CGA-Based Snake Robot Control Models

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Summary of the Abstract

The snake robot is a mechanism composed of links equipped with passive wheels connected by actuated joints whose movement is based on the locomotion of biological snakes. Planar control models have been obtained before, usually by means of differential geometry or geometric algebra. We present an extension of the CGA planar model to the full 3D case using CGA. A quick introduction to the 2D CGA model is given. A planar model is created using 3D CGA, whose geometry is then exploited in order to formulate the full 3D model. Two parametrisations of the joints' rotations as rotors are proposed and then utilised in the model's construction. The resulting motion based on the models is visualised.

2D CGA Model

The 2D CGA is the Clifford algebra $Cl_{3,1}$ with the basis $\{e_1, e_2, e_\infty, e_0\}$ along with the embedding of a point $(x, y) \in \mathbb{R}^2$ given by

$$(x, y) \mapsto x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + \frac{1}{2}(x^2 + y^2)\boldsymbol{e}_\infty + \boldsymbol{e}_0.$$

The snake robot consists of a series of links connected by actuated joints, usually revolute joints. Denote the configuration space of the mechanism as the manifold $Q \subset (\mathbb{R}^2 \times (S^1)^3)$ with point $\boldsymbol{q} = [x, y, \theta, \phi_1, \phi_2]$ representing a configuration of the mechanism at the time t.

We can represent euclidean transformations in CGA as exponentials of bivectors. The rotation by angle α about axis L is given by the rotor R:

$$R = e^{-\frac{1}{2}\alpha L},$$

a translation in the direction of vector $\boldsymbol{t} = t_1\boldsymbol{e}_1 + t_2\boldsymbol{e}_2$ is represented by the translator T:

$$T = e^{-\frac{1}{2}\ \boldsymbol{t}\boldsymbol{e}_\infty}.$$

We represent a general transformation M defined by bivector $L = L(\boldsymbol{q}(t))$ (depending on some of the state variables in order to parametrise the transformation) as

$$M = e^{-\frac{1}{2}L(\boldsymbol{q}(t))},$$

and thus the reverse of M is $\tilde{M} = e^{\frac{1}{2}L(\boldsymbol{q}(t))}$.
Let us now quickly go through the derivation of the control model as shown in [2]. The initial configuration of the i–th link of the mechanism is represented by point pairs $P_i^0 = A_i \wedge A_{i+1}$, where A_i are the edges of the links, see Figure 1.

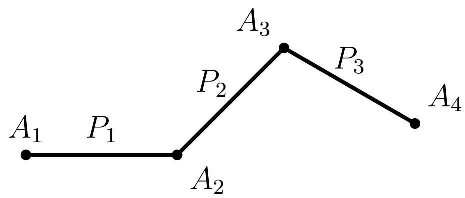

Figure 1: A three-link snake robot.

A general configuration is then represented as a sequence of transformations applied to the initial configuration. Then the configuration of the i–th link at time t is given by

$$P_i = \prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j, \tag{1}$$

where M_j is the j–th transformation.
To obtain the differential kinematics, we need to express the velocities of the state variables defining the mechanism's configuration - in the 2D case that is $\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}_1$ and $\dot{\phi}_2$.
The constraint imposed on snake robots is the non-slip nonholonomic constraint, which limits the velocity of the i–th link to the direction defined by the point pair P_i. In terms of CGA, we can express this constraint as

$$\dot{p}_i \wedge P_i \wedge \boldsymbol{e}_\infty = 0 \tag{2}$$

where $\dot{p}_i = \dot{x}\boldsymbol{e}_1 + \dot{y}\boldsymbol{e}_2 + \dot{z}\boldsymbol{e}_3$ is the velocity of the i–th point pair's centre p_i. The centre p_i is obtained by the decomposition

$$p_i = P_i \boldsymbol{e}_\infty \tilde{P}_i. \tag{3}$$

Taking the derivative w.r.t. time of Equation (3), we get

$$\dot{p}_i = \partial_t (P_i \boldsymbol{e}_\infty \tilde{P}_i) = \dot{P}_i \boldsymbol{e}_\infty \tilde{P}_i + P_i \boldsymbol{e}_\infty \dot{\tilde{P}}_i. \tag{4}$$

Assuming the state of P_i is represented by k transformations, expressing $\dot{P}_i$ we arrive at

$$\dot{P}_i = \partial_t (\prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j). \tag{5}$$

The derivative of the general transformation M is then given by

$$\partial_t M = -\frac{1}{2}(\partial_t L(\boldsymbol{q}(t))) e^{-\frac{1}{2}L(\boldsymbol{q}(t))} = -\frac{1}{2}\dot{L}(\boldsymbol{q}(t)) M$$

and thus the derivative of the reverse is $\partial_t \tilde{M} = \frac{1}{2}\dot{L}\tilde{M}$. By the chain rule

$$\dot{L} = \partial_t L(\boldsymbol{q}(t)) = \sum_{i=1}^{n} (\partial_{q_i} L)\dot{\boldsymbol{q}}_i.$$

Denoting $\partial_t M = \dot{M}$ and expanding Equation (5), we get

$$\begin{aligned} \dot{P}_i &= \partial_t (\prod_{j=k}^{1} M_j P_i^0 \prod_{j=1}^{k} \tilde{M}_j) = \\ &= \sum_{j=1}^{k} [P_i \cdot \dot{L}_j], \end{aligned} \tag{6}$$

utilising Lemma 1 from [3] in the last step (see [4] for the calculations). Substituting Equation (6) into Equation (4) we can write $\dot{p}_i$ in the form of

$$\dot{p}_i = \sum_{j=1}^{k} [p_i \cdot \dot{L}_j]. \tag{7}$$

Finally, substituting Equation (1) and Equation (7) into the nonholonomic condition Equation (2), we arrive at set of three differential equations with multivector coefficients:

$$\begin{aligned} &\Big(\dot{\theta} - 2\dot{x}\sin(\theta) + 2\dot{y}\cos(\theta)\Big)\boldsymbol{I} = 0, \\ &\Big(\dot{\phi}_1 + 2\dot{\theta}\cos(\phi_1) + \dot{\theta} - 2\dot{x}\sin(\phi_1 + \theta) + 2\dot{y}\cos(\phi_1 + \theta)\Big)\boldsymbol{I} = 0, \\ &\Big(2\dot{\phi}_1\cos(\phi_2) + \dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\theta}\cos(\phi_2) + 2\dot{\theta}\cos(\phi_1 + \phi_2) + \dot{\theta} - \\ &-2\dot{x}\sin(\phi_1 + \phi_2 + \theta) + 2\dot{y}\cos(\phi_1 + \phi_2 + \theta))\boldsymbol{I} = 0, \end{aligned} \tag{8}$$

where $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_0\boldsymbol{e}_\infty$ is the pseudoscalar. Since $\boldsymbol{I}$ is nonzero, it holds that

$$\begin{aligned} &\dot{\theta} - 2\dot{x}\sin(\theta) + 2\dot{y}\cos(\theta) = 0, \\ &\dot{\phi}_1 + 2\dot{\theta}\cos(\phi_1) + \dot{\theta} - 2\dot{x}\sin(\phi_1 + \theta) + 2\dot{y}\cos(\phi_1 + \theta) = 0, \\ &2\dot{\phi}_1\cos(\phi_2) + \dot{\phi}_1 + \dot{\phi}_2 + 2\dot{\theta}\cos(\phi_2) + 2\dot{\theta}\cos(\phi_1 + \phi_2) + \dot{\theta} - \\ &- 2\dot{x}\sin(\phi_1 + \phi_2 + \theta) + 2\dot{y}\cos(\phi_1 + \phi_2 + \theta) = 0, \end{aligned} \tag{9}$$

which are the kinematic equations of the 3–link robotic snake in the form as usually obtained using a differential geometric approach, see for example [1]. We have obtained only three equations, while the dimension of the tangent bundle TQ is 5, meaning two more equations will have to be added in order to define the control system of the mechanism. Denote $u_1 = u_1(t)$, $u_2 = u_2(t)$ as the control inputs at time t. Then by adding two equations $\dot{x} = u_1$, $\dot{y} = u_2$, the inverse kinematics are obtained. The forward kinematics would be obtained by adding the equations $\dot{\phi}_1 = u_1$, $\dot{\phi}_2 = u_2$ instead.

3D CGA Snake Robot Planar Motion Model

Having introduced the 2D model, we can now build on it in order to obtain a 3D model. In order to verify the modelling approach, the first step is to derive the model for planar motion (in the xy plane) in 3D space, see Figure 2. For this task, we will move from using $Cl_{3,1}$ to $Cl_{4,1}$ (3D CGA).

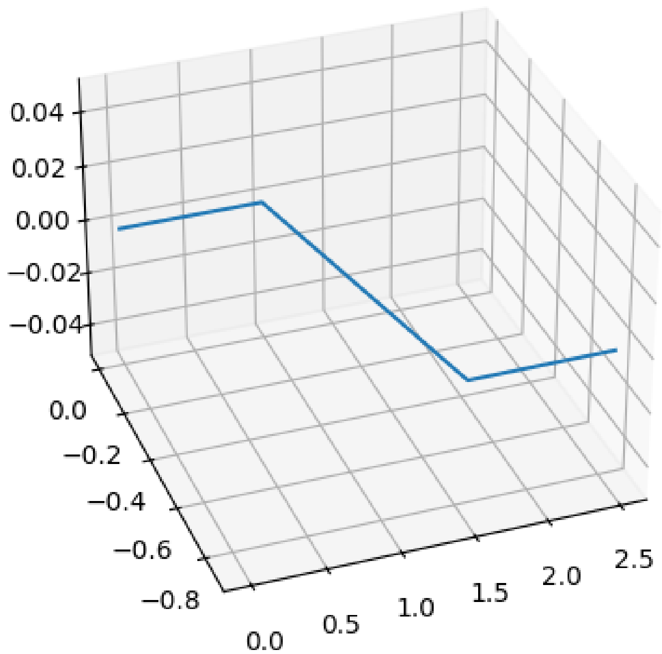

Figure 2: The configuration $\boldsymbol{q} = (0, 0, 0, 0, -\frac{\pi}{3}, \frac{\pi}{3})$.

Due to the properties of geometric algebra, the derivation of the differential kinematics remain the same. The new configuration space increases in dimension by one in order to account for the z–axis coordinate: $Q \subset (\mathbb{R}^3 \times (S^1)^3)$; and thus we denote a configuration at time t as $\boldsymbol{q} = [x, y, z, \theta, \phi_1, \phi_2]$. The rotations used in the kinematic chain are unchanged, while the translations gain an extra dimension: $T = e^{-\frac{1}{2}(t_1\boldsymbol{e}_1 + t_2\boldsymbol{e}_2 + t_3\boldsymbol{e}_3)\boldsymbol{e}_\infty}$. Expanding the nonholonomic constraint, we again arrive at a set of three differential equations with multivector coefficients. Due to the dimensionality of the problem, the respective equations consist of 4–vectors, with the respective blades being the basis blades constituting planes. As the kinematics must be invariant w.r.t. the x, y, z coordinates, evaluating the system in the origin of the coordinate system we get

$$\begin{aligned} &\left(\dot{\theta} - 2\dot{x}\sin(\theta) + 2\dot{y}\cos(\theta)\right) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty + 2\dot{z}\cos(\theta)\boldsymbol{e}_1 \\ &\wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty + 2\dot{z}\sin(\theta)\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_0 \wedge \boldsymbol{e}_\infty = 0, \end{aligned} \tag{10a}$$

$$
\begin{aligned}
&2\dot{z}\sin(\phi_1)\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_3\wedge\boldsymbol{e}_\infty\\
&\quad+\Big(\dot{\phi}_1+2\dot{\theta}\cos(\phi_1)+\dot{\theta}-2\dot{x}\sin(\phi_1+\theta)+2\dot{y}\cos(\phi_1+\theta)\Big)\,\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_0\wedge\boldsymbol{e}_\infty\\
&\quad+2\dot{z}\cos(\phi_1+\theta)\boldsymbol{e}_1\wedge\boldsymbol{e}_3\wedge\boldsymbol{e}_0\wedge\boldsymbol{e}_\infty+2\dot{z}\sin(\phi_1+\theta)\boldsymbol{e}_2\wedge\boldsymbol{e}_3\wedge\boldsymbol{e}_0\wedge\boldsymbol{e}_\infty=0,
\end{aligned}\tag{10b}
$$

$$
\begin{aligned}
&2\dot{z}\left(\sin(\phi_2)+\sin(\phi_1+\phi_2)\right)\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_3\wedge\boldsymbol{e}_\infty\\
&\quad+\Big(2\dot{\phi}_1\cos(\phi_2)+\dot{\phi}_1+\dot{\phi}_2+2\dot{\theta}\cos(\phi_2)+2\dot{\theta}\cos(\phi_1+\phi_2)\\
&\quad+\dot{\theta}-2\dot{x}\sin(\phi_1+\phi_2+\theta)+2\dot{y}\cos(\phi_1+\phi_2+\theta)\Big)\,\boldsymbol{e}_1\\
&\quad\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_0\wedge\boldsymbol{e}_\infty+2\dot{z}\cos(\phi_1+\phi_2+\theta)\boldsymbol{e}_1\wedge\boldsymbol{e}_3\wedge\boldsymbol{e}_0\\
&\quad\wedge\boldsymbol{e}_\infty+2\dot{z}\sin(\phi_1+\phi_2+\theta)\boldsymbol{e}_2\wedge\boldsymbol{e}_3\wedge\boldsymbol{e}_0\wedge\boldsymbol{e}_\infty=0.
\end{aligned}\tag{10c}
$$

Notice that the term including the blade $\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_\infty$ vanishes, as it was dependent on x, y or z in every term.

Lemma 1. *The projection of the kinematic equations into the coordinate planes is obtained as*

$$\dot{p}_i\wedge P_i\wedge\boldsymbol{e}_\infty\wedge\boldsymbol{e}_j=0,$$

where $\boldsymbol{e}_j$ is the normal vector defining the coordinate plane.

Example 1. Taking Equation (10a) to Equation (10c) and wedging them by $\boldsymbol{e}_3$, we get

$$\Big(\dot{\theta}-2\dot{x}\sin(\theta)+2\dot{y}\cos(\theta)\Big)\,\boldsymbol{I}=0,\tag{11a}$$

$$\Big(\dot{\phi}_1+2\dot{\theta}\cos(\phi_1)+\dot{\theta}-2\dot{x}\sin(\phi_1+\theta)+2\dot{y}\cos(\phi_1+\theta)\Big)\,\boldsymbol{I}=0,\tag{11b}$$

$$
\begin{aligned}
&\Big(2\dot{\phi}_1\cos(\phi_2)+\dot{\phi}_1+\dot{\phi}_2+2\dot{\theta}\cos(\phi_2)+2\dot{\theta}\cos(\phi_1+\phi_2)\\
&\quad+\dot{\theta}-2\dot{x}\sin(\phi_1+\phi_2+\theta)+2\dot{y}\cos(\phi_1+\phi_2+\theta)\Big)\,\boldsymbol{I},
\end{aligned}\tag{11c}
$$

which leads precisely to Equation (9), the kinematics system obtained for the 2D motion. Wedging by $\boldsymbol{e}_2$, we get

$$-2\dot{z}\cos(\theta)\boldsymbol{I}=0,\tag{12a}$$
$$-2\dot{z}\cos(\phi_1+\theta)\boldsymbol{I}=0,\tag{12b}$$
$$-2\dot{z}\cos(\phi_1+\phi_2+\theta)\boldsymbol{I}=0,\tag{12c}$$

and by wedging by $\boldsymbol{e}_1$, we get

$$2\dot{z}\sin(\theta)\boldsymbol{I}=0,\tag{13a}$$
$$2\dot{z}\sin(\phi_1+\theta)\boldsymbol{I}=0,\tag{13b}$$
$$2\dot{z}\sin(\phi_1+\phi_2+\theta)\boldsymbol{I}=0.\tag{13c}$$

Since the functions sin(...) and cos(...) in Equation (12a) to Equation (12c) and in Equation (13a) to Equation (13c) cannot be both zero at the same time, it follows that $\dot{z}=0$, which means that the mechanism remains in the xy plane.

Full 3D CGA Snake Robot Model

In order to obtain the control models in 3D, the kinematic equations must be derived first. Thanks to the properties of geometric algebra, the description of the mechanism remains the same as in the 2D case, with only an extra dimension for the z axis appearing in the configuration space and the parametrisations of the transformations used.

Currently, two models are planned: in the first model, we allow rotation between two links about an arbitrary axis; that is, the links are connected by a spherical joint. The rotation of the spherical joint about an axis L_α by angle α is expressed by the rotor $R_\alpha = e^{-\frac{1}{2}\alpha L_\alpha}$. Using this parametrisation, every rotation is described by 3 angles. In the case of a snake robot with 3 links, the dimension of the configuration space increases: we have $Q \subset (\mathbb{R}^2 \times (S^1)^9)$. Figure 3 depicts three different configurations of the mechanism using this parametrisation.

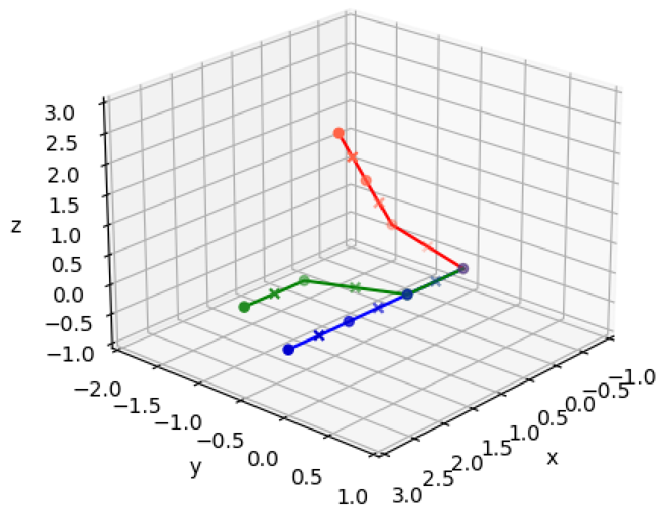

Figure 3: Three different configurations of the 3-link robot snake in 3D.

The axis of rotation L_α is parametrised by rotations about two axes, L_1 and L_2, see Figure 4. The first axis of rotation $L_1 = R_{\alpha_x}\boldsymbol{e}_{12}\tilde{R}_{\alpha_x}$ is obtained by rotating the z–axis L_0 by angle α_x about the x–axis. The second axis $L_2 = R_{\alpha_x}\boldsymbol{e}_{13}\tilde{R}_{\alpha_x}$ is the result of the same rotation applied to the y–axis. Thus

$$L_\alpha = R_{\alpha_y} L_1 \tilde{R}_{\alpha_y} = R_{\alpha_y} R_{\alpha_x} \boldsymbol{e}_{12} \tilde{R}_{\alpha_x} \tilde{R}_{\alpha_y}$$

where $R_{\alpha_x} = e^{-\frac{1}{2}\alpha_x \boldsymbol{e}_{12}}$ and $R_{\alpha_y} = e^{-\frac{1}{2}\alpha_y L_2}$. To obtain the kinematic equations, we express the derivative of R_α:

$$\partial_t R_\alpha = \frac{-1}{2} e^{-\frac{1}{2}\alpha L_\alpha} \partial_t(\alpha L_\alpha) = \frac{-1}{2} e^{-\frac{1}{2}\alpha L_\alpha} (\dot{\alpha} L_\alpha + \alpha \dot{L}_\alpha) \tag{14}$$

Thanks to the expression of $\dot{p}_i$ in the form of Equation (7), using Equation (14) we are now able to expand the nonholonomic condition Equation (2) and proceed as in the 3D planar motion model.

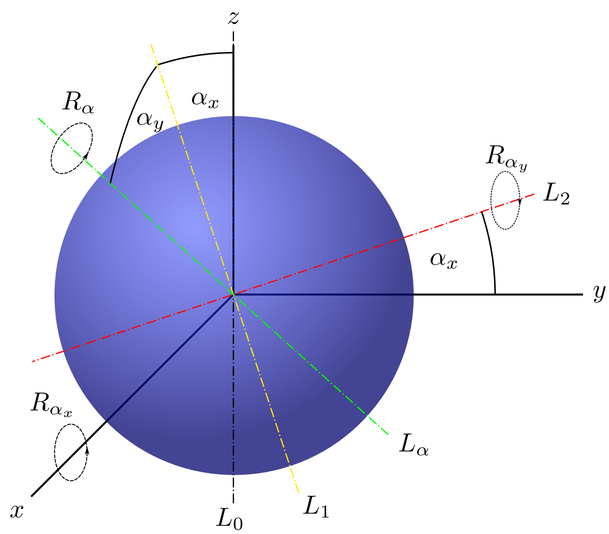

Figure 4: Model of a spherical joint.

This model would be appropriate for modelling the motion of snake robots in an aquatic environment. For the ground-based mechanisms, it makes more sense to restrict the possible rotations of the joints to two planes in order to avoid any of the links rotating about its own axis (while not impossible and not without use cases, allowing this motion would complicate the mechanism's mechanical construction), leading to the second model.

References

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