planes $\pi = x_1x_2$ and $\sigma = x_3x_4$ about an angle φ, i.e. $v = \boldsymbol{v}_{\pi||} + \boldsymbol{v}_{\pi\perp}$, where

$$\boldsymbol{v}_{\pi||} = v(\cos(t\omega)e_1 + \sin(t\omega)e_2),$$
$$\boldsymbol{v}_{\pi\perp} = -v(\cos(t\omega)e_3 + \sin(t\omega)e_4).$$

Note that, in general the angle doesn't have to be the same in both planes. By rotation about an angle φ in the plane π we get

$$\boldsymbol{v}'_\pi = \cos(\varphi)\boldsymbol{v}_{||} + \sin(\varphi)\boldsymbol{v}_\top + \boldsymbol{v}_\perp,$$

where

$$\boldsymbol{v}_{\pi\top} = v(\cos(t\omega)e_2 - \sin(t\omega)e_1).$$

So the rotation in plane π can be written explicitly as

$$\boldsymbol{v}'_\pi = \cos(\varphi)(v(\cos(t\omega)e_1+\sin(t\omega)e_2)))+\sin(\varphi)v(\cos(t\omega)e_2-\sin(t\omega)e_1)-v(\cos(t\omega)e_3+\sin(t\omega)e_4) =$$
$$= v(\cos(\varphi)\cos(t\omega) - \sin(\varphi)\sin(t\omega))e_1 + v(\cos(\varphi)\sin(t\omega) + \sin(\varphi)\cos(t\omega))e_2 + \boldsymbol{v}_{1\perp}$$
$$= v\cos(\varphi + t\omega)e_1 + v\sin(\varphi + t\omega)e_2 - v\cos(t\omega)e_3 - v\sin(t\omega)e_4$$

Analogously in plane σ we rotate the vector $\boldsymbol{v}'_\pi = \boldsymbol{v}_{\sigma||} + \boldsymbol{v}_{\sigma\perp}$, where

$$\boldsymbol{v}_{\sigma||} = -v(\cos(t\omega)e_3 + \sin(t\omega)e_4),$$
$$\boldsymbol{v}_{\sigma\perp} = v(\cos(t\omega + \varphi)e_1 + \sin(t\omega + \varphi)e_2).$$

Because

$$\boldsymbol{v}_{\sigma\top} = v(\sin(t\omega)e_3 - \cos(t\omega)e_4)$$

we get

$$v' = vcos(t\omega + \varphi)e_1 + vsin(t\omega + \varphi)e_2 - vcos(t\omega + \varphi)e_3 - vsin(t\omega + \varphi)e_4$$

as we desired.

2.1 Rotation in 4D done by four dimensional CGA

We want to transform points $V(t) = [v_1(t), v_2(t), v_3(t), v_4(t)]$ from circle(1.1) to points $V'(t) = [v'_1(t), v'_2(t), v'_3(t), v'_4(t)]$ from circle (1.2) for any time t. Using the ideas above, in CGA that transformation can be represented as composition of two commutative rotations

$$\mathcal{R} = \mathcal{R}_\sigma \mathcal{R}_\pi,$$

where the angle of rotation is φ:

$$\mathcal{R}_\pi = \cos(\varphi/2) - \sin(\varphi/2)e_{12},$$

$$\mathcal{R}_\sigma = \cos(\varphi/2) - \sin(\varphi/2)e_{34},$$

so

$$\begin{aligned}\mathcal{R} =& \mathcal{R}_\pi \mathcal{R}_\sigma = \mathcal{R}_\sigma \mathcal{R}_\pi = (\cos(\varphi/2) - \sin(\varphi/2)e_{12})(\cos(\varphi/2) - \sin(\varphi/2)e_{34}) \\ =& \cos^2(\varphi/2) - \cos(\varphi/2)\sin(\varphi/2)(e_{12} + e_{34}) + \sin^2(\varphi/2)e_{1234}.\end{aligned}$$

2.2 Note: rotation in 4D done by quaternions

Quaternions are widely used for representing rotations in three dimensional space. As it is shown in article [4] they can represent rotations in 4 dimensions as well. Consider unit quaternions $p, q \in \mathbb{H}$ and its conjugation $\bar{q} \in \mathbb{H}$ and v a vector in $\mathbb{R}^4$. We identify the vector $\boldsymbol{v} = v_1e_1 + v_2e_2 + v_3e_3 + v_4e_4$ with a quaternion $v = v_1 + v_2i + v_3j + v_4k \in \mathbb{H}$. Consider $p = p_{Re} + p_ii + p_jj + p_kk$ and $q = q_{Re} + q_ii + q_jj + q_kk$, then the left multiplication pv can be written:

$$A(p)\boldsymbol{v}^T = \begin{bmatrix} p_{Re} & -p_i & -p_j & -p_k \\ p_i & p_{Re} & -p_k & p_j \\ p_j & p_k & p_{Re} & -p_i \\ p_k & -p_j & p_i & p_{Re} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

And the right multiplication is $v\bar{q}$:

$$B(\bar{q})\boldsymbol{v}^T = \begin{bmatrix} q_{Re} & q_i & q_j & q_k \\ -q_i & q_{Re} & -q_k & q_j \\ -q_j & q_k & q_{Re} & -q_i \\ -q_k & -q_j & q_i & q_{Re} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

The idea in three dimensions is that qv is a rotation in two perpendicular planes about angles α and α and $v\bar{q}$ is a rotation in the same planes about angles $-\alpha$ and α. One of these planes is a subspace of $\mathbb{R}^3$, there we get rotation about angle $\alpha + \alpha$, in the other plane ($\alpha - \alpha = 0$) nothing changes.

We use this information for four dimensions. Any rotation in $\mathbb{R}^4 \cong \mathbb{H} = Re\mathbb{H} + Im\mathbb{H} \cong \mathbb{R} \times \mathbb{R}^3$ can be represented as $pv\bar{q}$, where p and q are unit quaternions. In our case σ is a plane that is a subspace of $\mathbb{R}^3 \cong Im\mathbb{H}$. Since it doesn't have to hold $q = \bar{p}$, we have two angles α (for p) and β (for q). So we want $\alpha + \beta = \varphi$ and $\alpha - \beta = \varphi$. So $\alpha = \varphi$ and $\beta = 0$, i.e. $p = (\cos(\varphi) + \sin(\varphi)i)$ and $q = 1$. In detail we have

$$\begin{aligned} pv\bar{q} =& (\cos(\varphi) + \sin(\varphi)i)(\cos(t\omega) + \sin(t\omega)i - \cos(t\omega)j - \sin(t\omega)k)1 \\ =& (\cos(t\omega)\cos(\varphi) - \sin(t\omega)\sin(\varphi)) + (\cos(t\omega)\sin(\varphi) + \sin(t\omega)\cos(\varphi))i + \\ &+ (-\cos(t\omega)\cos(\varphi) + \sin(t\omega)\sin(\varphi))j + (-\sin(t\omega)\cos(\varphi) - \cos(t\omega)\sin(\varphi))k \\ =& \cos(t\omega + \varphi) + \sin(t\omega + \varphi)i - \cos(t\omega + \varphi)j - \sin(t\omega + \varphi)k. \end{aligned}$$

In the terms of matrices whole transformation is easily recognised:

$$A(p)B(q)\boldsymbol{v}^T = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 & 0 \\ 0 & 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & 0 & \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}. \tag{2.1}$$

3 System with DC offset

Similar problem with DC offset looks like that: We want to transform the circle

$$\begin{aligned} v_1(t) =& V_n + vcos(t\omega), \qquad v_2(t) = V_n + vsin(t\omega), \\ v_3(t) =& V_n - vcos(t\omega), \qquad v_4(t) = V_n - vsin(t\omega) \end{aligned} \tag{3.1}$$

to the circle

$$\begin{aligned} v_1'(t) =& V_n' + v'cos(t\omega + \varphi), \qquad v_2'(t) = V_n' + v'sin(t\omega + \varphi), \\ v_3'(t) =& V_n' - v'cos(t\omega + \varphi), \qquad v_4'(t) = V_n' - v'sin(t\omega + \varphi). \end{aligned} \tag{3.2}$$

We saw the shift of φ can be dealt with by rotation. Now we need to move first circle to the second one. (Here the quaternions will not be sufficient.) In CGA for two circles in parallel planes it can be done easily using dilation and translation. The dilation gives us the circle with desired radius v':

$$\mathcal{D} = exp(-0.5ln(d)e_{+-}),$$

where $d = \frac{v'}{v}$. Then using translation we can move it, so center of first circle merges with center of second circle:

$$\mathcal{T} = 1 - 0.5\boldsymbol{t}e_\infty,$$

where $\boldsymbol{t} = (V_n' - V_n\frac{v'}{v})\boldsymbol{n}$, where $\boldsymbol{n} = e_1 + e_2 + e_3 + e_4$. Now we can apply it on $V(t)$:

$$V'(t) = \mathcal{T}\mathcal{D}\mathcal{R}V(t)\mathcal{R}^{-1}\mathcal{D}^{-1}\mathcal{T}^{-1}$$

and we have transformation that doesn't depend on time and holds for each t.

References

[1] Francisco Casado-Machado, José L. Martínez-Ramos, Manuel Barragán-Villarejo, José María Maza-Ortega, Reduced Reference Frame Transformation Assessment in Unbalanced Three-Phase Four-Wire Systems, IEEE Access, 10.1109/ACCESS.2023.3254299, 11, (24591-24603), (2023).

[2] Dorst, L., Fontijne, D. and Mann, S.: Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry. Morgan Kaufmann Publishers Inc. (2007)

[3] Ahmad H. Eid, Francisco G. Montoya, A Systematic and Comprehensive Geometric Framework for Multiphase Power Systems Analysis and Computing in Time Domain, IEEE Access, 10.1109/ACCESS.2022.3230915, 10, (132725-132741), (2022). 7/s00006-021-01196-7

[4] Gallier, J. (2001). The Quaternions and the spaces S3, SU(2), SO(3), and RP3. In Texts in applied mathematics (pp. 248–266). https://doi.org/10.1007/978-1-4613-0137-0_8

[5] Hildenbrand, D.: Foundations of Geometric Algebra Computing. Springer Science & Business Media (2013)

[6] Hitzer, E., Kamarianakis, M., Papagiannakis, G., and Vašík, P.: Survey of new applications of geometric algebra, Math. Meth. Appl. Sci. 1–17 (2023) DOI 10.1002/mma.9575.

[7] Lounesto, P.: Clifford Algebra and Spinors. 2nd edn. CUP, Cambridge (2006)

[8] Montoya, F. G., & Eid, A. H. (2022). Formulating the geometric foundation of Clarke, Park, and FBD transformations by means of Clifford's geometric algebra. Mathematical Methods in the Applied Sciences, 45(8), 4252-4277. https://doi.org/10.1002/mma.8038

[9] Francisco G. Montoya, Xabier Prado, Francisco M. Arrabal-Campos, Alfredo Alcayde, Jorge Mira, New mathematical model based on geometric algebra for physical power flow in theoretical two-dimensional multi-phase power circuits, Scientific Reports, 10.1038/s41598-023-28052-x, 13, 1, (2023).

[10] Perwass, Ch.: Geometric Algebra with Applications in Engineering (1st edn). Springer Verlag, (2009)

[11] Eduardo Viciana, Francisco M. Arrabal-Campos, Alfredo Alcayde, Raul Baños, Francisco G. Montoya, All-in-one three-phase smart meter and power quality analyzer with extended IoT capabilities, Measurement, 10.1016/j.measurement.2022.112309, 206, (112309), (2023)