Representation and Gauge Freedom in Electromagnetism and Acoustics

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In acoustic and electromagnetic phenomena, the concept of gauge freedom plays a pivotal role in understanding the underlying physics and its observable effects. Our recently published work extends traditional scalar and vector potential representations of measurable fields to multivector-valued gauge potentials, with each representation grade naturally coupling to distinct types of physical source. Notably, combining the possible potential representations into multi-graded representations further constrains the traditional gauge freedoms of both theories. This talk explores the physical relevance of each of the allowed potential fields in each theory and discusses the interplay between representational and gauge freedoms [1, 2].

In electromagnetism, the electromagnetic field F is traditionally represented by a dynamical vector potential A that must be varied in the Lagrangian to produce the equations of motion. This traditional representation of F by the (electric) vector potential A has the form,

$$F = \nabla \wedge A. \tag{1}$$

Gauge symmetries are precisely those transformations of the potential representation A that preserve the measurable field F. The form of the representation as a curl yields the familiar $U(1)$ gauge freedom of the spacetime vector potential

$$A \mapsto A + \nabla\phi, \tag{2}$$

because the curl of a curl vanishes,

$$\nabla \wedge (\nabla\phi) = 0. \tag{3}$$

However, Equation 1 is just one possible representation of an electromagnetic field F. Another possible choice of representation is a magnetic pseudovector potential BI, such that

$$F = \nabla \cdot (BI), \tag{4}$$

which remains invariant under

$$BI \mapsto BI + \nabla\phi I. \tag{5}$$

One's choice of representation, which is distinct from choice of particular gauge, has significant bearing on the physics it can describe. As a simple example, the electric vector potential can couple only to electric charges, while the magnetic vector potential can couple only to magnetic charges.

An important consequence of this multiplicity of distinct representations is that the structural symmetries of a system in the absence of electric charge are distinct from those in which charge is present. Specifically, helicity is a conserved quantity associated with a dual symmetry that exchanges the electric and magnetic potential representations—a vacuum symmetry which the traditional source-free electromagnetic Lagrangian does not possess. That is,

$$\mathcal{L}_{\text{em}} = \frac{\langle F\widetilde{F}\rangle}{2} \tag{6}$$

fails to be invariant under

$$F \mapsto Fe^{I\beta}, \tag{7}$$

which precludes helicity from being obtained as a Noether current of the theory.

A second and related role that representation plays is in its effect on the conserved stress-energy and angular momentum tensors of the theory. The spin angular momentum predicted by the traditional Lagrangian 6 under the representation $F = \nabla \wedge A$ is

$$\vec{S}_{\text{em}} = \epsilon\vec{A} \times \vec{E}, \tag{8}$$

while under the magnetic potential representation $F = \nabla \cdot (BI)$, the same Lagrangian yields instead

$$\vec{S}_{\text{em}} = \mu\vec{B} \times \vec{H}, \tag{9}$$

highlighting the representation dependence of spin, which leads to its gauge dependence. This asymmetry between electric and magnetic field contributions to spin points to an asymmetry in representation.

Recent progress in local spin angular momentum density measurements involving observation of the backaction of local force and torque on small probe particles in optical fields far from sources, e.g. Refs. [3, 4, 5, 6], require descriptions that respect dual symmetry to match experimental observations to theoretical predictions. These measurements of spin density include contributions from both electric and magnetic fields [7, 8, 9, 10], taking the form

$$\vec{S}_{\text{em}} = \frac{1}{2}\big(\epsilon\vec{A}_e \times \vec{E} + \mu\vec{A}_m \times \vec{H}\big). \tag{10}$$

The representation we proposed to solve these issues is given by[1, 2]

$$\mathcal{L}_{\text{em}} = \frac{\langle \nabla z_{\text{em}} \nabla \widetilde{z}_{\text{em}}\rangle}{2} \tag{11}$$

where $z_{\text{em}} = A_e + A_m I$ is an odd multivector-valued potential with electric (vector) and magnetic (trivector) parts, which transform under duality transformations as $z_{\text{em}} \mapsto z_{\text{em}}e^{I\beta}$, yielding the equations of motion

$$\nabla^2 z_{\text{em}} = 0. \tag{12}$$

Importantly, z_{em} is the quantity that transforms under duality transformations, rather than F, to ensure the Lagrangian is invariant and obtain the helicity as a Noether current. The full geometric structure additionally permits scalar and pseudoscalar parts of the measurable field $\psi = \nabla z_{\text{em}} = W_e/c^2 + F + W_m I/c$, where $W_e = c^2 \nabla \cdot A_e = \langle \psi \rangle_0$ and $IW_m = c\nabla \wedge (A_m I) = \langle \psi \rangle_4$ play the role of *power per unit charge*, which make direct contributions to the energy of the stress tensor [2].

Under this dual symmetric representation, the theory exhibits a distinct gauge symmetries. That is, transformations preserving ψ are more restrictive, consisting strictly of monogenic odd multivector fields

$$z_{\text{em}} \mapsto z_{\text{em}} + z \tag{13}$$

where $\nabla z = 0$. This representation respects dual symmetry of vacuum and predicts the correct form of spin angular momentum, Equation 10, as measured in the laboratory.

Parallel developments have been made experimentally and theoretically in acoustic field theory. With recent reports concerning the observation of acoustic spin [11, 12], standard acoustic theory required corrections. Acoustics is ordinarily considered as a scalar field theory with Lagrangian of the form

$$\mathcal{L}_{\text{ac}} = \frac{\langle p^2 \rangle}{2} \tag{14}$$

with pressure P and velocity $\vec{v}$ fields given by a gradient of a scalar field ϕ

$$p = (P/c + \rho \vec{v})\gamma_0 = \nabla \phi = \gamma_0 \left(\frac{1}{c} \partial_t - \vec{\nabla} \right) \phi \tag{15}$$

combined into a single four-vector in a spacetime algebra, where c is the constant speed of sound and the Lorentz structure reflects the symmetries of the acoustic wave equation

$$\nabla^2 \phi = 0. \tag{16}$$

However, Lagrangian 14 predicts zero spin:

$$\vec{S}_{\text{ac}} = 0. \tag{17}$$

Analogous to dual symmetry in the absence of charge in electromagnetism, the presence of directional sources, such as the speakers used in the spin experiments, alters the structural symmetries of the theory. The correct spin angular momentum was first calculated from microscopic arguments [11, 13, 14] to match experimental results. Field theoretic justification followed, as the appropriate description for the experimental conditions under which spin was observed required consideration of a larger class of potential representation of physical pressure and velocity fields [1, 2]. Under a timelike bivector potential representation

$$p = (P/c + \rho \vec{v})\gamma_0 = -\rho c \nabla \cdot \vec{x} = (-\rho c \vec{\nabla} \cdot \vec{x} + \rho \partial_t \vec{x})\gamma_0, \tag{18}$$

the Lagrangian above predicts a nonzero spin angular momentum density,

$$\vec{S}_{\text{ac}} = \vec{x} \times (\rho \vec{v}), \tag{19}$$

where the bivector $\vec{x}$ represents microscopic displacements. Notably, this prediction is twice the experimentally measured value. It is the *average* of the two representation predictions that correctly yields the measured result.

A geometric completion of the theory is obtained from the representation

$$\mathcal{L}_{\rm ac} = \frac{\langle \nabla \psi_{\rm ac} \nabla \widetilde{\psi}_{\rm ac} \rangle}{2} \tag{20}$$

where $\psi_{\rm ac} = \phi + \rho c \vec{x} + I\vec{J} + I\phi_w$ unifies the standard scalar ϕ and displacement potential $\vec{x}$ into a full even multivector, with the additional inclusion of a spacelike bivector potential $I\vec{J}$ representing an intrinsic angular momentum density of the field (analogous to A_e as momentum per unit charge), as well as a pseudoscalar potential ϕ_w. Under the introduction of a full spinor potential, the possible measurable fields extends to include a pseudovector part wI,

$$z_{\rm ac} = -\nabla \psi_{\rm ac} = p + wI, \tag{21}$$

which has components playing rotational analogs of the components of p, including rotational velocity and rotational work density. Interestingly, $z_{\rm ac}$ has the same odd-graded structure as the dual symmetric electromagnetic potential $z_{\rm em}$.

In the electromagnetic case, the benefits of a multivector representation were limited to the effects on fields and Noether currents, particularly in vacuum scenarios that obey dual symmetry. In the presence of sources, however, the geometrically complete theory admits magnetic charges, which has less utility due to the apparent lack of magnetic monopoles. In the acoustic case, the geometric completion of the representation admits a larger array of physically realizable sources, including scalar particle-density sources ν that couple to the scalar representation ϕ in the standard scalar theory, directional sources $\vec{F}$ like speakers which couple to the displacement bivector potential $\vec{x}$, vorticity sources $\vec{\Omega}I$ like a spinning propeller that couple to the angular momentum potential $\vec{J}I$, and pseudoscalar volume sources νI,

$$-\nabla^2 \psi_{\rm ac} = \nu + \vec{F}/c + \rho\vec{\Omega}I + \nu_w I. \tag{22}$$

Importantly, the gauge fields and symmetries have clear microscopic interpretations in the acoustic setting, allowing for the transport of intuition from acoustics to electromagnetism, where we lack such a concrete microscopic picture.

The geometrical completions of the potentials, fields, and sources in both electromagnetism and acoustics provides a setting for a rich investigation into the interplay of geometry, representation, and symmetry in field theories—particularly in consideration of the conserved quantities predicted by a Lagrangian and their realization in the laboratory. An explicit comparison of key quantities in the two theories, and their geometrically completed representations, is presented in Table 1. We anticipate that the insights we have gained from analyzing these electromagnetic and acoustic examples will affect our understanding of field theories more broadly.

	Electromagnetism	**Acoustics**
Potential fields	$z_{\rm em} = \lambda_- a_e + \lambda_+ a_m I$	$\psi_{\rm ac} = \lambda_-\,\phi + \lambda_+\, M/3 + \lambda_4\,\phi_w I$
Measurable fields	$\psi_{\rm em} = \nabla z_{\rm em} = W_e/c^2 + F + W_m I/c$	$z_{\rm ac} = -\nabla\psi_{\rm ac} = p + wI$
Symmetric Lagrangian	$\mathcal{L}_{\rm em} = c\dfrac{\langle \nabla z_{\rm em} \nabla \widetilde{z}_{\rm em}\rangle}{2}$	$\mathcal{L}_{\rm ac} = c\dfrac{\langle \nabla \psi_{\rm ac} \nabla \widetilde{\psi}_{\rm ac}\rangle}{2}$
Sources	$j = j_e + j_m I/c$	$\psi_N = \nu + \vec{F}/c + \rho\vec{\Omega} I + \nu_w I$
Equation of motion	$\nabla\psi_{\rm em} = \nabla^2 z_{\rm em} = \mu j$	$\nabla z_{\rm ac} = -\nabla^2\psi_{\rm ac} = -\psi_N$
Canonical Momentum	$T_{\rm em}(n) = \dot{\nabla}\langle \dot{z}_{\rm em}(\nabla z_{\rm em} n + n z_{\rm em}\nabla)\rangle$ $-n\mathcal{L}_{\rm em}$	$T_{\rm ac}(n) = \dot{\nabla}\langle \dot{\psi}_{\rm ac}(\nabla \psi_{\rm ac} n + n \psi_{\rm ac}\nabla)\rangle$ $-n\mathcal{L}_{\rm ac}$
Canonical Spin	$S_{\rm em}(n) = [z_{\rm em}, \nabla z_{\rm em} n + n z_{\rm em}\nabla]$	$S_{\rm ac}(n) = [\psi_{\rm ac}, \nabla \psi_{\rm ac} n + n \psi_{\rm ac}\nabla]$

Table 1: Summary of electromagnetic and acoustic structure, including all geometrically admissible fields and sources. Even in the absence of magnetic charge sources $j_m I$ in electromagnetism or rotational trivector fields wI in acoustics, the geometrical completion provides significant insight and utility in its account of fields in the absence of sources — particularly in consideration of the measurable potential-dependent canonical spin and momentum. In acoustics, the primary advantage of this completion is a full account of all types of sources, while in electromagnetism, the primary advantage of this completion is the correct account of canonical spin and momentum density in the dual symmetric, source-free regime. This complementarity provides opportunity for cross-pollination between the two theories and makes use of the full geometry of spacetime and the Clifford algebraic structure.

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