Characteristic multivectors of Coxeter transformations give novel insights into the geometry of root systems

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Abstract. There has been increased recent interest in novel Clifford geometric invariants of linear transformations. This motivates the investigation of such invariants for a certain type of geometric transformation of interest in the context of symmetry structures: the Coxeter transformations. We calculate the invariants for the bipartite Coxeter transformations for A_8, D_8, E_8 and A_6, D_6, E_6. We focus on bivector invariants in particular, and shed new light on the relationships with other well-known invariant planes, including the Coxeter plane, as well as recent work into orthogonal decomposition. I will also briefly present results from our recent paper that calculated invariants of all Coxeter elements exhaustively and analysed the resulting computational algebra dataset using data science techniques such as Neural Networks and Principal Component Analysis.

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1. Introduction

Recent work [5,6] has sparked great interest in Clifford geometric invariants of linear transformations, originally proposed in [4]. Orthogonal transformations, such as rotations, and their invariants are important in engineering, e.g. moving cameras, robots etc. The Coxeter transformations we are looking at in this work are also rotations, but in this context they are particularly interesting because of their symmetry structures. In Clifford algebras, algebraic objects have a clearer geometric interpretation than in the standard matrix approach. There is a systematic way of calculating multivector invariants of linear transformations via what are called 'simplicial derivatives'. These Clifford geometric invariants are then systematically related to geometric invariant spaces of the linear transformation and the coefficients in the characteristic polynomial and Cayley-Hamilton theorem. The decomposition of a linear transformation into orthogonal eigenspaces is also related to some interesting recent work by [7], the connection with which we will also explore. Coxeter elements have well-known invariant planes, and we will compare the bivectors found as Clifford invariants to the bivectors describing these eigenplanes. I will briefly present some results from a recent paper [2] that exhaustively computed the invariants for all such Coxeter transformations and explored the resulting data set using data science techniques.

2. Background

Thorough introductions to root systems and Clifford algebras are available elsewhere [3] so here we will be succinct. A root system lives in the arena of a vector space with a scalar product (which immediately

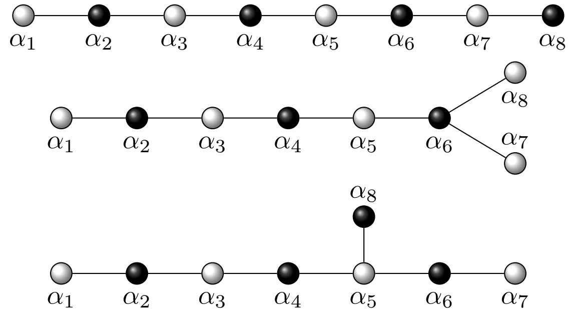

FIGURE 1. The diagrams of the 8-dimensional simply-laced root systems A_8, D_8 and E_8 (vertically downwards respectively), along with our labelling for the simple roots and a bipartite colouring.

allows one to consider the corresponding Clifford algebra). It is a collection of vectors (called 'roots', and customarily denoted α) in that vector space which is invariant under all the reflections in the hyperplanes to which the root vectors are perpendicular. We will only consider root systems with roots of the same length, which can be assumed to be normalised [1]. Such reflections in the normal hyperplanes are given by $x \to x - 2(x \cdot n)n$, where x is the vector to be transformed and n is a unit normal to the hyperplane.

A subset called 'simple roots' is sufficient to write all roots as (in our case) integer linear combinations of this basis of simple roots, whilst their corresponding reflections, the 'simple reflections', generate the reflection group. Taking these simple reflections all exactly once leads to interesting types of group elements called 'Coxeter elements'. They are of the same order h (the 'Coxeter number'), and have invariant planes, called 'Coxeter planes', which are useful for visualising root systems in any dimension (via projection into these planes). These reflection groups have interesting integer – in fact prime – invariants, that are characteristic of the geometry, called 'exponents' m. This name derives from the fact that Coxeter elements act on different invariant planes by h-fold rotations by m times $2\pi/h$, which is usually interpreted as a complex eigenvalue of the Coxeter element (even though we are by assumption in a real vector space). The root system geometry can also be encoded in diagrammatic form (called 'Coxeter-Dynkin diagram'), where each simple root corresponds to a node and orthogonal nodes are not linked, whilst roots at $2\pi/3$ angles are connected with a link (we will only be considering such 'simply-laced' examples, see Fig. 1). Likewise, our simply-laced examples are tree-like and admit an alternate colouring (or 'bipartite', e.g. black and white). This effectively means that all black roots are orthogonal to each other, and likewise for the white roots. This colouring means that there are distinguished types of Coxeter elements where first all the black reflections are taken, and then all the white (or the other way round). We will call these 'bipartite' Coxeter elements. This bipartite colouring also implies the existence of the Coxeter plane via a more complex argument, the details of which we will omit here, but which relies on the adjacency matrix of the Dynkin diagram having a distinguished largest eigenvalue and corresponding eigenvector, the Perron-Frobenius eigenvector (which *will* make an appearance below). In our labelling of the 8 simple roots for A_8, D_8 and E_8, α_1 to α_7 make one long string. The different diagrams arise depending on where the 8th root α_8 attaches: at the terminal node α_7 for A_8 (leading to bilateral symmetry), at the penultimate node α_6 for D_8 (leading to permutation symmetry of the terminal nodes), or α_5 for E_8.

As mentioned above, Clifford algebras can be constructed when one is working in an n-dimensional vector space with an inner product, giving rise to a 2^n-dimensional algebra of 'multivectors'. The scalar product is given as the symmetric part of the geometric product, i.e. $a \cdot b = \frac{1}{2}(ab + ba)$. The outer product $a \wedge b = \frac{1}{2}(ab - ba)$ is the antisymmetric part, is a bivector and determines the plane that two vectors generically span. Substituting this in the reflection formula above results in a cancellation

[1]Note this is different from the normalisation convention used in Lie theory

which leads to the uniquely simple 'sandwiching' reflection formula in Clifford algebras

$$x \to x - 2(x\cdot n)n = -nxn. \tag{2.1}$$

Both n and $-n$ doubly cover the same reflection. Via the Cartan-Dieudonné theorem orthogonal transformations are just products of such reflections so that one can build up

$$x \to \pm n_k \cdots n_1 x n_1 \cdots n_k = \pm \tilde{A} x A \tag{2.2}$$

such transformations via defining multivectors that are the products of normal vectors which encode the reflection hyperplanes, $A = n_1 \cdots n_k$ (called 'versors'), and a tilde denotes reversing the order of these vectors in the product. These versors again doubly cover the transformation.

We discuss here for a moment how this applies when the orthogonal transformation is a Coxeter element. In traditional root system notation, the simple reflections are denoted s_i such that a Coxeter element is denoted $w = s_1 \cdots s_n$. In the above versor framework, the reflections are encoded by the root vectors themselves (as a double cover), whilst the multivectors W that one gets from multiplying the simple roots together $\alpha_1 \cdots \alpha_n$ doubly cover w

$$wx \to \pm \alpha_k \cdots \alpha_1 x \alpha_1 \cdots \alpha_k = \pm \tilde{W} x W. \tag{2.3}$$

We return now to the setting of linear transformations in Clifford algebras more generally again. Let us denote this linear transformation by $f(x)$. In order to calculate the desired invariants of this linear transformation, we define the concept of 'simplicial derivatives'.

First, let $\{a_k\}, k = 1, \ldots, n$ denote a frame, i.e. a basis. Often we use either a Euclidean basis e_i or the basis of simple roots, α_i. We denote by $\{a^k\}$ its reciprocal frame such that $a^i \cdot a_j = \delta^i_j$. In a Euclidean basis this is effectively the basis itself; for a basis of simple roots the reciprocals are more commonly known as co-roots (up to a different conventional normalisation factor). We also define $b_k = f(a_k)$ as the transformation acting on the basis frame vectors. The rth simplicial derivative is then essentially defined as a combinatorial object

$$\partial_{(r)} f_{(r)} = \sum (a^{j_r} \wedge \cdots \wedge a^{j_1})(b_{j_1} \wedge \cdots \wedge b_{j_r}) \tag{2.4}$$

with sum over $0 < j_1 < \cdots < j_r \leq n$[2]. These simplicial derivatives are invariants of the linear transformation and are therefore 'characteristic multivectors' with geometric significance.

Now [4] showed that it is the scalar parts of these geometric invariants (denoted by $\partial_{(s)} * f_{(s)}$) that constitute the coefficients in the Cayley-Hamilton theorem

$$C_f(\lambda) = \sum_{s=0}^{m} (-\lambda)^{m-s} \partial_{(s)} * f_{(s)}$$

(where $\partial_{(0)} * f_{(0)}$ is interpreted as 1) and the characteristic polynomial

$$\sum_{s=0}^{m} (-1)^{m-s} \partial_{(s)} * f_{(s)} f^{m-s}(a) = 0$$

for any vector a (where $f^0(a)$ is interpreted as a).

One can explicitly perform these calculations for our examples uing the `galgebra` package, calculating Coxeter versors from the simple roots, and from that simplicial derivatives and geometric invariants. We will refer to the simplicial derivatives $\partial_{(r)} f_{(r)}$ as the invariant of order r or Inv_r. For our examples, the different grades of each invariant, which we could denote by Inv_r^k, are separately invariant under the Coxeter versor: $\tilde{W}\,\mathrm{Inv}_r^k\, W = \mathrm{Inv}_r^k$. So these Inv_r^k are eigenmultivectors of the Coxeter element of grade k, but they do not have to be k-blades (i.e. be able to be written as the outer product of k vectors[3]). In this work, we are particularly interested in the invariant bivector parts that arise thus. So amongst other multivector components, e.g. for E_8 we in particular have 4 invariant bivectors from the invariants. It turns out that these have vanishing commutator product.

[2]This is due to the original notion of a multivector derivative essentially being equivalent to a projection.
[3]Something also noticed in the example in [6].

We are interested in these in particular because of the following well-known invariant planes of the Coxeter element.

The Coxeter element is known to have an eigenplane, called the Coxeter plane, that one can construct by using the Perron-Frobenius eigenvector of the Cartan matrix. For our E_8 example it turns out that one can use any eigenvector of the Cartan matrix and likewise construct invariant plance. The Coxeter element thus acts on 4 invariant orthogonal bivectors (giving planes, and they are blades by construction). The way it acts in each plane has deep connections with some integer characteristic invariant of each symmetry structure: it rotates by some prime number of 'notches' $2\pi/h$, called the exponents. So there is an immediate question of how our characteristic bivectors relate to Coxeter bivectors, exponents and degrees. In fact, we will say here already that for E_8 one can show that the two sets of 4 eigenbivectors (from the simplicial derivatives and the Coxeter construction) span the same 4d-subspace of the 28d bivector space. So the two sets appear to be linear combinations of each other, but one set is properly orthogonal (Coxeter planes), whereas the other set only has vanishing commutator product (invariant bivectors). One is therefore naturally led to the question of how one can orthogonalise sets of bivectors.

This is a question that has been addressed in [4] and more recently in [7]. If one has a bivector B one wishes to decompose into orthogonal pieces b_i, one can essentially find the size of these pieces $|b_i|$ via the characteristic polynomial

$$0 = \sum_{m=0}^{k} \langle W_m^2 \rangle_0 (-\lambda_i)^{k-m}$$

$$\text{where } W_m = \frac{1}{m!}\langle B^m \rangle_{2m} = \frac{1}{m!} B \wedge B \wedge \cdots \wedge B$$

The orthogonal bivector pieces b_i can be found as

$$b_i = \begin{cases} \frac{\lambda_i^r W_0 + \lambda_i^{r-1} W_2 + \cdots + W_k}{\lambda_i^{r-1} W_1 + \lambda_i^{r-2} W_3 + \cdots + W_{k-1}} & k \text{ even} \\ \frac{\lambda_i^r W_1 + \lambda_i^{r-1} W_3 + \cdots + W_k}{\lambda_i^r W_0 + \lambda_i^{r-1} W_2 + \cdots + W_{k-1}} & k \text{ odd} \end{cases}$$

3. Results

We perform some explicit calculations for some ADE-type root systems.

3.1. E_8 (exponents 1, 7, 11, 13, 17, 19, 23, 29)

For E_8, we in consider the 4 invariant bivectors from the invariants. We omit printing them here but we calculate their characteristic polynomials.

$$Inv_{(2)}^{(1)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 8\lambda^3 + 14\lambda^2 + 7\lambda + 1$$

$$Inv_{(2)}^{(3)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 28\lambda^3 + 134\lambda^2 + 92\lambda + 1$$

One can show that the basis of the invariants spans the same 4-dimensional subspace of the 28-dimensional bivector space as the Coxeter bivectors. We write the linear combinations numerically here for succinctness (for some ordering).

$$-Inv_{(2)}^{(1)} = 1.98904 B_C + 0.415823 B_2 + 0.81347 B_3 + 1.4862 B_4$$

$$-Inv_{(2)}^{(2)} = -2.40486 B_C - 1.22929 B_2 + 0.67281 B_3 + 0.502754 B_4$$

$$-Inv_{(2)}^{(3)} = -1.4862 B_C + 1.98904 B_2 + 0.41582 B_3 - 0.813473 B_4$$

$$-Inv_{(2)}^{(4)} = 4.70463B_C - 2.2460B_2 + 0.90040B_3 - 0.105104B_4$$

One can find exact expressions in terms of eigenvectors of the Cartan matrix e.g.

$$-Inv_{(2)}^{(1)} = 2\cos\frac{\pi}{30}B_C + 2\cos\frac{13\pi}{30}B_2 + 2\cos\frac{11\pi}{30}B_3 + 2\cos\frac{7\pi}{30}B_4$$

$$-Inv_{(2)}^{(3)} = -2\cos\frac{7\pi}{30}B_C + 2\cos\frac{\pi}{30}B_2 + 2\cos\frac{13\pi}{30}B_3 - 2\cos\frac{11\pi}{30}B_4$$

which are explicitly in terms of the Coxeter number and the characteristic exponents. The sums of squares of these coefficients also add to $7, 8, 7, 28$, which is the first term in characteristic polynomials. This is just the size of the b_i, as one would expect from [4].

3.2. D_8 (exponents 1, 3, 5, 7, 7, 9, 11, 13)

One can find analogous results for D_8:

$$Inv_{(2)}^{(1)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 8\lambda^3 + 14\lambda^2 + 7\lambda + 1$$

$$Inv_{(2)}^{(3)}: \quad \lambda^4 + 7\lambda^3 + 14\lambda^2 + 8\lambda + 1$$

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 28\lambda^3 + 134\lambda^2 + 92\lambda + 1$$

Again, exact expressions of the linear combinations involve the exponents

$$-Inv_{(2)}^{(1)} = 2\cos\frac{\pi}{30}B_C + 2\cos\frac{13\pi}{30}B_2 + 2\cos\frac{11\pi}{30}B_3 + 2\cos\frac{7\pi}{30}B_4$$

$$-Inv_{(2)}^{(3)} = -2\cos\frac{7\pi}{30}B_C + 2\cos\frac{\pi}{30}B_2 + 2\cos\frac{13\pi}{30}B_3 - 2\cos\frac{11\pi}{30}B_4$$

and again the sums of squares of these coefficients add to first term in characteristic polynomials $7, 8, 7, 28$ (since the Coxeter blades are orthogonal).

3.3. A_8 (exponents 1, 2, 3, 4, 5, 6, 7, 8)

For A_8, the invariant polynomials are

$$Inv_{(2)}^{(1)}: \quad \lambda^4 + 9\lambda^3 + 27\lambda^2 + 30\lambda + 9$$

$$Inv_{(2)}^{(2)}: \quad \lambda^4 + 18\lambda^3 + 81\lambda^2 + 27\lambda$$

$$Inv_{(2)}^{(3)}: \quad \lambda^4 + 27\lambda^3 + 54\lambda^2 + 27\lambda$$

$$Inv_{(2)}^{(4)}: \quad \lambda^4 + 36\lambda^3 + 126\lambda^2 + 84\lambda + 9$$

and as expected the exponents feature in the linear combination

$$-Inv_{(2)}^{(1)} = 2\cos\frac{3\pi}{18}B_1 - 2\cos\frac{7\pi}{18}B_2 + 2\cos\frac{1\pi}{18}B_3 - 2\cos\frac{5\pi}{18}B_4$$

and the bivector norms b_i are $9, 18, 27, 36$ as given by the first term in characteristic polynomials.

3.4. E_6 (exponents 1, 4, 5, 7, 8, 11)

Calculations in 6D for the ADE cases works as expected

$$\lambda^3 + 5\lambda^2 + 7\lambda + 3$$
$$\lambda^3 + 8\lambda^2 + 4\lambda$$
$$\lambda^3 + 17\lambda^2 + 43\lambda + 3$$

and again one can find exact expressions of the invariant bivectors in terms of the Coxeter blade construction.

$$-Inv^{(1)}_{(2)} = 2\cos\frac{2\pi}{12}B_1 + 2\cos\frac{4\pi}{12}B_2 + 2\cos\frac{4\pi}{12}B_3$$

$$-Inv^{(2)}_{(2)} = (-1 + 2\cos\frac{2\pi}{12})B_2 + (-1 - 2\cos\frac{2\pi}{12})B_3$$

$$-Inv^{(3)}_{(2)} = -2\cos\frac{2\pi}{12}B_1 + (2 - 2\cos\frac{2\pi}{12})B_2 + (2 + 2\cos\frac{2\pi}{12})B_3$$

We again notice that the sum of the squares of these coefficients give exactly the first non-trivial coefficients in the characteristic polynomial: 5, 8 and 17.

4. Conclusion

This work ties together three pieces of previous work on 1) geometric invariants (Coxeter) 2) Clifford invariants/characteristic multivectors and 3) bivector orthogonalisation. The concrete calculations above show that indeed the Clifford invariants span exactly the same bivector subspace as the Coxeter blades, with the exponents featuring in the coefficients of the linear combinations. Bivector decomposition (of invariants) into orthogonal blades provides an alternative view of the usual Coxeter construction. The final paper will flesh out some more details and also present some results from the recent computational algebra and data science paper that calculated Coxeter elements and their invariants for all simple root permutations explicitly.

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