A multivector description of spacetime

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1. Introduction

A central aspect of the natural world is the presence of just three degrees of translational freedom. This is confirmed by the presence of exactly five regular solids, which only occurs in three dimensions. Three spatial dimensions also leads to the expectation of inverse square force laws, which for gravity and electromagnetism, has indeed been experimentally verified to very high precision. These preliminary observations regarding the nature of physical space provides a general conceptual structure within which to describe physical processes. The formalism of Clifford's geometric algebra (GA) of three dimensions $C\ell(\Re^3)$ is therefore natural to adopt as a mathematical framework to describe this space. The appropriateness of this choice of $C\ell(\Re^3)$ is confirmed by the fact that Minkowski spacetime is found to be embedded as a four-dimensional subspace, within the eight-dimensional space of $C\ell(\Re^3)$ [1, 2]. This shows that Minkoswki spacetime is a natural consequence of three spatial dimensions described by $C\ell(\Re^3)$ [1]. A fundamental insight that this provides, is that time is now identified as the scalar component of $C\ell(\Re^3)$, rather than as an extra time-like dimension. This gives it an advantage over other descriptions of spacetime, such as STA. The properties of light as well as Maxwell's equations also emerge directly from the algebra, without recourse to physical arguments [1]. The natural correspondence of $C\ell(\Re^3)$ with physical theory, is also illustrated by the fact that there are generally found to be four fundamental geometric types required to describe physical laws: scalars, vectors, pseudovectors and pseudoscalars, which correspond directly to the four algebraic grades found within $C\ell(\Re^3)$. Scalars naturally describe quantities such as energy or pressure, vectors relate to quantities such as velocity, acceleration, momentum and electric fields, pseudovectors describe torque, spin angular momentum and the magnetic field, and the pseudoscalars describe the property of helicity, such as in the magnetic helicity. Furthermore, as a significant aid to visualisation, the algebraic elements of scalars, vectors, bivector and trivectors, correspond with the common geometrical entities of points, lines, areas and volumes.

Now, while $C\ell(\Re^3)$ provides a natural representation for many common physical laws, for computational processes, such as in the human brain, it is natural to assume that it is able to implement more general higher dimensional Clifford algebras (CA). Therefore we wish to model neural networks (NN) with the multivectors in the larger space of $C\ell(p, q, r)$, which contains p basis vectors that square to $+1$, q multivectors that square to -1 and r basis vectors that square to zero. While NN's typically use the field of real, complex or perhaps quaternionic numbers to describe the values of the inputs, outputs and weights, significant advantages have been found for the Clifford numbers [3], allowing faster and more accurate training. GA is also advantageous for physics informed neural networks (PINN), which incorporate the laws of physics into the neural net at a foundational level [4, 5].

2. The n-dimensional Clifford algebra

A Clifford algebra of dimension n, generates a graded structure with a total of 2^n dimensions, with $n+1$ distinct geometric types. A general multivector can be represented as

$$M = c_0 + c_i e_i + c_{ij} e_i e_j + c_{ijk} e_i e_j e_k \ldots c_{1\ldots n} e_1 \cdots e_n, \tag{1}$$

where $e_1, e_2 ... e_n$ is an n-dimensional orthonormal basis and $c_i, c_{ij} \cdots$ are real scalars. The number of components for each grade r element $e_{1\cdots r}$ is C^n_r. The basis orthogonality is simply enforced by stipulating the anti-commutativity of the basis vectors, $e_i e_j = -e_j e_i$. We typically specify $e_i^2 = 1$ for physical space, but for more general spaces $e_i^2 \in \{-1, 0, +1\}$.

For planar geometry, we can use $C\ell(2,0,1)$. If we select the basis vectors e_1, e_2 with $e_1^2 = e_2^2 = 1$, and e_0, with $e_0^2 = 0$. Rotations about this point allow translations to be written as a rotation operator. For 3D geometry, we can use $C\ell(3,0,1)$. If we select the basis vectors e_1, e_2, e_3 with $e_1^2 = e_2^2 = e_3^2 = 1$, and e_0, with $e_0^2 = 0$. The e_0 basis vector effectively represents a point at infinity. Rotations about this point allow translations to be written as a rotation operator, unifying these two transformations.

Now, the Cartan-Dieudonné theorem states that every orthogonal transformation of a n dimensional space can be decomposed into, at most, n reflections in hyper planes. In CA, the reflection operation is implemented with the sandwich product $M' = \boldsymbol{v} M \boldsymbol{v}$, which reflects the multivector about the vector $\boldsymbol{v}$. With $C\ell(2,0,1)$ we need $n = 3$ reflections and in $C\ell(3,0,1)$, $n = 4$ four reflections. As is well known, two reflections create a rotation, hence this is a general unified form for all orthogonal transformations, reflections, rotations and translations.

3. Transformations in $C\ell(\Re^3)$

We now look at the special case of $C\ell(\Re^3)$, which allow a general set of transformations that embody the Lorentz group, thus generalising the orthogonal transformations. Due to the close connections between invariants and the laws of Nature, we seek the invariants in the space of $C\ell(\Re^3)$, after applying the most general transformation rules.

We firstly define *Clifford conjugation* of a multivector M as

$$\bar{M} = t - \boldsymbol{x} - j\boldsymbol{n} + jb. \tag{2}$$

Clifford conjugation is an involution that is an anti-automorphism, so that for a product MN of two multivectors $M, N \in C\ell(\Re^3)$, $\overline{MN} = \bar{N}\bar{M}$. Clifford conjugation can be written algebraically as $\bar{M} = \frac{1}{2}(-M + e_1 M e_1 + e_2 M e_2 + e_3 M e_3)$. Clifford conjugation is equivalent to a time reversal.

We define the *amplitude squared* of a multivector M through Clifford conjugation, giving the bilinear form

$$|M|^2 = M\bar{M} = t^2 - \boldsymbol{x}^2 + \boldsymbol{n}^2 - b^2 + 2j(tb - \boldsymbol{x}\cdot\boldsymbol{n}) \tag{3}$$

forming a complex-like number $\in \mathbb{C}$, and thus commuting with the rest of the algebra.

We refer to this as a "complex-like" number because the trivector j is analogous to the unit imaginary and all other quantities are real scalars. The square root is therefore well defined from complex number theory and so we can define the multivector *amplitude* as $|M| = \sqrt{|M|^2}$. We can therefore write a norm relation

$$|M_1 M_2| = \pm |M_1||M_2|. \tag{4}$$

We can ensure a positive sign if the appropriate branch is used when finding the complex square roots. We can view this complex distance measure as combining a real distance and a phase, analogous to a distance measurement using a photon. Alternatively, this also naturally describes quantum amplitudes and phases. We thus have a distance measure between two multivectors M_1, M_2, of $|M_1 - M_2|$, making $C\ell(\Re^3)$ a metric space.

We define a general bilinear transformation on a multivector M as

$$M' = KML, \tag{5}$$

where $M, K, L \in C\ell(\Re^3)$. We then find the transformed multivector amplitude

$$|M'|^2 = KML\,\overline{KML} = KML\bar{L}\bar{M}\bar{K} = |K|^2|L|^2|M|^2. \tag{6}$$

We can specify a unitary condition $|K|^2|L|^2 = 1$ for these transformations, so that the amplitude $|M|$ will be invariant. The selection of the involution of Clifford conjugation is not arbitrary, as it is the only involution producing a commuting complex-like number allowing these invariants to form, according to Eq. (5).

For transformations that are continuous with the identity, we can use the power series expansion of the exponential function [6] to produce

$$M' = \mathrm{e}^{\boldsymbol{p}+j\boldsymbol{q}} M \mathrm{e}^{\boldsymbol{r}+j\boldsymbol{s}}. \tag{7}$$

This generalises the conventional Lorentz group, which now appear as special cases. The most general transformation in Eq. (5) also describing reflections and other transformations. If we consider the transformation

$$M' = \mathrm{e}^{j\boldsymbol{v}/2} M \mathrm{e}^{j\boldsymbol{w}/2}, \tag{8}$$

where we have used two distinct rotation axes $\boldsymbol{v}$ and $\boldsymbol{w}$. This operation acts separately on two four-dimensional subspaces $t + j\boldsymbol{n}$ and $\boldsymbol{x} + jb$, with each of the two rotations being isomorphic to a rotation in a four-dimensional Cartesian space.

Now, since $M\bar{M}$ is invariant, then $(A+B)(\overline{A+B})$ is also invariant, where $A, B \in C\ell(\Re^3)$. We have $(A+B)(\overline{A+B}) = A\bar{A} + B\bar{B} + A\bar{B} + B\bar{A}$. Hence, as $A\bar{A}, B\bar{B}$ are known to be invariant, then we can define a multivector dot product with the final two terms

$$A \cdot \bar{B} = \frac{1}{2}\left(A\bar{B} + B\bar{A}\right) = B \cdot \bar{A}. \tag{9}$$

The invariant dot product thus provides a mechanism to combine two distinct multivectors, as in the electromagnetic Lagrangian $A \cdot \bar{J}$, for example.

Now, multivectors formed from a product of two multivectors $\bar{A}B$ transform as

$$\bar{A}'B' = \overline{KAL}KBL = \bar{L}\bar{A}\bar{K}KBL = \bar{L}\bar{A}BL. \tag{10}$$

Hence multivectors formed as a product $F = \bar{A}B$ form a distinct class of multivectors with a distinct transformation law

$$F' = \bar{L}FL. \tag{11}$$

We refer to such quantities as "fields", as we find this transformation applies to the electromagnetic field, for example. We find that the product of two fields $F_1'F_2' = \bar{L}F_1L\bar{L}F_2L = \bar{L}(F_1F_2)L$, also transforms as a field. This implies that polynomials of such multivector fields are also invariant, and so can utilized to approximate some unknown function. The field transformation

$$F' = \mathrm{e}^{-\boldsymbol{r}-j\boldsymbol{s}}F\mathrm{e}^{\boldsymbol{r}+j\boldsymbol{s}}, \tag{12}$$

turns out to be the standard transformation for the electromagnetic field [2]. The transformation incorporates the Lorentz boost transformation, and so will not be grade preserving. Now, the product of a multivector with a field XF will transform the same as a general multivector. That is $X'F' = KXL\bar{L}FL = K(XF)L$. Hence, we can write an invariant equation $XF = Y$, where X, Y transform as multivectors, defined in Eq. (7), and $F = \bar{B}A$ transforms as a field. Hence, in GA, both spacetime and the fields arise from the same abstract background structure $C\ell(\Re^3)$, thus giving a more unified approach to spacetime.

We can write a spacetime event X, in differential form, as

$$dX = dt + d\boldsymbol{x} + jd\boldsymbol{n} + jdb, \tag{13}$$

where the special case $dX = dt + d\boldsymbol{x}$ is isomorphic to the conventional Minkowski four vector $dX = [dt, d\boldsymbol{x}]$. The magnitude of the invariant interval is commonly defined equal to $d\tau^2$, which defines the proper time. Dividing through by this invariant, from Eq. (13), we produce the velocity multivector

$$\begin{aligned} V = \frac{dX}{d\tau} &= \frac{dt}{d\tau} + \frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau} + j\frac{d\boldsymbol{n}}{dt}\frac{dt}{d\tau} + j\frac{db}{dt}\frac{dt}{d\tau} \\ &= \gamma\left(1 + \boldsymbol{v} + j\boldsymbol{w} + jh\right), \end{aligned} \tag{14}$$

where $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$, $\boldsymbol{w} = \frac{d\boldsymbol{n}}{dt}$ and $h = \frac{db}{dt}$. As we defined $|dX|^2 = d\tau^2$ then we have $|V|^2 = \frac{|dX|^2}{d\tau^2} = 1$, a dimensionless number. This leads to the energy-momentum-spin-helicity multivector

$$P = E + \boldsymbol{p} + \boldsymbol{s} + h, \tag{15}$$

each of which component is conserved, with $P\bar{P} = m^2$, also invariant.

3.1. Physical principles encoded by the algebra

From Eq. (13), lightlike particles satisfy the condition $dt^2 - d\boldsymbol{x}^2 = 0$, and so the general condition for null lightlike particles to be

$$\boldsymbol{v} \cdot \hat{\boldsymbol{n}} = \pm c, \tag{16}$$

where for clarity we introduce the speed of light. Hence, due to the nature of the dot product, we can see that it is only satisfied by a velocity $||\boldsymbol{v}|| = c$, parallel to the spin axis $\hat{\boldsymbol{n}}$. That is, based on the eight-dimensional structure of $C\ell(\Re^3)$ alone, we find that a null particle, if traveling at the speed of light c, is required to have its spin axis parallel to its direction of motion, exactly as observed for electromagnetic radiation.

3.2. Projectile motion

GA allows a purely vector based approach to projectile motion. We can form the general expression in GA governing projectile motion

$$2\boldsymbol{a}\boldsymbol{s} = v^2 - u^2 + 2\boldsymbol{v} \wedge \boldsymbol{u}, \tag{17}$$

an equation that relates $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{s}$ and $\boldsymbol{a}$, where $\boldsymbol{a}$ is the acceleration due to gravity vector and $\boldsymbol{s}$ is the vector to the target allowing sloping ground and $\boldsymbol{u}, \boldsymbol{v}$ are the initial and final velocity vectors, respectively. For energy efficient trajectories we can find directly an expression for the initial velocity vector

$$\boldsymbol{u} = \hat{\boldsymbol{s}}\sqrt{\frac{as}{2}} - \hat{\boldsymbol{a}}\sqrt{\frac{as}{2}}. \tag{18}$$

3.3. Null fields

$C\ell(\Re^3)$ also can naturally describe free fields. Using $F = \nabla \wedge \boldsymbol{A}$, we define $A = \alpha \nabla \beta$, in terms of two complex potentials α and β [7]. This then automatically satisfies Maxwell's equation without sources. We only require two complex scalar functions α, β to define the electric $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, as we have two less degrees of freedom because of the constraint $\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{E} = 0$. The condition for divergenceless flow of the field. Importantly, it has been shown that Bateman's construction can describe all possible null electromagnetic fields [8]. This then gives

$$F = \nabla \wedge \boldsymbol{A} = \nabla\alpha \wedge \nabla\beta. \tag{19}$$

This approach allows radiating fields to be defined with conserved energy and helicity.

3.4. The action

The invariant distance provides a suitable action integral

$$S = \int |dX|, \tag{20}$$

where the distance $|dX|$ is given by the amplitude of the spacetime multivector, given by Eq. (3). Now, as shown previously, with the assumption of a proper time in a rest frame we have $|dX| = d\tau$ and so we have the spacetime distance

$$|dX|^2 = \left(\dot{t}^2 - \dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{n}}^2 - \dot{b}^2\right) d\tau^2, \tag{21}$$

where we define $\dot{t} = \frac{dt}{d\tau}$, $\dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{d\tau}$, $\dot{\boldsymbol{n}} = \frac{d\boldsymbol{n}}{d\tau}$ and $\dot{b} = \frac{db}{d\tau}$. We can then write the action as $S = \int \frac{|dX|}{d\tau} d\tau$ that implies a Lagrangian

$$\mathcal{L} = \frac{|dX|}{d\tau} = |V| = \sqrt{\dot{t}^2 - \dot{\boldsymbol{x}}^2 + \dot{\boldsymbol{n}}^2 - \dot{b}^2} = 1, \tag{22}$$

where we now extremize $S = \int \mathcal{L} d\tau$.

Using the Euler-Lagrange equation we find the four fundamental conservation laws for inertial particles, of energy, momentum, spin and helicity are reproduced [9]. A simple extension of this Lagrangian is $\mathcal{L} = |V+U|$, where the multivector U conceptually represents a 'flow' in the background spacetime, perturbing particle inertial motion V. We can also add the known invariant of $V \cdot \bar{A}$ to the Lagrangian. We can thus produce a generalised Lagrangian

$$\mathcal{L} = \frac{1}{2}|V + U|^2 + A \cdot \bar{V}, \tag{23}$$

describing a particle moving in a electromagnetic potential A. We naturally incorporate physical laws through using the action of a physical process [1].

The formalism of Clifford's geometric algebra (GA) of three dimensions $C\ell(\Re^3)$ is thus a natural framework to describe spacetime, and physical laws, and provides insights into their fundamental nature.

References

[1] Chappell J M, Hartnett J G, Iannella N, Iqbal A, Berkahn D L and Abbott D 2023 *Journal of Physics Communications* **7** 065001

[2] Chappell J M, Drake S P, Seidel C L, Gunn L J, Iqbal A, Allison A and Abbott D 2014 *Proceedings of the IEEE* **102** 1340–1363

[3] Ruhe D, Gupta J K, De Keninck S, Welling M and Brandstetter J 2023 *arXiv preprint arXiv:2302.06594*

[4] Lagaris I, Likas A and Fotiadis D 1998 *IEEE Transactions on Neural Networks* **9** 987–1000

[5] Raissi M, Perdikaris P and Karniadakis G E 2019 *Journal of Computational physics* **378** 686–707

[6] Hestenes D 1966 *Spacetime Algebra* (New York: Gordon and Breach)

[7] Bateman H 1915 *The Mathematical Analysis of Electrical and Optical Wave-motion on the Basis of Maxwell's Equations: By H. Bateman...* vol 16 (University Press)

[8] Hogan P 1984 *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **396** 199–204

[9] Lehmkuhl D 2014 *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* **46** 316–326